4.4 Representable Functors

A foundational lemma in category theory demonstrates that every object can be characterized by a *universal property*.

Certain classes of universal properties define blueprints which specify how a new object may be build out of a collection of existing ones.

4.4.1 Definition. Let F be covariant or contravariant functor from a locally small category C to Set. A ***representation*** of the functor F is a choice of object X in C together with a natural isomorphism $\operatorname{Hom}_{\mathsf{C}}(X, -) \cong F$ when F is covariant, or $\operatorname{Hom}_{\mathsf{C}}(-, X) \cong F$ when F is contravariant. One says that F is ***represented by*** X. A functor is ***representable*** if there exists a representation.

A universal property of an object X in the locally small category C is a description of the covariant functor $\operatorname{Hom}_{\mathsf{C}}(X, -)$ or the contravariant functor $\operatorname{Hom}_{\mathsf{C}}(-, X)$.

4.4.2 Example. The identity $\mathrm{id}_{\mathsf{Set}} : \mathsf{Set} \to \mathsf{Set}$ is represented by the singleton set $\{\varnothing\}$. For any set X, there exists a natural isomorphism $\operatorname{Hom}_{\mathsf{Set}}(\{\varnothing\}, X) \cong X$ that defines a bijection between the elements $x \in X$ and maps $x : \{\varnothing\} \to X$ carrying the singleton element to x. Naturality says that, for any $f : X \to Y$, the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathsf{Set}}(\{\varnothing\}, X) & \xrightarrow{\cong} & X \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Hom}_{\mathsf{Set}}(\{\varnothing\}, Y) & \xrightarrow[\cong]{} & Y \end{array}$$

commutes. The composite function $\{\varnothing\} \xrightarrow{x} X \xrightarrow{f} Y$ corresponds to the element $f(x)$. ◇

4.4.3 Example. The forgetful functor $U : \mathsf{Top} \to \mathsf{Set}$ is represented by the singleton space. There is a natural bijection between elements of a topological space and continuous functions from the one-point space. ◇

4.4.4 Example. The forgetful functor $U : \mathsf{Grp} \to \mathsf{Set}$ is represented by the group $\mathbb{Z}$. For any group G, there is a natural isomorphism $\operatorname{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G) \cong U(G)$ that associates, to every element $g \in U(G)$, the unique group homomorphism $\mathbb{Z} \to G$ that maps the integer 1 to g. This defines a bijection because every group homomorphism $\mathbb{Z} \to G$ is determined by the image of 1. In other words, $\mathbb{Z}$ is the free group on a single generator. This bijection is natural because the composite group homomorphism $\mathbb{Z} \xrightarrow{g} G \xrightarrow{\varphi} H$ carries the integer 1 to $\varphi(g) \in H$. ◇

4.4.5 Example. For any commutative ring R, the forgetful functor $U : \mathsf{Mod}_R \to \mathsf{Set}$ is represented by the R-module R^1. There exists a natural bijection between R-module homomorphisms $R \to V$ and the elements of the underlying set of V; $v \in U(V)$ is associated to the unique R-module homomorphism that carries the multiplicative identity of R to v. In other words, R is the free R-module on a single generator. ◇

4.4.6 Example. The functor $U : \mathsf{CRng} \to \mathsf{Set}$ is represented by the ring $\mathbb{Z}[x]$. A ring homomorphism $\mathbb{Z}[x] \to R$ is uniquely determined by the image of x. In other words, $\mathbb{Z}[x]$ is the free commutative ring on a single generator. ◇

The adjective "free" is reserved for universal properties expressed by covariant represented functors.

4.4.7 Example. For any $n \in \mathbb{N}$, the functor $U(-)^n : \mathsf{Grp} \to \mathsf{Set}$ that sends a group G to the set of n-tuples of elements of G is represented by the free group on n generators. For any commutative ring R, the functor $U(-)^n : \mathsf{Mod}_R \to \mathsf{Set}$ is represented by the free R-module R^n. The functor $U(-)^n : \mathsf{CRng} \to \mathsf{Set}$ is represented by the polynomial algebra $\mathbb{Z}[x_1, x_2, \ldots, x_n]$. ◇

