NONNEGATIVITY CERTIFICATES ON REAL ALGEBRAIC SURFACES

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ABSTRACT. We introduce tools for transferring nonnegativity certificates for global sections between line bundles on real algebraic surfaces. As applications, we improve Hilbert's degree bounds on sum-of-squares multipliers for nonnegative ternary forms, give a complete characterization of nonnegative real forms of del Pezzo surfaces, and establish quadratic upper bounds for the degrees of sum-of-squares multipliers for nonnegative forms on real ruled surfaces.

1. OVERVIEW

Characterizing nonnegativity is a fundamental problem in both real algebraic geometry and optimization. We develop a *transfer* approach to testing and certifying nonnegativity of polynomials. The key to this approach is the following simple observation: If f, g are multivariate polynomials satisfying the equation $fg = s$ where s is a sum-of-squares of polynomials then nonnegativity of f is equivalent to nonnegativity of g. The identity $fg = s$ thus *transfers* the problem of testing the nonnegativity of f to that of g. We are interested in transferring nonnegativity certificates between *classes of functions*: if we can show that every nonnegative function f in a certain class has a multiplier g in a different class, such that fg is a sum of squares, then we have transferred nonnegativity testing from the class of f to the class of g. If the class of g is simpler in an appropriate sense, then we can iteratively apply the transfer procedure aiming to reduce the problem to a class of functions where nonnegativity is well understood.

We carry out the program outlined above for forms on *real projective surfaces*. Our method transfers nonnegativity certificates for sections of a certain line bundle on a surface, to nonnegativity certificates of sections of a "simpler" line bundle, provided the bundles satisfy certain cohomological inequalities. These inequalities hold for line bundles on a wide array of algebraic surfaces, including rational and more generally ruled surfaces, and allow us to fully characterize nonnegativity sections in several situations.

One of the least understood aspects of the relationship between nonnegative polynomials and sums of squares is the question of *degree bounds* when writing nonnegative polynomials as *sums of squares of rational functions*. Hilbert's 17th problem asked whether every globally nonnegative polynomial f can be written as a sum of squares of rational functions. It is easy to see that being able to write a nonnegative polynomial f in this way is equivalent to the existence of a sum-of-squares multiplier g and a sum-of-squares s such that $fg = s$ as above. Artin's affirmative solution to Hilbert's 17th problem in [Art27] thus transfers nonnegativity certification into a sum-of-squares feasibility problem. However, our understanding of bounds on the degree of the available multipliers g (both upper and lower) remains quite poor.

Prior to posing the 17th problem, Hilbert showed in 1893 [Hil93] that the result holds for ternary forms (homogeneous polynomials in three variables). Hilbert's original proof of the case of ternary forms came with upper bounds on the degree of the sum-of-squares multipliers, and these bounds remained unimproved until our current work. His proof iteratively lowers the degree of the ternary form for which nonnegativity has to be certified. Hilbert's iterative approach is an inspiration for

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the approach introduced in this article: we extend Hilbert's methods from the case of $\mathbb{P}^2$ to general algebraic surfaces, developing a general transfer theory. We then specialize our results to several types of surfaces. First, we focus on toric surfaces and provide a combinatorial interpretation for our transfer Theorem in terms of lattice points in polygons. The freedom to choose different line bundles allowed by the transfer Theorem leads us to revisit Hilbert's bounds for ternary forms, obtaining the first improvement since the original paper in 1893. Furthermore, in the first case where degree bounds for multipliers of ternary forms are not known, namely ternary forms of degree 10, we prove a tight upper bound; this is the first new instance since Hilbert's work of an exact multiplier degree bound for his 17th problem.

We then focus on real del Pezzo surfaces where we take advantage of the classification of their real structures [Rus02]. The resulting understanding of the real Picard group allows us to fully carry out the transfer program outlined above, obtaining a complete classification of nonnegative sections for all line bundles. The del Pezzo case is particularly interesting since it illustrates the role that real structures play in determining the available multipliers and the final form of the nonnegativity certificates. Furthermore, this is the first example of concrete geometric bounds for nonnegativity certificates that apply to a family of surfaces having a non-trivial moduli space.

