Problems 12

Due: Friday, 3 April 2026 before 23:59 ET

P12.1. Let f be the minimal polynomial of $\alpha := \sqrt[5]{\sqrt[3]{17} + \sqrt[4]{37}} \in \mathbb{R}$ over $\mathbb{Q}$, where all of the indicated radicals are real. Prove that f is solvable by radicals over $\mathbb{Q}$.

P12.2. Determine the cyclotomic polynomial $\Phi_{105} \in \mathbb{Q}[x]$.

P12.3. Let K be an algebraically closed field of positive characteristic p, and let $M := K(t)$ where t is an indeterminant. Consider $f := x^p - x + t \in M[x]$ and let L be a splitting field of f over M. Fix a root $\alpha \in L$ of the polynomial f.

i. Show that the roots of f are $\alpha, \alpha + 1, \ldots, \alpha + p - 1$.

ii. Prove that there exists a group isomorphism

$$\mathrm{Gal}(L/M) \to \frac{\mathbb{Z}}{\langle p \rangle}$$

such that the automorphism $\sigma \in \mathrm{Gal}(L/M)$ maps to the class $[j]_p \in \mathbb{Z} \big/ \langle p \rangle$ if and only if $\sigma(\alpha) = \alpha + j$.