Problems 11

Due: Friday, 27 March 2026 before 23:59 ET

P11.1. **i.** Show that the subset $H := \{\mathrm{id}_{[n]}, (2\ \ 1)(4\ \ 3), (3\ \ 1)(4\ \ 2), (3\ \ 2)(4\ \ 1)\}$ is a normal subgroup of the symmetric group $\mathfrak{S}_4$.

ii. Show that the alternating group A_4 and the symmetric group $\mathfrak{S}_4$ are solvable.

P11.2. Let L be the splitting field of $x^3 + x^2 - 2x - 1$ over $\mathbb{Q}$, and set $\zeta := \exp(2\pi\mathrm{i}/7) \in \mathbb{C}$.

i. Verify that the roots of x^3+x^2-2x-1 are $2\cos(2\pi j/7) := \zeta^j + \zeta^{-j}$ for all $1 \leqslant j \leqslant 3$.

ii. Confirm that $\mathbb{Q} \subset L \subset \mathbb{Q}(\zeta)$ and explain why $\mathbb{Q} \subset \mathbb{Q}(\zeta)$ is radical.

P11.3. Consider finite field extensions $K \subseteq M \subseteq L$ where $K \subseteq M$ is radical. For every $\sigma \in \mathrm{Gal}(L/K)$, prove that the field extension $K \subseteq \sigma(M)$ is also radical.