Solutions 12

1. Consider the integer matrix

$$\mathbf{B} := \begin{bmatrix} 1 & 3 & 4 & 2 \\ 5 & 15 & -4 & 34 \\ -3 & -3 & 36 & -24 \end{bmatrix}.$$

i. Find a (3×3)-matrix $\mathbf{Q}$ over $\mathbb{Z}$ and a (4×4)-matrix $\mathbf{P}$ over $\mathbb{Z}$ such that $\mathbf{Q}\,\mathbf{B}\,\mathbf{P}$ is in Smith normal form.

ii. Let G be an abelian group with generators g_1, g_2, g_3 and g_4, and the relations

$$\begin{aligned} 1g_1 + \ 3g_2 + \ 4g_3 + \ 2g_4 &= 0 \\ 5g_1 + 15g_2 - \ 4g_3 + 34g_4 &= 0 \\ -3g_1 - \ 3g_2 + 36g_3 - 24g_4 &= 0\,. \end{aligned}$$

Show that $G \cong \mathbb{Z}/\langle 6\rangle \oplus \mathbb{Z}/\langle 24\rangle \oplus \mathbb{Z}$. In addition, find new generators h_1, h_2 and h_3 such that $6\,h_1 = 0$, $24\,h_2 = 0$ and h_3 has infinite order.

iii. Find all integer (3×1)-matrices $\mathbf{X}$ such that $\mathbf{B}\,\mathbf{X} = \mathbf{C}$ where

$$\mathbf{C} := \begin{bmatrix} 7 \\ 11 \\ -9 \end{bmatrix}.$$

Solution.

i. We reduce the matrix $\mathbf{B}$ to its Smith normal form by a sequence of row and column operations. We simultaneously record the effect of these operations by performing them on the matrix $\mathbf{I}_3$ (for row operations) and the matrix $\mathbf{I}_4$ (for column operations);

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 5 & 15 & -4 & 34 \\ -3 & -3 & 36 & -24 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 0 & -24 & 24 \\ 0 & 6 & 48 & -18 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ -5 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 6 & 48 & -18 \\ 0 & 0 & -24 & 24 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 5 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 6 & 48 & -18 \\ 0 & 0 & 24 & -24 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 5 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 4 & 6 \\ 0 & 6 & 48 & 30 \\ 0 & 0 & 24 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 5 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -20 & -9 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 24 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -8 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 5 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 24 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -3 & 20 & 9 \\ 0 & 1 & -8 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Setting

$$\mathbf{Q} := \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 5 & -1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P} := \begin{bmatrix} 1 & -3 & 20 & 9 \\ 0 & 1 & -8 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we have $\mathbf{Q}\,\mathbf{B}\,\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 24 & 0 \end{bmatrix}$ which is in Smith normal form.

ii. Let $\varphi_{\mathbf{B}} : \mathbb{Z}^4 \to \mathbb{Z}^3$ be the linear map whose matrix with respect to the standard bases is $\mathbf{B}$. By definition, we have $G \cong \mathbb{Z}^4 / \operatorname{Ker}(\varphi_{\mathbf{B}})$. The invertible matrix $\mathbf{P}$ from part *i* defines an automorphism of $\mathbb{Z}^4$. Since

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 8 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the elements $h_0 := g_1 + 3g_2 + 4g_3 + 2g_4$, $h_1 := g_2 + 8g_3 - 3g_4$, $h_2 := g_3 - g_4$, and $h_3 := g_4$ generate G. The matrix $\mathbf{Q}$ from part *i* implies that

$$\begin{aligned} h_0 &= g_1 + 3g_2 + 4g_3 + 2g_4 = 0 \\ 6h_1 &= 6g_2 + 48g_3 - 18g_4 \\ &= 3(g_1 + 3g_2 + 4g_3 + 2g_4) + (-3g_1 - 3g_2 + 36g_3 - 24g_4) = 0 \\ 24h_2 &= 24g_3 - 24g_4 \\ &= 5(g_1 + 3g_2 + 4g_3 + 2g_4) - (5g_1 + 15g_2 - 4g_3 + 34g_4) = 0 \end{aligned}$$

and h_3 has infinite order in G. The Smith normal form for $\mathbf{B}$ given in part *i* implies that $G \cong \mathbb{Z}/\langle 6\rangle \oplus \mathbb{Z}/\langle 24\rangle \oplus \mathbb{Z}$ as required.

