Problems 04

Due: Friday, 30 January 2026 before 23:59 ET

P4.1. Let K be a field and fix an irreducible polynomial $f \in K[x]$. Consider a nonzero coset $g + \langle f \rangle$ in the quotient ring $L := K[x] \big/ \langle f \rangle$.

- **i.** Show that f and g are relative prime and deduce that there exists $p, q \in K[x]$ such that $pf + qg = 1$.
- **ii.** Show that $q + \langle f \rangle$ is the multiplicative inverse of $g + \langle f \rangle$ in L.
- **iii.** Find a multiplicative inverse for $(1+x)+\langle x^2 + x + 1\rangle$ in the field $\mathbb{Q}[x]/\langle x^2 + x + 1\rangle$.

P4.2. Prove that the following are equivalent:

- ($\mathbb{C}$) Every nonconstact polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.
- ($\mathbb{R}$) Every nonconstant polynomial with coefficients in $\mathbb{R}$ is a product of linear and quadratics factors with real coefficients.

P4.3. Let α be a root of the irreducible polynomial $x^3 + 7x + 1 \in \mathbb{Q}[x]$.

- **i.** Find the minimal polynomial for $\alpha + 3$ over $\mathbb{Q}$.
- **ii.** Find the minimal polynomial for $\alpha^2 + 1$ over $\mathbb{Q}$.
- **iii.** Find the minimal polynomial for $\alpha^2 - 2\alpha + 3$ over $\mathbb{Q}$.