4.4.8 Example. The functor $(-)^\times : \mathsf{CRng} \to \mathsf{Set}$ that sends a ring to its set of units is represented by the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$. A ring homomorphism $\mathbb{Z}[x, x^{-1}] \to R$ may be defined by sending x to any unit of R and is completely determined by this assignment. No ring homomorphism carries x to a non-unit. ◇

4.4.9 Example. The contravariant power set functor $P : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ is represented by the set $\Omega := \{0, 1\}$ with two elements. The natural isomorphism $\mathrm{Hom}_{\mathsf{Set}}(X, \Omega) \cong P(A)$ is defined by the bijection that associates a map $X \to \Omega$ with the subset that is the preimage of 1. Reversing perspectives, a subset $X' \subseteq X$ is identify with its indicator function $\chi_{X'} : X \to \Omega$ which sends exactly the elements of X' to 1. The naturality condition stipulates that, for any map $f : X \to Y$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{Set}}(Y, \Omega) & \xrightarrow{\cong} & P(X) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^{-1}} \\ \mathrm{Hom}_{\mathsf{Set}}(X, \Omega) & \xrightarrow[\cong]{} & P(Y) \end{array}$$

commutes. Given an indicator function $\chi_{Y'} : Y \to \Omega$, the composite function $X \xrightarrow{f} Y \xrightarrow{\chi_{Y'}} \Omega$ determines the subset $f^{-1}(Y') \subset X$. ◇

4.4.10 Example. For any field K, the functor $U(-)^* : \mathsf{Vect}_K^{\mathrm{op}} \to \mathsf{Vect}_K$ that sends a vector space to the set of vectors in its dual space is represented by the vector space K; linear functionals $V \to K$ are, by definition, precisely the vectors in the dual space V^*. ◇

4.4.11 Example. For any two set Y and Z, the functor

$$\mathrm{Hom}(- \times Y, Z) : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$$

that sends a set X to the set of functions $X \times Y \to Z$ is represented by the set Z^Y of functions from Y to Z. Hence, there exists a natural bijection between functions $X \times Y \to Z$ and functions $X \to Z^Y$. This natural isomorphism is referred to as ***currying*** in computer science. By fixing a variable in a two-variable function, one obtains a family of functions in a single variable. ◇

4.5 The Yoneda Lemma

The previous section suggests that a representation encodes some sort of universal property of its representing object. If two objects represent the same functor, are they isomorphic?

This eponymous lemma was baptized by Saunders Mac Lane after learning about it from Nobuo Yoneda in 1954.

4.5.1 Definition. For any categories C and D, the ***functor category*** D^{C} has the functors $F : \mathsf{C} \to \mathsf{D}$ as the objects and natural transformations between them as morphisms. Given an object F in D^{C}, its identity morphism $\mathrm{id}_F : F \Rightarrow F$ in D^{C} is the natural transformation determined by $(\mathrm{id}_F)_X := \mathrm{id}_{F(X)}$ for all objects X in C. To describe composition in D^{C}, consider three parallel functors $E, F, G : \mathsf{C} \to \mathsf{D}$ and two natural transformations $\alpha : E \Rightarrow F$ and $\beta : F \Rightarrow G$. The composite natural transformation $\beta\,\alpha : E \Rightarrow G$ is determined, for all objects X in C, by $(\beta\,\alpha)_X := \beta_X\,\alpha_X$. Naturality of α and β implies that, for any morphism $f : X \to Y$ in C, each square in the diagram

$$\begin{array}{ccccc} E(X) & \xrightarrow{\alpha_X} & F(X) & \xrightarrow{\beta_X} & G(X) \\ {\scriptstyle E(f)}\big\downarrow & & {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ E(Y) & \xrightarrow{\alpha_Y} & F(Y) & \xrightarrow{\beta_Y} & G(Y) \end{array}$$

commutes, so the composite rectangle also commutes. To remains to verify that composition is associative and unital. It suffices to verify these properties for all objects X in C. Therefore, they follow from the associativity and unitality of composition in D.

The next result is arguably the most important result in category theory. Its takes time to appreciate this deceptively deep statement.