Finally we develop an asymptotic theory of degree bounds for surfaces. The basic question is the following: given a fixed real algebraic surface X, can we bound k such that every form f of degree $2d$ has a multiplier g of degree $2k$ so that fg is a sum of squares? We show that for most nonsingular ruled surfaces and d large enough, the degree of multipliers k is bounded from above by a quadratic function in d. This is the first result on multiplier degree bounds that applies to non-rational surfaces.

Main results. Let X be a totally-real variety. We say that a divisor E *supports multipliers* for a divisor D if, for any nonnegative global section f in $H^0\big(X,\mathscr{O}_X(2D)\big)$, there exists a nonzero global section g in $H^0\big(X,\mathscr{O}_X(2E)\big)$ such that the product fg in $H^0\big(X,\mathscr{O}_X(2D+2E)\big)$ is a sum of squares. Equivalently, we can transfer testing nonnegativity of global sections in $H^0\big(X,\mathscr{O}_X(2D)\big)$, to testing nonnegativity of global sections in $H^0\big(X,\mathscr{O}_X(2E)\big)$.

Our main technical contribution is the following theorem.

Theorem 4.2. *Assume that X is a totally-real geometrically-integral projective surface. Let D and E be divisors on X with D free (equivalently, the line bundle $\mathscr{O}_X(D)$ is globally generated), $D+E$ very ample, and $H^0\big(X,\mathscr{O}_X(E-D)\big) = H^1\big(X,\mathscr{O}_X(D+E)\big) = H^1\big(X,\mathscr{O}_X(2E)\big) = 0$. The inequality*

$$h^0(X,D+E) > 1 + \left\lceil \frac{h^0(X,2D+2E) - h^0(X,2E) - h^0(X,D+E) - h^1(X,E-D)}{2} \right\rceil$$

implies that the divisor E supports multipliers for the divisor D.

The main inequalities and cohomological vanishing conditions can sometimes be simplified. We first consider toric surfaces, where we develop a criterion for transferring nonnegativity of Laurent polynomials with support in a lattice polygon $2P$ to Laurent polynomials with support in a lattice polygon $2Q$. We need the following terminology: If $A \subseteq \mathbb{R}^2$ then the number of *reduced connected components of* A is one less than the number of connected components of A and a *lattice translate of* A is a set of the form $A+m$ for $m \in \mathbb{Z}^2$. We write $\#A$ for the number of lattice points contained in A and A° for the interior of A.

Theorem 5.1. *Assume that P and Q are convex lattice polygons such that no lattice translate of P is contained in Q. Let h be the total number of reduced connected components of the set differences*

$P \setminus Q'$ as Q' ranges over all lattice translates of Q. The inequality

$$\#(2Q) + h > \#\big((P+Q)^\circ\big)$$

implies that Q supports multipliers for P (i.e. for every nonnegative Laurent polynomial f with monomial support in $2P$ there exists a Laurent polynomial g with monomial support in $2Q$ such that fg is a sum of squares of Laurent polynomials with monomial support in $2(P+Q)$).

This allows us prove sharp degree bounds for degree 10 ternary forms in Example 5.9, and improve Hilbert's bounds for ternary forms in Example 5.10.

The inequality in Theorem 4.2 can be rewritten geometrically, in terms of the adjoint bundle as

$$h^0(X,2E) + h^1(X,E-D) > h^0(X,K_X+D+E)\,.$$

A negative canonical bundle makes the right hand side of this inequality smaller, making it natural to focus on surfaces with large anticanonical divisor. Therefore, we look at del Pezzo surfaces in detail. Our main result in this direction is the following:

Theorem 6.1. *Let X be a totally-real del Pezzo surface having degree at least 3 and canonical divisor K_X. For any nonzero real effective divisor D on X, there exists a finite sequence $D_0, D_1, \dots, D_k$ of effective divisors on X with $D_0 = D$ such that $-K_X \cdot D_i < -K_X \cdot D_{i-1}$, D_i supports multipliers for D_{i-1} for any $1 \leqslant i \leqslant k$, and D_k is either zero or a positive multiple of a conic bundle. In particular, the length k of the sequence is bounded above by $-K_X \cdot D$.*

This theorem allows us to find certificates of nonnegativity on del Pezzo surfaces as explained in Remark 6.7.

