Problems 10

Due: Friday, 20 March 2026 before 23:59 ET

P10.1. Consider the field extension $K := \mathbb{C}(t^4) \subset L := \mathbb{C}(t)$ where t is an indeterminant.

- **i.** Show that L is the splitting field of $x^4 - t^4 \in K[x]$ over K.
- **ii.** Show that $x^4 - t^4$ is irreducible over K.
- **iii.** Show that $\mathrm{Gal}(L/K) \cong \mathbb{Z} \big/ \langle 4 \rangle$.
- **iv.** Determine all the subgroups of $\mathrm{Gal}(L/K)$ and the corresponding intermediate fields between K and L.

P10.2. Let K be a field of characteristic different from 2. For every finite field extension $K \subset L$, prove that the following are equivalent:

- **(a)** The field extension $K \subset L$ is Galois with $\mathrm{Gal}(L/K) \cong \mathbb{Z}/\langle 2 \rangle \times \mathbb{Z}/\langle 2 \rangle$.
- **(b)** The field L is the splitting field of a polynomial $(x^2 - a)(x^2 - b)$ where $a, b \in K$ but $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{ab}$ do not lie in K.

P10.3. Compute the Galois groups of the following cubic polynomials.

- **i.** $x^3 - 4x + 2$ over $\mathbb{Q}$.
- **ii.** $x^3 - 4x + 2$ over $\mathbb{Q}(\sqrt{37})$.
- **iii.** $x^3 - 3x + 1$ over $\mathbb{Q}$.
- **iv.** $x^3 - t$ over $\mathbb{C}(t)$ where t is an indeterminant.
- **v.** $x^3 - t$ over $\mathbb{Q}(t)$ where t is an indeterminant.