4.5.2 Theorem (Yoneda lemma). *Let C be a locally small category. For any functor $F : \mathsf{C} \to \mathsf{Set}$ and any object X in C, there is a bijection*

The statement of the dual form of the Yoneda Lemma is left as an exercise.

$$\mathrm{Hom}_{\mathsf{Set}^{\mathsf{C}}}\big(\mathrm{Hom}_{\mathsf{C}}(X, -), F\big) \cong F(X)$$

that associates a natural transformation $\alpha : \mathrm{Hom}_{\mathsf{C}}(X, -) \Rightarrow F$ to the element $\alpha_X(\mathrm{id}_X) \in F(X)$. This bijection is natural in both X and F. ■

A special case of the Yoneda lemma characterizes the natural transformations between representable functors. Each object X in the category C represents a functor $\mathrm{Hom}_{\mathsf{C}}(X, -) : \mathsf{C} \to \mathsf{Set}$ and each morphism $f : X \to Y$ in C corresponds to a natural transformation $f^* : \mathrm{Hom}_{\mathsf{C}}(Y, -) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(X, -)$ determined, for all objects Z in C, by the pre-composition map $f^* : \mathrm{Hom}_{\mathsf{C}}(Y, Z) \to \mathrm{Hom}_{\mathsf{C}}(X, Z)$. This data determines a functor $H^- : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{C}}$.

4.5.3 Corollary (Yoneda embedding). *The functor $H^- : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{C}}$ is full and faithful.*

Proof. By Definition 4.3.6, the functor $H^- : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{C}}$ is full and faithful provided it defines local bijections between hom-sets

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) \cong \mathrm{Hom}_{\mathsf{Set}^{\mathsf{C}}}\big(\mathrm{Hom}_{\mathsf{C}}(Y, -), \mathrm{Hom}_{\mathsf{C}}(X, -)\big).$$

Rhe definition of H^- ensures that this map is injective: distinct morphism induces distinct natural transformations. The Yoneda lemma implies that a natural transformation

$$\alpha : \mathrm{Hom}_{\mathsf{C}}(Y, -) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(X, -)$$

corresponds to elements in $\mathrm{Hom}_{\mathsf{C}}(X, Y)$; the morphism $f : X \to Y$ in C is $f = \alpha_Y(\mathrm{id}_Y)$. Defined as pre-composition with $f : X \to Y$, the natural transformation $f^* : \mathrm{Hom}_{\mathsf{C}}(Y, -) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(X, -)$ sends id_Y to f. Thus, the bijection implies that $\alpha = f^*$. □

The Yoneda lemma is a generalization of Theorem 1.5.8.

4.5.4 Corollary. *Every group is isomorphic to a subgroup of a symmetric group.*

A map $\alpha : Y \to X$ is G-***equivariant*** if, for all $g \in G$, the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & X \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ Y & \xrightarrow[\alpha]{} & X \end{array}$$

commutes.

Proof. For any group G, Example 4.2.11 establishes that a functor $X : \mathsf{B}G \to \mathsf{Set}$ corresponds to a set X with an action of G. A natural transformation $\alpha : X \Rightarrow Y$ consists of a single G-equivariant map $\alpha : Y \to X$ of sets. The G-equivariant maps $G \to X$ correspond bijectively to elements of X: identify a map with the image of $e \in G$. Hence, the image of the Yoneda embedding $\mathsf{B}G^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{B}G}$ is the set G under left translation. Corollary 4.5.3 implies that the only G-equivariant endomorphisms of G are those defined by right multiplication with an element of G. In particular, any G-equivariant endomorphism of G must be an automorphism.

Thus, the Yoneda embedding defines an isomorphism between G and the automorphism group of the G, regarded as an object in $\mathsf{Set}^{\mathsf{B}G}$. Composing with the faithful forgetful functor $\mathsf{Set}^{\mathsf{B}G} \to \mathsf{Set}$, we obtain an isomorphism between G and a subgroup of the automorphism group $\mathfrak{S}_G$ of the set G. □

4.5.5 Corollary. *Let X and Y be objects in a locally small category C. If the functors represented by X and Y are naturally isomorphic, then X and Y are isomorphic. In particular, if X and Y represent the same functor, then X and Y are isomorphic.*

Sketch of Proof. The full and faithful Yoneda embedding $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{C}}$ creates isomorphisms: for any two objects in the source category, whose images are isomorphic in the target, are isomorphic in the source. Thus, an isomorphism between represented functors is induced by a unique isomorphism between their representing objects. Finally, given a functor represented by both X and Y, the representing natural isomorphisms compose to demonstrate that X and Y are representably isomorphic. □