4.2 Functoriality

Within category theory, mathematical objects should be considered together with a suitable notion of structure-preserving morphism. Categories are themselves mathematical objects. What are their morphisms?

4.2.1 Definition. A ***functor*** $F : \mathsf{C} \to \mathsf{D}$ consists of two assignments:
- for each X in C, an object $F(X)$ in D.
- for each $f : X \to Y$ in C, a morphism $F(f) : F(X) \to F(Y)$ in D,

such that
- for each object X in C, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and
- for any composable pair f, g in C, we have $F(g)\,F(f) = F(g\,f)$.

4.2.2 Example. There is a functor $P : \mathsf{Set} \to \mathsf{Set}$ that sends a set X to its power set $P(X)$ consisting of all subsets of X. This functor sends map $f : X \to Y$ to the direct-image map $f_* : P(X) \to P(Y)$ defined, for all $X' \subseteq X$, by $f_*(X') := \{f(x) \mid x \in X'\} \subseteq Y$. ◇

4.2.3 Example. A ***forgetful functor*** is a term used for any functor that forgets structure. For instance, the functor $U : \mathsf{Grp} \to \mathsf{Set}$ sends a group to its underlying set and a group homomorphism to its underlying map of sets. The functor $U : \mathsf{Top} \to \mathsf{Set}$ sends a space to its set of points. There are two natural forgetful functors $V : \mathsf{Graph} \to \mathsf{Set}$ and $E : \mathsf{Graph} \to \mathsf{Set}$ that send a graph to its vertex and edge sets respectively. These mappings are functorial because in each case a morphism in the source category has an underlying map of sets. ◇

4.2.4 Example. There are functors $\mathsf{Mod}_R \to \mathsf{Ab}$ and $\mathsf{CRng} \to \mathsf{Ab}$ that forget some, but not all, of the algebraic structure. The canonical functors $\mathsf{Ab} \to \mathsf{Grp}$ and $\mathsf{Field} \to \mathsf{CRng}$ may be regarded as forgetful. The latter two, but neither of the former, are injective on objects: a group is either abelian or not, but an abelian group might admit the structure of a ring in multiple ways. ◇

4.2.5 Example. The chain rule articulates the functoriality of the derivative. Let Euclid_* be the category whose objects are pointed finite-dimensional Euclidean spaces $(\mathbb{R}^n, a)$ and whose morphisms are functions differentiable at a. The total derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $a \in \mathbb{R}^n$, gives rise to the Jacobian matrix. This defines a functor $D : \mathsf{Euclid}_* \to \mathsf{Mat}_{\mathbb{R}}$. On objects, D assigns a pointed Euclidean space its dimensions. For a function $g : \mathbb{R}^m \to \mathbb{R}^k$ that is differentiable at $f(a)$ and carries the designated basepoint $f(a) \in \mathbb{R}^m$ to $(g\,f)(a) \in \mathbb{R}^k$, the functoriality of D is equivalent to saying that the product of the Jacobian of f at a with the Jacobian of g at $f(a)$ equals the Jacobian of $g\,f$ at a. This is the chain rule. ◇

The functors in Definition 4.2.1 are called ***covariant*** to distinguish them from another type of functor.

4.2.6 Definition. A ***contravariant functor*** F from the category C to the category D is a functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$. For each $f : X \to Y$ in C, there is a morphism $F(f) : F(Y) \to F(X)$ in D.

4.2.7 Example. The contravariant functor $P : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ sends a set X to its power set $P(A)$ and a map $f : X \to Y$ of sets to the inverse-image map $f^{-1} : P(Y) \to P(X)$ defined, for all $Y' \subseteq Y$, by $f^{-1}(Y') := \{x \in X \mid f(x) \in Y'\} \subseteq X$. ◇

4.2.8 Example. There is functor $(-)^* : \mathsf{Vect}_K^{\mathrm{op}} \to \mathsf{Vect}_K$ that carries a K-vector space V to its dual space $V^* := \mathrm{Hom}(V, K)$. A vector in V^* is a linear functional on V. This functor is contravariant because the linear map $\varphi : V \to W$ is sent to the linear map $\varphi^* : W^* \to V^*$ that pre-composes a linear functional $\omega : W \to K$ with φ to obtain the linear functional $(\omega\,\varphi) : V \to K$. ◇

4.2.9 Example. The functor $\mathcal{O} : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Poset}$ that carries a space X to its poset $\mathcal{O}(X)$ of open subsets is contravariant. A continuous map $f : X \to Y$ gives rise to a function $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ that sends an open subset $U \subseteq Y$ to its preimage $f^{-1}(U)$. ◇

