Problems 07
Due: Tuesday, 03 November 2020

1. (i) Describe the kernel of the $\mathbb{C}$-linear ring homomorphism $\psi : \mathbb{C}[x, y, z] \to \mathbb{C}[t]$ defined by $\psi(x) := t$, $\psi(y) := t^2$, and $\psi(z) := t^3$.

(ii) Describe the kernel of the map $\varphi : \mathbb{Z}[x] \to \mathbb{R}$ defined by $\varphi(x) := 1 + \sqrt{2}$.

2. Euclid proves that there are infinitely many prime integers in the following way: if $p_1, p_2, \dots, p_k$ are prime numbers, then any prime factor of $1 + p_1 p_2 \cdots p_k$ must be different from p_i for all $1 \leqslant i \leqslant k$.

(i) Adapt this argument to show that, for any field K, there are infinitely many monic irreducible polynomials in $K[x]$.

(ii) Explain why the argument fails for the formal power series ring $K[[x]]$ over a field K.

(iii) Adapt this argument to show that the set of prime integers of the form $4n - 1$ is infinite.

3. *Existence of Partial Fraction Decompositions.* Let R be a principal ideal domain and let K be its field of fractions.

(i) Suppose $R = \mathbb{Z}$. Write $r = 7/24 \in \mathbb{Q}$ in the form $r = a/8 + b/3$.

(ii) Let $g := pq \in R$ where p and q are relatively prime. Prove that every fraction $f/g \in K$ can be written in the form

$$\frac{f}{g} = \frac{a}{q} + \frac{b}{p}$$

where $a, b \in R$.

(iii) Let $g := p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \in R$ be the factorization of g into irreducible elements p_i, for all $1 \leqslant i \leqslant k$, such that the relation $p_i = u\, p_j$ for some unit $u \in R$ implies that $i = j$. Prove that every fraction $r = f/g \in K$ can be written in the form

$$r = \sum_{i=1}^{k} \frac{h_i}{p_i^{m_i}}$$

where $h_i \in R$ for all $1 \leqslant i \leqslant k$.