Inverting (3×3)-matrices

There is *long-hand* method for computing (3×3)-inverses which is often quicker and more reliable than either row reduction or the general formula for inverses.

Bayer Algorithm. To find the inverse of a (3×3)-matrix, do the following:

(1) Transpose the original matrix, leaving plenty of space between the rows and columns.
(2) Duplicate the two leftmost columns on the right.
(3) Duplicate the top two two rows on the bottom.
(4) Cross-out the top row and left column.
(5) In the 9 spaces between each (2×2)-block of entries, write the (2×2)-determinant and circle it.
(6) Copy out the nine circled numbers as new matrix and multiply by the original matrix. If no errors have been made, then the answer is scalar multiple of the identity matrix and the scalar equals the determinant of the original matrix.
(7) Divide the new matrix by the determinant of the original matrix to obtain the inverse.

Here is the general pattern:

$$\mathbf{M} := \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{array}{c|cccccccc} a & d & & g & & a & & d \\ \hline b & e & & h & & b & & e \\ & & A & & B & & C & \\ c & f & & i & & c & & f \\ & & D & & E & & F & \\ a & d & & g & & a & & d \\ & & G & & H & & I & \\ b & e & & h & & b & & e \end{array}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = \begin{bmatrix} \det(\mathbf{M}) & 0 & 0 \\ 0 & \det(\mathbf{M}) & 0 \\ 0 & 0 & \det(\mathbf{M}) \end{bmatrix} \qquad \mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}$$

Example.

$$\mathbf{M} := \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \begin{array}{c|cccccccc} 1 & -1 & & 0 & & 1 & & -1 \\ \hline 0 & 1 & & -1 & & 0 & & 1 \\ & & 2 & & -1 & & -1 & \\ 1 & 1 & & 1 & & 1 & & 1 \\ & & 1 & & 1 & & -2 & \\ 1 & -1 & & 0 & & 1 & & -1 \\ & & 1 & & 1 & & 1 & \\ 0 & 1 & & -1 & & 0 & & 1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \mathbf{M}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$