Problems 10
Due: Tuesday, 24 November 2020

1. Let $\mathbb{F}_q$ be a finite field with q elements.

(i) For any nonnegative integer n, determine the number of elements in the $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$.

(ii) Let $\mathrm{GL}(n, \mathbb{F}_q)$ denote the group of all invertible $(n \times n)$-matrices over the field $\mathbb{F}_q$. Determine the order of the group $\mathrm{GL}(n, \mathbb{F}_q)$.

(iii) Let $\mathrm{SL}(n, \mathbb{F}_q)$ denote the subgroup of $\mathrm{GL}(n, \mathbb{F}_q)$ consisting of matrices having determinant 1. Find the order of the group $\mathrm{SL}(n, \mathbb{F}_q)$.

2. Let $R := \mathbb{Z}/\langle 30 \rangle$. Show that in the matrix

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix} \in \mathrm{Mat}(2, 3, R)\,,$$

the rows are linearly independent, but any two columns are linearly dependent.

3. Let U, V and W be three vector spaces and $\varphi : U \to V$, $\psi :\ V \to W$ two linear maps. A linear map φ has ***finite index*** if both $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are finite-dimensional. The number $\mathrm{ind}(\varphi) := \dim \mathrm{Ker}(\varphi) - \dim \mathrm{Coker}(\varphi)$ is the ***index*** of φ.

(i) Prove that U decomposes into a direct sum of $\mathrm{Ker}(\varphi)$ and two subspaces U', U'' such that $\mathrm{Ker}(\psi \circ \varphi) = \mathrm{Ker}(\varphi) \oplus U'$ and $\mathrm{Im}(\psi \circ \varphi) = \psi(\varphi(U''))$.

(ii) Prove that if two of the three linear maps φ, ψ and $\psi \circ \varphi$ are of finite index, then so is the third and $\mathrm{ind}(\psi \circ \varphi) = \mathrm{ind}(\varphi) + \mathrm{ind}(\psi)$.

Do not assume that U, V, and W are finite-dimensional.