Next we also establish asymptotic degree bounds for some embedded surfaces. Let X in $\mathbb{P}^n$ be a totally-real surface with canonical divisor K_X. Let A be the divisor defined by the hyperplane section of the embedding. Our goal is to prove degree bounds for certifying nonnegativity of global sections of $H^0(X,2dA)$ for large d. Our main result is that nonnegativity transfer is possible via the surface Z obtained by blowing-up X at a real point when $-K_X \cdot A > 0$, which implies that the surface X must be ruled. More precisely, we prove the following:

Theorem 7.3. *Assume that X is a totally-real smooth surface with a very ample divisor A satisfying $-K_X \cdot A > 0$. Let $\pi \colon Z := \mathrm{Bl}_p(X) \to X$ be the blow-up of X at a real point p and set $H := \pi^*(A)$. Fix s to be the smallest positive integer such that $s(-K_X \cdot A) > A \cdot (A + K_X)$ and choose a positive integer t such that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{t+1} > 2(1+\sqrt{s})$. For all sufficiently large integers d, there exists an $(t+1)$-step transfer on Z from dH to $(d-1)H$.*

It follows that the degree of sum of square multipliers can be bounded by a quadratic polynomial in d, i.e. there exists a sum of squares $g \in H^0(X,2kA)$, such that fg is a sum of squares, and k can be bounded by a quadratic function in d (see Corollary 7.5).

Relationship with earlier work. The statement that a globally nonnegative polynomial f is a sum of squares of rational functions is equivalent (by clearing denominators) to saying that there exists a real polynomial r such that fr^2 is a sum of squares. As mentioned earlier this is in fact equivalent to saying that there exists a sum of squares g, such that fg is a sum of squares. When referring to degree bounds, we always refer to the bounds on the degree of the sum of squares multiplier g. It is worth noting that Artin's original proof did not produce any degree bounds. Currently, the best known upper bounds due to Lombardi, Perrucci and Roy are a multiple tower of exponentials [LPR20], while the best known lower bounds are linear [BGP16].

The results of [LPR20] imply multiple tower of exponentials degree bounds for real varieties and, more generally, for semialgebraic sets. In [BSV16] tight degree bounds were proved for real curves, which significantly improve on the general bounds from real algebraic geometry. However, the method does not transfer to varieties of higher dimension, making surfaces a very interesting natural next step for a quantitative understanding of nonnegativity on varieties.

Finally, we would like to mention a related line of work which uses powers of a fixed polynomial as a multiplier. A result of Reznick [Rez95] shows that if f is a strictly positive homogeneous polynomial (form), then for some large enough r we have that $(x_1^2+\cdots+x_n^2)^r f$ is a sum of squares. The paper includes an upper bound on r in terms of the degree, the number of variables of f, and the minimum value of f on the unit sphere. It is crucial to note that dependence of r on the minimum of f cannot be removed, and so-called “uniform denominators" cannot work for all forms of degree $2d$, as demonstrated in [Rez05]. Reznick's result was later generalized by Scheiderer [Sch12], who showed that if f and g are both strictly positive forms, then for k large enough we have fg^k is a sum of squares. However, in this generality, there is no estimate on the size of k. There is also a large literature on Positivstellensatz theorems on compact affine varieties which do not use multipliers [Mar08]. However, it is a feature of these theorems that uniform degree bounds are simply not possible in general [Sch00].

Structure of the paper. Let $Y \subset \mathbb{P}^n$ be a real, projective, linearly normal curve with graded coordinate ring R. One of our main technical tools is establishing bounds so that a linear functional ℓ on R_2 can be written as a sum of few evaluations on points of Y, and the points of Y are chosen in a conjugate-invariant way. We call the least number of conjugate-invariant point evaluations the *conjugate invariant length* of ℓ. The main result of Sections 2 and 3 is a bound on the (maximal typical) conjugate invariant length. If the curve Y has no real points, then there is a trivial bound of $\lceil \frac{1}{2}\dim R_2 \rceil$ of complex pairs of evaluations. We modestly improve it to $\lceil \frac{1}{2}(\dim R_2 - \dim R_1)\rceil + 1$, but this improvement is crucial. The basic idea is simple: given a generic linear functional $\ell \in R_2^*$, consider the associated quadratic form φ_ℓ. We can make φ_ℓ drop rank by 1 by adding a multiple of some complex point evaluation. The resulting linear functional ℓ' can be written in terms of point evaluations of $\operatorname{Ker}\varphi_{\ell'} \cap Y$ (this uses linear normality of Y). Then we just apply the trivial bound to $\operatorname{Ker} Q_{\ell'} \cap Y$, so the rank of ℓ' is at most $\lceil \frac{1}{2}(\dim R_2 - \dim R_1)\rceil$, and so the length of ℓ is at most $\lceil \frac{1}{2}(\dim R_2 - \dim R_1)\rceil + 1$.