iii. Since

$$\mathbf{Q}\,\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 5 & -1 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \\ -9 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 24 \end{bmatrix},$$

the system $\mathbf{B}\,\mathbf{X} = \mathbf{C}$ is transformed into $(\mathbf{Q}\,\mathbf{B})\,\mathbf{Y} = \mathbf{Q}\,\mathbf{C}$ or

$$\begin{cases} y_1 = 7 \\ 6y_2 = 12 \\ 24y_3 = 24 \end{cases}$$

This new system has the solutions $y_1 = 7$, $y_2 = 2$, $y_3 = 1$, and $y_4 = m \in \mathbb{Z}$, so the solutions to the original equation $\mathbf{B}\,\mathbf{X} = \mathbf{C}$ are given by $\mathbf{X} = \mathbf{P}\,\mathbf{Y}$ or

$$\mathbf{P}\,\mathbf{Y} = \begin{bmatrix} 1 & -3 & 20 & 9 \\ 0 & 1 & -8 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \\ 1 \\ m \end{bmatrix} = \begin{bmatrix} 21 + 9m \\ -6 - 5m \\ 1 + m \\ m \end{bmatrix}$$

where m is an arbitrary integer. □

2. Consider the matrix

$$\mathbf{A} := \begin{bmatrix} 2 & 0 & -1 & -5 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & -1 \\ 1 & 0 & -1 & -4 \end{bmatrix}.$$

i. Find the minimal polynomial of $\mathbf{A}$.
ii. Find the rational canonical form of $\mathbf{A}$.
iii. Find the Jordan canonical form of $\mathbf{A}$.

Solution.

i. The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} \det(t\,\mathbf{I} - \mathbf{A}) &= \det \begin{bmatrix} t-2 & 0 & 1 & 5 \\ 0 & t-1 & 0 & 0 \\ -2 & 0 & t+1 & 1 \\ -1 & 0 & 1 & t+4 \end{bmatrix} = (t-1)\det \begin{bmatrix} t-2 & 1 & 5 \\ -2 & t+1 & 1 \\ -1 & 1 & t+4 \end{bmatrix} \\ &= (t-1)\det \begin{bmatrix} t-1 & 0 & -t+1 \\ -2 & t+1 & 1 \\ -1 & 1 & t+4 \end{bmatrix} = (t-1)^2 \det \begin{bmatrix} 1 & 0 & -1 \\ -2 & t+1 & 1 \\ -1 & 1 & t+4 \end{bmatrix} \\ &= (t-1)^2 \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & t+1 & -1 \\ 0 & 1 & t+3 \end{bmatrix} = (t-1)^2 \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & t+2 & t+2 \\ 0 & 1 & t+3 \end{bmatrix} \\ &= (t-1)^2 (t+2) \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & t+3 \end{bmatrix} = (t-1)^2\,(t+2) \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & t+2 \end{bmatrix} \\ &= (t-1)^2\,(t+2)^2\,. \end{aligned}$$

The minimal polynomial divides the characteristic polynomial. Since

$$(\mathbf{A} - \mathbf{I})(\mathbf{A} + 2\,\mathbf{I}) = \begin{bmatrix} -3 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -3 & -6 \\ -3 & 0 & 3 & 6 \end{bmatrix} \neq 0$$

and $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 2\,\mathbf{I})^2 = \mathbf{0}$, we deduce that the minimal polynomial of the matrix $\mathbf{A}$ is $(t-1)(t+2)^2 = t^3 + 3t^2 - 4$.

