Solutions 02

P2.1. Consider the cubic equation $x^3 + x - 2 = 0$.

i. Use Cardano's formulas (carefully) to derive the surprising formula

$$1 = \sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}} + \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}}\,.$$

ii. Show that

$$1 + \frac{2}{3}\sqrt{\frac{7}{3}} = \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{7}{3}}\right)^3,$$

and use this to explain part **i**.

Solution.

i. This reduced cubic as a unique real root because its discriminant is negative:

$$\Delta = -4(1^3) - 27(-2)^2 = -112 < 0\,.$$

Moreover, the equation $(1)^3 + (1) - 2 = 0$ shows that this real root equals 1.

On the other hand, the product of the real cube roots

$$z_1 = \sqrt[3]{\frac{1}{2}\left(2 + \sqrt{4 + \frac{4(1)^3}{27}}\right)} = \sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}}\,,$$

$$z_2 = \sqrt[3]{\frac{1}{2}\left(2 - \sqrt{4 + \frac{4(1)^3}{27}}\right)} = \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}}\,,$$

is $1/3 = p/3$, so Cardano's formula shows that the unique real root of this reduced cubic is

$$1 = \sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}} + \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}}\,.$$

ii. The binomial theorem gives

$$\begin{aligned}\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{7}{3}}\right)^3 &= \left(\frac{1}{2}\right)^3 \pm 3\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\sqrt{\frac{7}{3}}\right) + 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\sqrt{\frac{7}{3}}\right)^2 \pm \left(\frac{1}{2}\sqrt{\frac{7}{3}}\right)^3 \\ &= \frac{1}{8}\left(1 \pm 3\sqrt{\frac{7}{3}} + 7 \pm \frac{7}{3}\sqrt{\frac{7}{3}}\right) = \frac{1}{8}\left(8 \pm \frac{16}{3}\sqrt{\frac{7}{3}}\right) = 1 \pm \frac{2}{3}\sqrt{\frac{7}{3}}\,.\end{aligned}$$

It follows that

$$\sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}} + \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}} = \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{7}{3}}\right) + \left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{7}{3}}\right) = 1\,.$$

□

P2.2. **i.** Show that $\sqrt[3]{4 + \mathrm{i}\sqrt{11}} \in \mathbb{C}$ is not of the form $a + \mathrm{i}b\sqrt{11}$ for some $a, b \in \mathbb{Z}$.

ii. Find a cubic polynomial of the form $y^3 + py + q$ with $p, q \in \mathbb{Z}$ which has

$$\sqrt[3]{4 + \mathrm{i}\sqrt{11}} + \sqrt[3]{4 - \mathrm{i}\sqrt{11}}$$

as a root.

Solution.

i. Suppose that there exists $a, b \in \mathbb{Z}$ such that $4+\mathrm{i}\sqrt{11} = \left(a + \mathrm{i}b\sqrt{11}\right)^3$. The binomial theorem would give

$$\begin{aligned}\left(a + \mathrm{i}b\sqrt{11}\right)^3 &= a^3 + 3a^2\left(\mathrm{i}b\sqrt{11}\right) + 3a\left(\mathrm{i}b\sqrt{11}\right)^2 + \left(\mathrm{i}b\sqrt{11}\right)^3 \\ &= (a^3 - 33ab^2) + \mathrm{i}\sqrt{11}(3a^2b - 11b^3) = 4 + \mathrm{i}\sqrt{11}\,.\end{aligned}$$

It would follow that $4 = a^3 - 33ab^2 = a(a^2 - 33b^2)$ and $1 = b(3a^2 - 11b^2)$. Since a and b are integers, we would deduce that a divides 4 and b divides 1. In particular, we have a is either ± 1, ± 2, or ± 4, and $b = \pm 1$. Hence, the integer $a^2 - 33b^2$ would also divide 4 and satisfy $a^2 - 33b^2 = a^2 - 33 \leqslant 16 - 33 \leqslant -17$ which is absurd. We conclude that the $4 + \mathrm{i}\sqrt{11} = (a + \mathrm{i}b\sqrt{11})^2$ has no solutions with $a, b \in \mathbb{Z}$.

ii. Consider the reduced cubic polynomial $y^3 - 9y - 8 \in \mathbb{Z}[y]$. The product of the cube roots

$$\begin{aligned} z_1 &= \sqrt[3]{\frac{1}{2}\left(8 + \sqrt{64 + \frac{4(-9)^3}{27}}\right)} = \sqrt[3]{4 + \sqrt{16 - 27}} = \sqrt[3]{4 + \mathrm{i}\sqrt{11}}\,, \\ z_2 &= \sqrt[3]{\frac{1}{2}\left(8 - \sqrt{64 + \frac{4(-9)^3}{27}}\right)} = \sqrt[3]{4 - \sqrt{16 - 27}} = \sqrt[3]{4 - \mathrm{i}\sqrt{11}}\,, \end{aligned}$$

is $-3 = -9/3$, so Cardano's formula shows a root of this reduced cubic is

$$z_1 + z_2 = \sqrt[3]{4 + \mathrm{i}\sqrt{11}} + \sqrt[3]{4 - \mathrm{i}\sqrt{11}}\,.$$

□

P2.3. Consider the reduced cubic polynomial $y^3 + py + q$ with real coefficients. Assume that its discriminant is positive: $\Delta := -(4p^3 + 27q^2) > 0$.

i. Explain why $p < 0$.

ii. For a positive real number λ, the substitution $y = \lambda t$ transforms the reduced cubic equation into $\lambda^3 t^3 + \lambda p t + q = 0$, which can be expressed as

$$4t^3 - \left(\frac{-4p}{\lambda^2}\right)t - \left(\frac{-4q}{\lambda^3}\right) = 0\,.$$