4.2.10 Lemma. *Functors preserve isomorphisms*

Proof. Let $f : X \to Y$ be an isomorphism in a category C and let $g : Y \to X$ be its inverse. For any $F : \mathsf{C} \to \mathsf{D}$, functoriality implies that $F(g)\,F(f) = F(g\,f) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$. By symmetry, we see that $F(g) : F(Y) \to F(X)$ is the inverse of $F(f) : F(X) \to F(Y)$. □

4.2.11 Example. For any group G, let $\mathsf{B}G$ be its one-object category. A functor $X : \mathsf{B}G \to \mathsf{C}$ specifies an object X in C and a morphism $g_* : X \to X$ for each $g \in G$. Two conditions hold:

- For all $g, h \in G$, we have $h_*\,g_* = (h\,g)_*$.
- For the identity element $e \in G$, we have $e_* = \mathrm{id}_X$.

The functor $\mathsf{B}G \to \mathsf{C}$ defines an action of the group G on the object X in C. When $\mathsf{C} = \mathsf{Set}$, the object X is endowed with a group action. When $\mathsf{C} = \mathsf{Vect}_K$, the object X is a representation of G. Because the elements $g \in G$ are isomorphisms in $\mathsf{B}G$, their images under any functor must also be isomorphisms in the target category. ◇

4.2.12 Example. A category is ***locally small*** if between any pair of objects there is only a set's worth of morphisms. The category CAT has locally small categories as objects and functors between then as morphisms. The categories Set, Grp, Poset are objects in CAT. The functor $(-)^{\mathrm{op}} : \mathsf{CAT} \to \mathsf{CAT}$ defines a non-trivial automorphism of the category of categories. ◇

In a locally small category C, it is traditional to write $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ for the set of morphisms from X to Y.

4.3 Naturality

Categories and functors were first conceived as auxiliary concepts needed to give a precise meaning to the concept of naturality.

In practice, a natural transformation is defined by saying "the morphisms x are natural" which means that the collection defines a natural tranformation. Although the correct choice of the source and target functors and the course and target categories may be implicit, the naturality condition refers to every object and every morphism in the source category and is described using the images in the target category under the action of both functors.

4.3.1 Definition. Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{C} \to \mathsf{D}$ be functors. A ***natural transformation*** $\alpha : F \Rightarrow G$ consists of a morphism $\alpha_X : F(X) \to G(X)$ in D for each object X in C such that, for any morphism $f : X \to Y$ in C, the following square in D

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & F(X) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\alpha_Y]{} & G(Y) \end{array}$$

commutes.

4.3.2 Example. There is natural transformation $\eta : \mathrm{id}_{\mathsf{Set}} \Rightarrow P$ from the identity to the covariant power set functor where $\eta_X : X \to P(X)$ sends $x \in X$ to the singleton $\{x\} \in P(X)$. ◇

4.3.3 Example. For any K-vector space V, the map $\mathrm{ev} : V \to V^{**}$ that sends a vector $v \in V$ to the linear function $\mathrm{ev}_v : V^* \to K$ defines a natural transformation from the identity functor on Vect_K to the double dual functor. To check that, for any linear map $\varphi : V \to W$, the natural square

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{ev}} & V^{**} \\ {\scriptstyle \varphi}\big\downarrow & & \big\downarrow{\scriptstyle \varphi^{**}} \\ W & \xrightarrow[\mathrm{ev}]{} & W^{**} \end{array}$$

commutes, it suffices to consider a generic vector $v \in V$. The map $\mathrm{ev}_{\varphi(v)} : W^* \to K$ carries a linear functional $f : W \to K$ to $f(\varphi(v))$. Example 4.2.8 shows that the morphism sends $\varphi^{**}(\mathrm{ev}_v) : W^* \to K$ carries a linear function $f : W^* \to K$ to $f(\varphi(v))$. ◇

The identity functor and the single dual functor on finite-dimensional vectors spaces are not naturally isomorphic. One obstruction is technical: the identify functor is covariant while the dual functor is contravariant. The essential failure of naturality is more significant. The isomorphisms between a vector space and its dual require a choice of basis, which is perserved by no non-identity linear transformations.

A familiar isomorphism arising from the classification of finitely generated abelian groups is not natural.