Cohomological conditions on X allow us to pass to (and count dimensions in) the normalization of $Y = V(f) \cap X$. Interestingly, passing to the normalization Y' of Y increases the dimension of R_1, while keeping the dimension of R_2 the same, which improves the effectiveness of our bound on conjugate symmetric length $\lceil \frac{1}{2}(\dim R_2(Y') - \dim R_1(Y'))\rceil + 1$

In Section 4, we prove Theorem 4.2 by applying results on conjugate-invariant length, which allow us to count signs in a suitably defined real quadratic form. We derive a contradiction to non-existence of sum of squares multipliers in some degree, since the quadratic form is supposed to be positive definite, and yet the results on the conjugate-invariant length allow us to show that it doesn't have enough positive eigenvalues.

In the second half of the paper we focus on the applications of Theorem 4.2. In Section 5, we apply it to toric surfaces and derive the sharpest known degree bounds for sum of squares multiplies for ternary forms, improving Hilbert's 1893 result. In Section 6 we work with real del Pezzo surfaces, and in Section 7 we focus on asymptotic degree bounds for ruled surfaces and prove the existence of asymptotic quadratic multiplier bounds for them.

2. Point Evaluations

This section introduces a numerical invariant of a real projective subvariety called the maximum typical conjugation-invariant length; see Definition 2.4. It is derived from the conjugation-invariant length of linear functionals, which depends on expressing linear functionals as conjugation-invariant linear combinations of point evaluations. This subtle invariant allows one to bound the number of positive and negative of the eigenvalues for quadratic forms on the subvariety.

Fix a nonnegative integer n and consider an $(n+1)$-dimensional real vector space V. Let $x_0, x_1, \dots, x_n$ be a basis for its real dual space V^* and set $S := \operatorname{Sym}(V^*) \cong \mathbb{R}[x_0, x_1, \dots, x_n]$ where $\deg(x_j) = 1$ for all $0 \leqslant j \leqslant n$. A real projective subvariety X of $\mathbb{P}^n := \operatorname{Proj}(S)$ is an integral closed subscheme over $\mathbb{R}$ such that the structure morphism $X \to \operatorname{Spec}(\mathbb{R})$ is separated and of finite type. The saturated homogeneous ideal of the subscheme X in the polynomial ring S is denoted by I and the homogeneous coordinate ring of X is the quotient $R := S/I$. For all integers j, the graded component R_j, consisting of all homogeneous elements in R having degree j, is a finite-dimensional real vector space.

We begin by describing a correspondence between the linear functionals $\ell \colon R_2 \to \mathbb{R}$ and certain quadratic forms. Since $R_2 = S_2/I_2$, the pullback of the canonical surjection $\eta \colon S_2 \to R_2$, which sends a linear functional $\ell \colon R_2 \to \mathbb{R}$ to the composite map $\ell \circ \eta \colon S_2 \to \mathbb{R}$, is injective and defines an isomorphism between R_2^* and $I_2^{\perp} := \{\psi \in S_2^* \mid \psi(g) = 0 \text{ for all } g \in I_2\}$. The corresponding quadratic form $\varphi_\ell \colon S_1 \to \mathbb{R}$ is defined, for all f in S_1, by $\varphi_\ell(f) = (\ell \circ \eta)(f^2)$. The kernel of the quadratic form φ_ℓ is the kernel of its associated symmetric matrix: it is the linear subspace consisting of all polynomials f in S_1 such that $(\ell \circ \eta)(f\,S_1) = 0$. The *corank* of φ_ℓ is the dimension of its kernel.

