Problems 01
Due: Tuesday, 15 September 2020

1. Let X and Y be two sets. The identity map $\mathrm{id}_X : X \to X$ is defined by $x \mapsto x$ and the map $\pi_1 : X \times Y \to X$ is defined by $(x, y) \mapsto x$. Given two maps $\varphi : X \to X$ and $\psi : X \to X$, the map $\varphi \,||\, \psi : X \to X \times X$ is defined by $x \mapsto \big(\varphi(x), \psi(x)\big)$ and the map $\varphi \times \psi : X \times X \to X \times X$ is defined by $(x, y) \mapsto \big(\varphi(x), \psi(y)\big)$. For the one-element set $\{\varnothing\}$, there exists a unique map $\eta : X \to \{\varnothing\}$ defined by $\eta(x) = \varnothing$.

Suppose that X is nonempty and consider the maps $\beta : X \times X \to X$, $\varepsilon : \{\varnothing\} \to X$, and $\iota : X \to X$ satisfying the following three conditions:

(i) $\beta \circ (\beta \times \mathrm{id}_X) = \beta \circ (\mathrm{id}_X \times \beta)$
(ii) $\beta \circ (\mathrm{id}_X \times \varepsilon) = \mathrm{id}_X \circ \pi_1$
(iii) $\beta \circ (\mathrm{id}_X \,||\, \iota) = \varepsilon \circ \eta$.

Prove that the quadruple $(X, \beta, \varepsilon, \iota)$ defines a group.

2. For all nonnegative integers n, the ***sign function*** $\operatorname{sgn} : \mathfrak{S}_n \to \mu_2 = \{-1, 1\}$ is defined by $\operatorname{sgn}(\sigma) := (-1)^{n-c}$ where the permutation σ is the product of c disjoint cycles.

(i) For any permutation $\sigma \in \mathfrak{S}_n$ and any transposition $\varpi \in \mathfrak{S}_n$, prove that

$$\operatorname{sgn}(\varpi\, \sigma) = -\operatorname{sgn}(\sigma)\,.$$

(ii) For any two permutations $\sigma, \tau \in \mathfrak{S}_n$, prove that $\operatorname{sgn}(\sigma\,\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$.
(iii) When σ is the product of m transpositions, prove that $\operatorname{sgn}(\sigma) = (-1)^m$.

3. The ***quaternion group*** is the subgroup of $\mathrm{SL}(2, \mathbb{C})$ generated by the eight matrices:

$$e := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad i := \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}, \qquad j := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad k := \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix},$$

$$\bar{e} := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \bar{\imath} := \begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix}, \qquad \bar{\jmath} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \bar{k} := \begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}.$$

(i) Determine the order of the quaternion group.
(ii) Find a minimal set of generators for the quaternion group.
(iii) Show that the quaternion group is not isomorphic to the dihedral group D_4 of order 8.