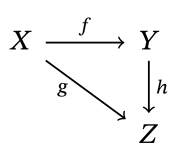
4
Category Theory

Category theory provides a language for mathematics, designed to capture general phenomena and enable the transfer of ideas. This approach supplies simplifying abstraction, isolating those results that hold for formal reasons from those that require methods from a particular branch of mathematics. Category theory also formulates new proof techniques.

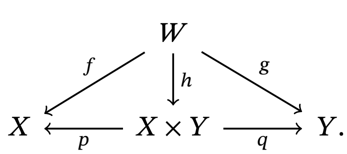
4.0 Mathematical Analogies

Category theory emphasizes the maps between objects. From this perspective, mathematical structures are defined or described by diagrams of arrows. Each arrow $f : X \to Y$ represents a map or function. A typical diagram



is commutative when $g = h \circ f$. The same diagram applies in many contexts: sets and functions, topological spaces and continuous maps, groups and homomorphisms, etc.

Many structural properties may be viewed as universal features of diagrams. Consider the cartesian product $X \times Y$ of two sets X and Y, which consists of all order pairs (x, y) of elements $x \in X$ and $y \in Y$. The projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$ of the product on its factors are the maps $p : X \times Y \to X$ and $q : X \times Y \to Y$. Any map $h : W \to X \times Y$ is uniquely determined by $p \circ h$ and $q \circ h$.



Conversely, given two functions $f : W \to X$ and $g : W \to Y$, there exists a unique function $h: W \to X \times Y$ which makes the diagram commute: $h(w) = \big(f(w), g(w)\big)$. Thus, given the sets X and Y, the pair (p, q) is "universal" among the functions from some set to X and Y, because any other such pair (f, g) factors uniquely (via h) through the pair (p, q). This property describes the cartesian product $X \times Y$ uniquely (up to a bijection). The same diagram, within the category of groups or topological spaces, describes the direct product of groups or the cartesian product of spaces.

The "cartesian product" construction is called a functor because it applies to sets and the maps between them. Two maps $k: X \to X'$ and $\ell: Y \to Y'$ have a map $k \times \ell: X \times Y \to X' \times Y'$, defined by $(x, y) \mapsto \big(k(x), \ell(y)\big)$ as their cartesian product. Observe that the one-point set pt := $\{0\}$ serves as an identity under the operations "cartesian product", in view of the bijections

$$\mathrm{pt} \times X \xrightarrow{\ i\ } X \xleftarrow{\ j\ } X \times \mathrm{pt}.$$

given by $i(0, x) = x$ and $j(x, 0) = x$.

A monoid M (semigroup with identity) may be described as a set M together with two maps $\mu: M \times M \to M$ and $\eta: \mathrm{pt} \to M$ such that the following two diagrams commute:

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{\mathrm{id}_M \times \mu} & M \times M \\ {\scriptstyle \mu \times \mathrm{id}_M}\downarrow & & \downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow[\mu]{} & M \end{array} \qquad \begin{array}{ccccc} \mathrm{pt} \times M & \xrightarrow{\eta \times \mathrm{id}_M} & M \times M & \xleftarrow{\mathrm{id}_M \times \eta} & M \times \mathrm{pt} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle j} \\ M & = & M & = & M \end{array}$$

To say that these diagrams commute means

$$\mu \circ (\mathrm{id}_M \times \mu) = \mu \circ (\mu \times \mathrm{id}_M), \quad \mu \circ (\eta \times \mathrm{pt}) = i, \quad \text{and} \quad \mu \circ (1 \times \eta) = j.$$

Rewriting these diagrams with elements gives

$$\begin{array}{ccc} (x, y, z) & \longmapsto & (x, yz) \\ \downarrow & & \downarrow \\ (xy, z) & \longmapsto & (xy)z = x(yz) \end{array} \qquad \begin{array}{ccc} (0, x) & \longmapsto & (e, x) \\ \downarrow & & \downarrow \\ x & = & ex \end{array} \qquad \begin{array}{ccc} (x, e) & \longleftarrow & (x, 0) \\ \downarrow & & \downarrow \\ xe & = & x \end{array}$$

where $\eta(0) = e \in M$. These are the familar axioms: multiplication is associative and the element e is a left and right identity. The same process applies to other axioms.

Because the diagrams make no mention of elements, they apply in other circumstances. When applied to topological spaces and continuous maps, they define topological groups. For differentiable manifolds and smooth maps, they define a Lie group.

4.1 Categories

A category is a context for studying a class of mathematical objects. Importantly, a category has both 'nouns' and 'verbs' with specified collections of objects and maps between them.

4.1.1 Definition. A ***category*** C consists of
- a collection of ***objects*** $X, Y, Z, \ldots$
- a collection of ***morphisms*** $f, g, h, \ldots$

such that:
- Every morphism has a specified source and target objects. The morphism f with source X to target Y is denoted by $f : X \to Y$.
- Each object X has a designated ***identity morphism*** $\mathrm{id}_X : X \to X$.
- For any pair of morphisms f, g such that the target of f is equal to the source of g, there exists a specified ***composite morphism*** $g\,f$ whose source is equal to the source of f and whose target is equal to the target of g.