Example 2.1. Let C be the complete intersection curve in $\mathbb{P}^3$ whose homogeneous coordinate ring is

$$R := \mathbb{R}[x_0, x_1, x_2, x_3] / \langle x_0^2 + x_1^2, x_1 x_2 - x_2^2 - x_3^2 \rangle .$$

Any linear functional $\ell \colon R_2 \to \mathbb{R}$ can be represented as a quadratic form $\varphi_\ell \colon S_1 \to \mathbb{R}$. Since $\dim R_2 = 8$, the associated symmetric matrix of the corresponding quadratic form, relative to the ordered basis dual to x_0, x_1, x_2, x_3 in S_1, is a real matrix of the form

$$\begin{bmatrix} -a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_5 + a_6 & a_7 \\ a_3 & a_5 + a_6 & a_5 & a_8 \\ a_4 & a_7 & a_8 & a_6 \end{bmatrix} . \qquad \diamond$$

We record some basic features of projective space. A closed point p in $\mathbb{P}^n$ is an equivalence class consisting of nonzero elements in $V \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{n+1}$ up to multiplication by a nonzero complex number. Any element $\widehat{p}$ in this equivalence class is an affine representative of the point p. Complex conjugation on $\mathbb{C}$ induces involutions on both the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ and the closed points in $\mathbb{P}^n$. A closed point p is real if and only if it is fixed under conjugation: $\overline{p} = p$. The set of real points in $\mathbb{P}^n$ is denoted by $\mathbb{P}^n(\mathbb{R})$ and a nonreal point on $\mathbb{P}^n$ is any closed point in the complement $\mathbb{P}^n \setminus \mathbb{P}^n(\mathbb{R})$. Every real point in $\mathbb{P}^n$ admits real affine representatives and any two real affine representatives coincide up to multiplication by a nonzero real number. Hence, there is a bijection between $\mathbb{P}^n(\mathbb{R})$ and the equivalence classes of nonzero elements of V up to multiplication by a real number. The set $\mathbb{P}^n(\mathbb{R})$ is a differentiable manifold endowed with a Euclidean topology induced by any norm on V under the canonical quotient map $V \setminus \{0\} \to \mathbb{P}^n(\mathbb{R})$.

We first analyze the geometric properties of the quadratic forms having corank 1. Although its proof is elementary, we were unable to find a suitable reference. A subvariety X in $\mathbb{P}^n = \operatorname{Proj}(S)$ is *nondegenerate* if it is not contained in a hyperplane or, equivalently, if we have $R_1 = S_1$.

Lemma 2.2. *Let X be a nondegenerate real subvariety of $\mathbb{P}^n$ with homogeneous coordinate ring R.*

(i) *The set* $M := \bigl\{ \ell \colon R_2 \to \mathbb{R} \bigm| \text{the quadratic form } \varphi_\ell \text{ has corank } 1 \bigr\}$ *is a differentiable manifold of dimension* $\dim R_2 - 1$.

(ii) *For any quadratic form φ in M, the tangent space $T_\varphi(M)$ is the linear subspace of $I_2^\perp$ consisting of the linear maps $\psi \colon S_2 \to \mathbb{R}$ such that $\psi(g^2) = 0$ for all polynomials g in the kernel of φ.*

(iii) *The map $\Phi \colon M \to \mathbb{P}^n(\mathbb{R})$, which sends a quadratic form φ having corank 1 to the equivalence class of its kernel, has a surjective differential at all points in M. In particular, the image of any Euclidean open set in M has nonempty Euclidean interior in $\mathbb{P}^n(\mathbb{R})$.*

Proof. Every quadratic polynomial g in S_2 corresponds to a unique real symmetric matrix $\mathbf{A}$ where $g = \mathbf{x}^\mathsf{T} \mathbf{A} \mathbf{x}$ and $\mathbf{x} := [x_0 \; x_1 \; \cdots \; x_n]^\mathsf{T}$. For any two integers j and k satisfying $0 \leqslant j \leqslant k \leqslant n$, we write $a_{j,k}$ for the linear functional $a_{j,k} \colon S_2 \to \mathbb{R}$ that determines the (j,k)-entry in the associated symmetric matrix $\mathbf{A}$. Let Z be the closed affine subscheme of S_2 defined by the determinant of $\mathbf{A}$. The Jacobi formula $d\det(\mathbf{A}) = \operatorname{tr}\bigl(\operatorname{adj}(\mathbf{A})\, d\mathbf{A}\bigr)$ expresses the derivative of this determinant in terms of the adjugate of $\mathbf{A}$ and the derivative of $\mathbf{A}$. Consider a point φ in the determinantal hypersurface Z corresponding to a quadratic form having corank 1. As every symmetric matrix is orthogonally diagonalizable, we may choose coordinates so that the point φ is represented by a diagonal matrix $\mathbf{Q} := \operatorname{diag}(\lambda_0, \lambda_1, \dots, \lambda_{n-1}, 0)$ where $\lambda_j \neq 0$ for all $0 \leqslant j \leqslant n-1$. Hence, the differential $d\det(\mathbf{A})$ at the point φ equals $\lambda_0 \lambda_1 \cdots \lambda_{n-1}\, da_{n,n} \neq 0$. It follows that φ is a nonsingular point on Z.

