Problems 11

Due: Tuesday, 1 December 2020

1. Consider the ring $\mathbb{Q}[x]$. Find a basis for the submodule of $\mathbb{Q}[x]^3$ generated by

$$f_1 = \begin{bmatrix} 2x-1 \\ x \\ x^2+3 \end{bmatrix}, \qquad f_2 = \begin{bmatrix} x \\ x \\ x^2 \end{bmatrix}, \qquad \text{and} \qquad f_3 = \begin{bmatrix} x+1 \\ 2x \\ 2x^2-3 \end{bmatrix}.$$

2. Consider the integer matrix

$$\mathbf{B} := \begin{bmatrix} -1 & 5 & -8 & 7 \\ 2 & 8 & -2 & 4 \\ 1 & -11 & 32 & -13 \end{bmatrix}.$$

(i) Find matrices $\mathbf{Q} \in \mathrm{Mat}(3,3,\mathbb{Z})$ and $\mathbf{P} \in \mathrm{Mat}(4,4,\mathbb{Z})$ such that $\mathbf{Q}\,\mathbf{B}\,\mathbf{P}$ is in Smith normal form.

(ii) Let G be an abelian group with four generators g_1, g_2, g_3 and g_4 subject to the relations

$$\begin{aligned} -g_1 + 5g_2 - 8g_3 + 7g_4 &= 0 \\ 2g_1 + 8g_2 - 2g_3 + 4g_4 &= 0 \\ g_1 - 11g_2 + 32g_3 - 13g_4 &= 0\,. \end{aligned}$$

Show that $G \cong \mathbb{Z}/\langle 6\rangle \oplus \mathbb{Z}/\langle 54\rangle \oplus \mathbb{Z}$. In addition, find new generators h_1, h_2 and h_3 such that $6\,h_1 = 0$, $54\,h_2 = 0$ and h_3 has infinite order.

(iii) Find all $\mathbf{X} \in \mathrm{Mat}(3,1,\mathbb{Z})$ such that $\mathbf{B}\,\mathbf{X} = \mathbf{C}$ where $\mathbf{C} := \begin{bmatrix} -5 \\ 100 \\ -7 \end{bmatrix}$.

3. Consider the matrix

$$\mathbf{A} := \begin{bmatrix} -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & -2 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Find the minimal polynomial of $\mathbf{A}$.

(ii) Find the rational canonical form of $\mathbf{A}$.

(iii) Find the Jordan canonical form of $\mathbf{A}$.