ii. As the minimal polynomial is $t^3 + 3t^2 - 4$ and the characteristic polynomial is $(t-1)^2(t+2)^2$, it follows that

$$\mathbb{Q}^4 \cong \frac{\mathbb{Q}[t]}{\langle t-1 \rangle} \oplus \frac{\mathbb{Q}[t]}{\langle t^3 + 3t^2 - 4 \rangle}$$

and the rational canonical form of the matrix $\mathbf{A}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}.$$

iii. As the minimal polynomial is $(t-1)(t+2)^2$ and the characteristic polynomial is $(t-1)^2(t+2)^2$, it follows that

$$\mathbb{C}^4 \cong \frac{\mathbb{C}[t]}{\langle t-1 \rangle} \oplus \frac{\mathbb{C}[t]}{\langle t-1 \rangle} \oplus \frac{\mathbb{C}[t]}{\langle (t+1)^2 \rangle}$$

and the Jordan canonical form of the matrix $\mathbf{A}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

□

3. A ***poset*** is a set P with a binary relation $\leqslant_P$ that is reflexive, anitsymmetric, and transitive. For any elements x, y, and z in P, we have the following:

(reflexivity) The relation $x \leqslant_P x$ holds.
(antisymmetry) The relations $x \leqslant_P y$ and $y \leqslant_P x$ imply that $x = y$.
(transitivity) The relations $x \leqslant_P y$ and $y \leqslant_P z$ imply that $x \leqslant_P z$.

A function $f : P \to Q$ between posets is ***order-preserving*** if, for any x and y in P satisfying $x \leqslant_P y$, we have $f(x) \leqslant_Q f(y)$.

i. Show that the collection of posets together with order-preserving functions forms a category.
ii. In the category of posets, exhibit a bijective morphism between nonisomorphic posets.

Solution.

i. For any poset P, the identity map $\mathrm{id}_P : P \to P$ is order-perserving. Suppose that $f : P \to Q$ and $g : Q \to R$ are order-perserving functions between posets. For any elements x and y in P such that $x \leqslant_P y$, we have $f(x) \leqslant_Q f(y)$ because f is order-preserving. We also have

$$(g \circ f)(x) = g\big(f(x)\big) \leqslant_R g\big(f(y)\big) = (g \circ f)(y)$$

because g is order-preserving. Hence, the composite map $g \circ f : P \to R$ is also order-preserving. The identity property and the associativity of composition are inherited from the category of sets.

ii. Let P be the set $2^{\{1,2\}}$ consisting of all subsets of $\{1,2\}$ ordered by inclusion and let Q be the set $\{0,1,2,3\} \subset \mathbb{N}$ with the canonical order. Consider the map $f : P \to Q$ defined by $f(X) := \sum_{x \in X} x$. It follows that

$$\varnothing \subseteq \{1\} \mapsto 0 \leqslant 1 \qquad \varnothing \subseteq \{2\} \mapsto 0 \leqslant 2 \qquad \varnothing \subseteq \{1,2\} \mapsto 0 \leqslant 3$$
$$\{1\} \subseteq \{1,2\} \mapsto 1 \leqslant 3 \qquad \{2\} \subseteq \{1,2\} \mapsto 2 \leqslant 3$$

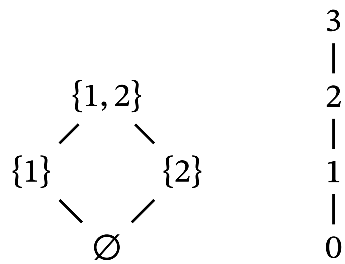

FIGURE 1. Hasse diagrams for the posets P and Q.

so f is an order-preserving bijection. The poset P does not contain a totally ordered subset of cardinality four. Hence, there is no injective order-preserving map from Q to P. In particular, the posets P and Q are not isomorphic. □