To establish that the point φ is a nonsingular point on $Z \cap I_2^\perp$, we show that the tangent space $T_\varphi(Z)$ and the linear subspace $I_2^\perp$ intersect transversely, which proves that M is nonsingular and $T_\varphi(M) = T_\varphi(Z) \cap I_2^\perp$. By hypothesis, the subvariety X is nondegenerate, so the polynomial x_n^2 does not belong to I_2. Hence, the real vector space $\langle x_n^2 \rangle + I_2$ has dimension $1 + \dim I_2$. Since $\bigl(T_\varphi(Z) \cap I_2^\perp\bigr)^\perp = T_\varphi(Z)^\perp \cap I_2 = \langle x_n^2 \rangle + I_2$, we see that $T_\varphi(Z)$ and $I_2^\perp$ intersect transversely. Because this calculation is invariant under orthogonal transformations, we deduce that parts (i) and (ii) hold.

To understand the map $\Phi \colon M \to \mathbb{P}^n(\mathbb{R})$, recall that the adjugate matrix of $\mathbf{A}$ satisfies the equation $\mathbf{A} \operatorname{adj}(\mathbf{A}) = \det(\mathbf{A})\, \mathbf{I}$. Hence, in a neighbourhood of the point φ in M represented by the diagonal matrix $\mathbf{Q}$, the map Φ sends the associated symmetric matrix $\mathbf{A}$ to the equivalence class spanned by the last column of the adjugate matrix $\operatorname{adj}(\mathbf{A})$, whose entries are polynomials in the $a_{j,k}$. From this local description, we see that the map $\Phi \colon M \to \mathbb{P}^n(\mathbb{R})$ is differentiable.

Finally, we compute the differential $d\Phi$ at the point φ in M. For all $0 \leqslant j \leqslant n$ and all $0 \leqslant k \leqslant n$, let $\mathbf{E}_{j,k}$ be the $\bigl((n+1) \times (n+1)\bigr)$-matrix whose (j,k)-entry is 1 and all other entries are 0. By identifying points in dual space S_2^* with their associated symmetric matrices, we see that the differentiable curves $\sigma_{j,k} \colon \mathbb{R} \to S_2^*$ defined by

$$\sigma_{j,k}(t) := \begin{cases} \mathbf{Q} + t(\mathbf{E}_{j,k} + \mathbf{E}_{k,j}) & \text{for all } 0 \leqslant j \leqslant k \leqslant n-1 \\ \mathbf{Q} + t(\mathbf{E}_{j,n} + \mathbf{E}_{n,j}) + (t^2/\lambda_j)\mathbf{E}_{n,n} & \text{for all } 0 \leqslant j \leqslant n-1 \text{ and } k = n \end{cases}$$

lie in the hypersurface Z and their tangent directions at $t=0$ span the tangent space $T_{\varphi}(Z)$. It follows that $\Phi(\varphi) = [0:0:\cdots:0:1]$ and

$$\Phi\big(\sigma_{j,k}(t)\big) = \begin{cases} [0:0:\cdots:0:1] & \text{for all } 0 \leqslant j \leqslant k \leqslant n-1 \\ [0:0:\cdots:0:t:0:\cdots:0:-\lambda_j] & \text{for all } 0 \leqslant j \leqslant n-1 \text{ and } k=n. \end{cases}$$

Differentiating with respect to t establishes that the differential $d\Phi$ surjects onto the tangent space $T_{\Phi(q)}\big(\mathbb{P}^n(\mathbb{R})\big)$ and its kernel K is spanned by forms that vanish at $x_j x_n$ for all $0 \leqslant j \leqslant n$. The map $\mu_{x_n} \colon R_1 \to R_2$ is injective, so the dimension of real vector space $\langle x_j x_n \mid 0 \leqslant j \leqslant n \rangle + I_2$ is equal to the sum of the dimensions of its summands. Since $\big(K \cap I_2^{\perp}\big)^{\perp} = \langle x_j x_n \mid 0 \leqslant j \leqslant n \rangle + I_2$, we see that K and $I_2^{\perp}$ intersect transversely. Therefore, the differential of the map $\Phi \colon M \to \mathbb{P}^n(\mathbb{R})$ is surjective at all points in M. The final assertion follows from the implicit function theorem. □