Show that this coincides with $4t^3 - 3t - \cos(3\theta) = 0$ if and only if

$$\lambda = 2\sqrt{\frac{-p}{3}} \qquad \text{and} \qquad \cos(3\theta) = \frac{3\sqrt{3}q}{2p\sqrt{-p}}\,.$$

iii. Prove that

$$\left|\frac{3\sqrt{3}q}{2p\sqrt{-p}}\right| < 1\,.$$

iv. Explain how part **iii** implies that the last equation in part **ii** can be solved for θ.

v. Show that $4t^3 - 3t - \cos(3\theta)$ has roots $\cos(\theta)$, $\cos(\theta + \frac{2\pi}{3})$, and $\cos(\theta + \frac{4\pi}{3})$.

vi. Show that the roots of $y^3 + py + q$ are

$$y_1 = 2\sqrt{\frac{-p}{3}}\cos(\theta)\,, \qquad y_2 = 2\sqrt{\frac{-p}{3}}\cos(\theta + \tfrac{2\pi}{3})\,, \qquad y_3 = 2\sqrt{\frac{-p}{3}}\cos(\theta + \tfrac{4\pi}{3})\,.$$

Solution.

i. Since the square of any real number is nonnegative, we see that $q^2 \geqslant 0$. The hypothesis $\Delta > 0$ implies that $-4p^3 = \Delta + 27q^2 > 0$, so $p^3 < 0$. The function $x \mapsto x^3$ is increasing when $x < 0$ because its derivative $3x^2$ is positive. We deduce that $p < 0$.

ii. Comparing coefficients gives

$$\frac{-4p}{\lambda^2} = 3 \qquad \Rightarrow \qquad \lambda = \sqrt{\frac{4(-p)}{3}} = 2\sqrt{\frac{-p}{3}}\,,$$

$$\frac{-4q}{\lambda^3} = \cos(3\theta) \qquad \Rightarrow \qquad \cos(3\theta) = \frac{-4q}{\lambda^3} = \left(\frac{-4q}{1}\right)\left(\frac{3}{-4p}\right)\left(\frac{1}{2}\sqrt{\frac{-p}{3}}\right) = \frac{3\sqrt{3}q}{2p\sqrt{-p}}\,.$$

iii. The assumption that $\Delta > 0$ implies that $4p^3 + 27q^2 < 0$ and $0 \leqslant 27q^2 < -4p^3$. Since $p < 0$, it follows that

$$0 \leqslant \frac{27q^2}{-4p^3} < 1\,.$$

The function $x \mapsto \sqrt{x}$ is increasing when $x > 0$ because its derivative $1/2\sqrt{x}$ is positive. Hence, taking the positive square roots gives

$$0 \leqslant \sqrt{\frac{27q^2}{-4p^3}} = \frac{3\sqrt{3}\,|q|}{2\,|p|\sqrt{-p}} = \left|\frac{3\sqrt{3}q}{2p\sqrt{-p}}\right| < 1\,.$$

iv. Combining the principal inverse of the cosine function with Part **iii**, it follows that there there exists a real number θ such that $0 \leqslant \theta \leqslant \pi$ and

$$\theta := \frac{1}{3}\arccos\left(\frac{3\sqrt{3}q}{2p\sqrt{-p}}\right).$$

v. Combining Euler's formula and the binomial theorem gives

$$\begin{aligned}\cos(3\theta) + \mathrm{i}\sin(3\theta) = \exp(3\theta\mathrm{i}) &= \big(\exp(\theta\mathrm{i})\big)^3 \\ &= \big(\cos(\theta) + \mathrm{i}\sin(\theta)\big)^3 \\ &= \cos^3(\theta) + 3\mathrm{i}\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - \mathrm{i}\sin^3(\theta)\,.\end{aligned}$$

Comparing real parts and using the identity $\cos^2(\theta) + \sin^2(\theta) = 1$, we obtain

$$\begin{aligned}\cos(3\theta) &= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) \\ &= \cos^3(\theta) - 3\cos(\theta)\big(1 - \cos^2(\theta)\big) = 4\cos^3(\theta) - 3\cos(\theta)\,.\end{aligned}$$

In particular, the equation $4\cos^3(\theta) - 3\cos(\theta) - \cos(3\theta) = 0$ implies that $\cos(\theta)$ is a root of $4t^3 - 3t - \cos(3\theta)$. Since

$$\cos\Big(3\big(\theta + \tfrac{2\pi}{3}\big)\Big) = \cos(3\theta) \qquad \text{and} \qquad \cos\Big(3\big(\theta + \tfrac{4\pi}{3}\big)\Big) = \cos(3\theta)\,,$$

it follows that $\cos\big(\theta + \frac{2\pi}{3}\big)$ and $\cos\big(\theta + \frac{4\pi}{3}\big)$ are also roots.

vi. Since $t = y/\lambda$, we conclude that the roots of $y^3 + py + q$ are

$$y_1 = 2\sqrt{\frac{-p}{3}}\cos(\theta)\,, \qquad y_2 = 2\sqrt{\frac{-p}{3}}\cos\big(\theta + \tfrac{2\pi}{3}\big)\,, \qquad y_3 = 2\sqrt{\frac{-p}{3}}\cos\big(\theta + \tfrac{4\pi}{3}\big)\,.$$

where the real number θ is defined by

$$\theta := \frac{1}{3}\arccos\left(\frac{3\sqrt{3}q}{2p\sqrt{-p}}\right). \qquad \square$$