Problems 09

Due: Friday, 13 March 2026 before 23:59 ET

P9.1. For each of the following field extensions, determined whether it is Galois.

- **i.** $\mathbb{Q} \subset \mathbb{Q}(\sqrt{7}, \sqrt[3]{7})$.
- **ii.** $\mathbb{Q} \subset \mathbb{Q}(\alpha, \beta)$ where α and β are distinct roots of $x^3 + x^2 + 2x + 1$.
- **iii.** $\mathbb{F}_p(t^p) \subset \mathbb{F}_p(t)$ where p is a prime integer and t is an indeterminant.
- **iv.** $\mathbb{C}(t + t^{-1}) \subset \mathbb{C}(t)$ where t is an indeterminant.

P9.2. Consider the field $\mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{4})$.

- **i.** Find the minimal polynomial of $\sqrt[3]{2} + \sqrt[3]{4}$ over $\mathbb{Q}$.
- **ii.** Show that the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{4})$ is not Galois.
- **iii.** Determine the Galois closure of the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{4})$.

P9.3. Consider the intermediate field $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2} + \sqrt{5}) \subset \overline{\mathbb{Q}}$.

- **i.** Find the minimal polynomial of $\sqrt{2} + \sqrt{5}$ over $\mathbb{Q}$.
- **ii.** Determine the corresponding conjugate fields.
- **iii.** Show that all conjugate fields are equal.