Our second lemma relates linear functionals to point evaluations. A closed point in the subvariety X is a closed point in $\mathbb{P}^n$ at which the polynomials in the homogeneous ideal I vanish. The set of real points in X is denoted by $X(\mathbb{R})$. Given any closed point p in X, any choice $\widehat{p}$ of affine representative defines a ring homomorphism $\operatorname{ev}_p \colon R \to \mathbb{C}$ by sending the coset f in R to the evaluation $\widehat{f}(\widehat{p})$, where $\widehat{f}$ is a polynomial in S that maps to f under the canonical surjection. Since the point p lies on X, the complex number $\widehat{f}(\widehat{p})$ is independent of the choice of the polynomial $\widehat{f}$. The closed point p in X determines the point evaluation $\operatorname{ev}_p \colon R \to \mathbb{C}$ up to multiplication by a nonzero complex number. The affine representatives of real points are always chosen to be real, so a point p in $X(\mathbb{R})$ determines the map $\operatorname{ev}_p \colon R \to \mathbb{R}$ up to multiplication by a nonzero real number.

Lemma 2.3. *Let X be a real subvariety of $\mathbb{P}^n$ with homogeneous coordinate ring R. For any linear functional $\ell \colon R_2 \to \mathbb{R}$, there are nonnegative integers r and c, real numbers $a_1, a_2, \dots, a_r$, real points $p_1, p_2, \dots, p_r$ in X, complex numbers $z_1, z_2, \dots, z_c$, and nonreal points $q_1, q_2, \dots, q_c$ in X such that*

$$\ell = a_1 \operatorname{ev}_{p_1} + a_2 \operatorname{ev}_{p_2} + \cdots + a_r \operatorname{ev}_{p_r} + (z_1 \operatorname{ev}_{q_1} + \overline{z_1} \operatorname{ev}_{\overline{q_1}}) + (z_2 \operatorname{ev}_{q_2} + \overline{z_2} \operatorname{ev}_{\overline{q_2}}) + \cdots + (z_c \operatorname{ev}_{q_c} + \overline{z_c} \operatorname{ev}_{\overline{q_c}}) \,.$$

Proof. To start, we claim that any linear functional from R_2 to $\mathbb{C}$ is a $\mathbb{C}$-linear combination of point evaluations. The point evaluations span a linear subspace of the linear functionals R_2^*. Suppose that this linear subspace is contained in a hyperplane. It follows that the corresponding element f in R_2 vanishes at every closed point in X. Since X is a subvariety, its homogeneous ideal I is radical, so any polynomial $\widehat{f}$ in S_2 that maps to f under the canonical surjection belongs to I. Hence, we have $f=0$ and the linear subspace spanned by point evaluations is not contained in a nonzero hyperplane.

It remains to describe the real-valued linear functionals $\ell \colon R_2 \to \mathbb{R}$. The previous paragraph implies that there exists nonnegative integers r and c, complex numbers $a_1, a_2, \dots, a_r$, real points $p_1, p_2, \dots, p_r$ in X, complex numbers $z_1, z_2, \dots, z_c$, and nonreal points $q_1, q_2, \dots, q_c$ in X such that

$$\ell = a_1 \operatorname{ev}_{p_1} + a_2 \operatorname{ev}_{p_2} + \cdots + a_r \operatorname{ev}_{p_r} + z_1 \operatorname{ev}_{q_1} + z_2 \operatorname{ev}_{q_2} + \cdots z_c \operatorname{ev}_{q_c} \,.$$

Since ℓ is real-valued and $p_j = \overline{p_j}$ for all $1 \leqslant j \leqslant r$, we see that $\ell = \overline{\ell}$ and

$$\begin{aligned} \ell = \tfrac{1}{2}(\ell + \overline{\ell}) &= \operatorname{Re}(a_1) \operatorname{ev}_{p_1} + \operatorname{Re}(a_2) \operatorname{ev}_{p_2} + \cdots + \operatorname{Re}(a_r) \operatorname{ev}_{p_r} \\ &\quad + (z_1 \operatorname{ev}_{q_1} + \overline{z_1} \operatorname{ev}_{\overline{q_1}}) + (z_2 \operatorname{ev}_{q_2} + \overline{z_2} \operatorname{ev}_{\overline{q_2}}) + \cdots (z_c \operatorname{ev}_{q_c} + \overline{z_c} \operatorname{ev}_{\overline{q_c}}) \,. \end{aligned}$$ □

Building on this lemma, we introduce two numerical invariants.