This data satisfies the following two axioms:
- For any $f : X \to Y$, we have $f\ \mathrm{id}_X = f = \mathrm{id}_Y\ f$.
- For any triple $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$, we have $h\,(g\,f) = (h\,g)\,f$, so the notation $h\,g\,f$ is unambiguous.

Russell's paradox implies that there is no set whose elements are "all sets". For this reason, we use the vague word "collection" in Definition 4.1.1. The set-theoretical foundations for category theory are a separate topic.

As in Definition 1.3.3, a morphism $f : X \to Y$ is an ***isomorphism*** if there exists a morphism $g : Y \to X$ such that $g\,f = \mathrm{id}_X$ and $f\,g = \mathrm{id}_Y$.

4.1.2 Example. The category Set has sets as its objects and maps, with specified source and target, as its morphisms. ◇

Despite downplaying the significance of the morphisms, it is traditional to name a category after its objects.

4.1.3 Example. The category Grp has groups as objects and group homomorphisms as morphisms. ◇

4.1.4 Example. The category Ab has abelian groups as objects and group homomorphisms as morphisms. ◇

4.1.5 Example. The category CRng has commutative rings as its objects and ring homomorphisms as its morphisms. ◇

4.1.6 Example. For a ring R, the category Mod_R has R-modules as objects and R-module homomorphisms as morphisms. Thus, the category Ab is the same as the category $\mathsf{Mod}_{\mathbb{Z}}$. ◇

4.1.7 Example. The category Top has topological spaces as objects and continuous functions as morphisms. ◇

4.1.8 Example. The category Top_* has pointed topological spaces (some point in the space is chosen to be the basepoint) as objects and basepoint-preserving continuous functions as morphisms. ◇

4.1.9 Example. The category Htpy has topological spaces as objects and homotopy classes of continuous maps as morphisms. ◇

4.1.10 Example. The category Graph has (undirected simple) graphs as objects and graph morphisms (a pair of maps on vertices and edges preserving incidence relations) as morphisms. ◇

The first examples are "concrete" categories—objects have underlying sets and the morphisms are maps between the underlying sets. However, "abstract" categories are also common.

4.1.11 Example. The category Poset has partially-ordered sets as its objects and order-perserving maps as its morphisms. ◇

4.1.12 Example. For any ring R, the category Mat_R has nonnegative integers as objects. For any two nonnegative integers n and m, the morphisms from n to m are $(m \times n)$-matrices with entries in R. Composition is by matrix multiplication with identity matrices serving as the identity morphisms. ◇

4.1.13 Example. For any group G, the category $\mathsf{B}G$ has a just one object. Each group element gives a distinct endomorphism of the single object with composition given by multiplication. The identity $e \in G$ acts as the identity morphism for the unique object. ◇

4.1.14 Example. Any poset $(\mathsf{P}, \leqslant)$ forms a category. The elements in P are the objects and there is a unique morphism $x \to y$ if and only if $x \leqslant y$. Transitivity implies that the required composite morphisms exists. Reflexivity implies that identity morphisms exist. ◇

This duality has a very important in the development of category theory. Any theorem containing a universal quantification of the form "for all categories C" applies to the opposites of these categories. Interpreting the result in the dual context leads to a dual theorem, in which the direction of each arrow is reversed.

4.1.15 Definition. Let C be a category. The ***opposite category*** has the same objects as C. There is a morphism $f^{\mathrm{op}} : X \to Y$ in C^{op} if there is a morphism $f : Y \to Y$ in C. A pair of morphisms $f^{\mathrm{op}}, g^{\mathrm{op}} \in \mathsf{C}^{\mathrm{op}}$ is composable when the pair g, f is composable in C. We define $g^{\mathrm{op}} f^{\mathrm{op}}$ to be $(f\,g)^{\mathrm{op}}$.

4.1.16 Definition. A morphism $f : X \to Y$ in a category is
- a ***monomorphism*** if for any morphisms $h : W \to X$ and $k : W \to X$, the relation $f\,h = f\,k$ implies that $h = k$, or
- a ***epimorphism*** if for any morphisms $h : Y \to Z$ and $k : Y \to Z$, the relation $h\,f = k\,f$ implies that $h = k$,

4.1.17 Lemma. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms.*
(i) *If f and g are monomorphisms, then $g\,f : X \to Z$ also is.*
(ii) *If $g\,f : X \to Z$ is a monomorphism, then f is a monomorphism.*
(iii) *If f and g are epimorphisms then $g\,f : X \to Z$ also is.*
(iv) *If $g\,f : X \to Z$ is a epimorphism, then g is a epimorphism.*

Proof. Consider morphisms $h : W \to X$ and $k : W \to X$.
(i) Since g is monomorphism, the relation $g\,f\,h = g\,f\,k$ implies that $f\,h = f\,k$. Since f is also a monomorphism, we deduce that $h = k$, so $g\,f$ is a monomorphism.
(ii) The relation $f\,h = f\,k$ implies that $g\,f\,h = g\,f\,k$. Since $g\,h$ is a monomorphism, we see that $h = k$, so f is a monomorphism.
(iii) The notions of monomorphism and epimorphism are dual, so it follows from part (i).
(iv) By duality, this follows from part (ii). □