4.3.4 Proposition. *Let G be a finitely generated abelian group and let $\tau(G)$ be its torsion subgroup. The isomorphism $G \cong \tau(G) \oplus \big(G/\tau(G)\big)$ is not natural.*

Proof. Suppose that the isomorphism $G \cong \tau(G) \oplus \big(G/\tau(G)\big)$ were natural in G. The composite $G \to G/\tau(G) \to \tau(G) \oplus \big(G/\tau(G)\big) \cong G$ of the canonical quotient map, the canonical inclusion map, and this isomorphism would define a natural transformation on the identity functor on $\mathsf{Ab}^{\mathrm{fg}}$. We claim that this is impossible.

We first show that any natural transformation $\alpha : \mathrm{id}_{\mathsf{Ab}^{\mathrm{fg}}} \to \mathrm{id}_{\mathsf{Ab}^{\mathrm{fg}}}$ is multiplication by some integer n. A homomorphism $\varphi : \mathbb{Z} \to G$

of abelian groups is determined by the image $g := \varphi(1_{\mathbb{Z}}) \in G$. Thus, the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\alpha_{\mathbb{Z}}} & \mathbb{Z} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ G & \xrightarrow[\alpha_G]{} & G \end{array}$$

forces $\alpha_G(g) = n\,g$.

From the case $G = \mathbb{Z}$, it follows that the natural transformation defined by $G \to G/\tau(G) \to \tau(G) \oplus \big(G/\tau(G)\big) \cong G$ is multiplication by a nonzero integer n. Consider $G = \mathbb{Z}/\langle 2n\rangle$. This is a torsion group, so any map, such as $\alpha_{\mathbb{Z}/\langle 2n\rangle}$ which factors through the quotient, is zero. However, we have $n \neq 0 \in \mathbb{Z}/\langle 2n\rangle$ which in a contradiction. □

Natural isomorphisms lead to the equivalence of categories.

As the nomenclature suggests, the notion of equivalent categories defines an equivalence relation.

4.3.5 Definition. An ***equivalence of categories*** consists of functors $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{C}$ and natural isomorphisms $\eta :\ \mathrm{id}_{\mathsf{C}} \xrightarrow{\cong} G\,F$ and $\epsilon :\ \mathrm{id}_{\mathsf{D}} \xrightarrow{\cong} F\,G$. The categories C and D are ***equivalent***, written $\mathsf{C} \simeq \mathsf{D}$, if there exists an equivalence between them.

There is a useful characterization of the functors in an equivalence of categories. Its statement requires new terminology.

4.3.6 Definition. A functor $f : \mathsf{C} \to \mathsf{D}$ is

- ***full*** if, for any pair X, Y of objects in the category C, the morphism map $\mathrm{Hom}_{\mathsf{C}}(X,Y) \to \mathrm{Hom}_{\mathsf{D}}\big(F(X), F(Y)\big)$ is surjective.
- ***faithful*** if, for any pair X, Y of objects in the category C, the map $\mathrm{Hom}_{\mathsf{C}}(X,Y) \to \mathrm{Hom}_{\mathsf{D}}\big(F(X), F(Y)\big)$ is injective, and
- ***essentially surjective on objects*** if, for every object Z in the category D, there exists an object X in C such that Z is isomorphic to $F(X)$.

The proof of Theorem 4.3.7 involves a lengthy diagram chase. The second part uses the axiom of choice.

4.3.7 Theorem. *A functor defining an equivalence of categories is full, faithful, and essentially surjective on objects. Moreover, any functor with these properties defines an equivalence of categories.* ■

4.3.8 Example. For any field K, the categories Mat_K and $\mathsf{Vect}_K^{\mathrm{fd}}$ are equivalent. Consider an intermediate category $\mathsf{Vect}_K^{\mathrm{bs}}$ whose objets are finite-dimensional vector spaces with a chosen basis and whose morphisms are arbitrary linear maps. These three categories are related by four functors: $U : \mathsf{Vect}_K^{\mathrm{bs}} \to \mathsf{Vect}_K^{\mathrm{fd}}$ is the forgetful functor. The functor $K^{(-)} : \mathsf{Mat}_K \to \mathsf{Vect}_K^{\mathrm{bs}}$ sends n to the vector space K^n, equipped with the standard basis. An $(m\times n)$-matrix, relative to the standard bases on K^n and K^m, defines a linear map $K^n \to K^m$. The functor $H : \mathsf{Vect}_K^{\mathrm{bs}} \to \mathsf{Mat}_K$ carries a vector space to its dimension and a linear map $\varphi : V \to W$ to the matrix relative to the chosen bases. The functor $C : \mathsf{Vect}_K^{\mathrm{fd}} \to \mathsf{Vect}_K^{\mathrm{bs}}$ is defined by choosing a basis for each vector space. One verifies that functors are full, faithful, and essentially surjective on objects. ◇