Problems 06

Due: Friday, 13 February 2026 before 23:59 ET

P6.1. Suppose that the elements α and β are algebraic over the field K having minimal polynomials $f \in K[x]$ and $g \in K[x]$ respectively. Prove that f is irreducible over $K(\beta)$ if and only if g is irreducible over $K(\alpha)$.

P6.2. **i.** Prove that $\left[\overline{\mathbb{Q}} : \mathbb{Q}\right] = \infty$.

ii. Hermite (1874) establishes that the real number e is transcendental over $\mathbb{Q}$, and Lindermann (1882) shows that the real number π is transcendental over $\mathbb{Q}$. It is unknown whether $\pi + e$ and $\pi - e$ are transcendental. Prove that at least one of these numbers is transcendental over $\mathbb{Q}$.

P6.3. Let $\mathbb{F}_3 := \mathbb{Z}\big/\langle 3\rangle$ be the field with three elements and consider $f := x^3 - x + 1 \in \mathbb{F}_3[x]$.

i. Show that f is irreducible over $\mathbb{F}_3$.

ii. Let L be the splitting field of f over $\mathbb{F}_3$. Prove that $[L : \mathbb{F}_3] = 3$.

iii. Explain why L is a field with 27 elements.