Definition 2.4. For a linear functional $\ell \colon R_2 \to \mathbb{R}$, the *conjugation-invariant length* $\operatorname{clen}(\ell)$ is the minimum of the sum $r + c$ among all expressions for ℓ appearing in Lemma 2.3. The *maximum* *typical conjugation-invariant length of* X, denoted by $\operatorname{mclen}(X)$, is the smallest integer k such that the subset of the linear functionals $\ell \colon R_2 \to \mathbb{R}$ having conjugation-invariant length at most k is dense in the Euclidean topology on R_2^*. Equivalently, the numerical invariant $\operatorname{mclen}(X)$ is the smallest integer k such that the subset of the linear functionals $\ell \colon R_2 \to \mathbb{R}$ having conjugation-invariant length greater than k does not contain a nonempty open neighbourhood in the Euclidean topology on R_2^*.

The next proposition links the eigenvalues of a quadratic form to the conjugation-invariant length. A set of points in $\mathbb{P}^n$ is *in linear general position* if they impose independent conditions on linear forms, meaning that the set of linear forms vanishing at any $k \leqslant n+1$ of the points is a linear subspace of codimension k.

Proposition 2.5. *Consider a linear function $\ell \colon R_2 \to \mathbb{R}$ of the form*

$$\ell = a_1 \operatorname{ev}_{p_1} + a_2 \operatorname{ev}_{p_2} + \cdots + a_r \operatorname{ev}_{p_r} + (z_1 \operatorname{ev}_{q_1} + \overline{z_1} \operatorname{ev}_{\overline{q_1}}) + (z_2 \operatorname{ev}_{q_2} + \overline{z_2} \operatorname{ev}_{\overline{q_2}}) + \cdots + (z_c \operatorname{ev}_{q_c} + \overline{z_c} \operatorname{ev}_{\overline{q_c}}),$$

where $a_1, a_2, \ldots, a_r$ are real numbers, $p_1, p_2, \ldots, p_r$ are real points in $\mathbb{P}^n$, $z_1, z_2, \ldots, z_c$ are complex numbers, and $q_1, q_2, \ldots, q_c$ are nonreal points in $\mathbb{P}^n$. Setting r_+ and r_- to be the number of positive and negative numbers in the set $\{a_1, a_2, \ldots, a_r\}$ respectively, the corresponding quadratic form φ_ℓ has at most $r_+ + c$ positive eigenvalues and $r_- + c$ negative eigenvalues. In particular, φ_ℓ has at most $\operatorname{clen}(\ell)$ *positive eigenvalues.*

Proof. For any real point p in $\mathbb{P}^n(\mathbb{R})$, the quadratic form corresponding to $\operatorname{ev}_p \colon R_2 \to \mathbb{R}$ has rank one and one positive eigenvalue because $\operatorname{ev}_p(f^2) \geqslant 0$ for any f in R_1. For any complex number z and any nonreal point q in $\mathbb{P}^n$, we claim that the quadratic form corresponding to $z \operatorname{ev}_q + \bar{z} \operatorname{ev}_{\bar{q}}$ has rank at most two, and at most one positive and one negative eigenvalue. Since $g(\bar{q}) = \overline{g(q)}$ for any g in R_2, we have $(\operatorname{ev}_q + \operatorname{ev}_{\bar{q}})(f^2) = 2\operatorname{Re}\big(f^2(q)\big)$ for any f in R_1. Writing $q = a + \mathrm{i}b$ for some real points a and b in $\mathbb{P}^n$ and using that f is linear, we see that $2\operatorname{Re}\big(f^2(q)\big) = 2\big(f^2(a) - f^2(b)\big)$. By the first part, we see that this quadratic form has rank at most two, and at most one positive and one negative eigenvalue. As $z \operatorname{ev}_q$ is the same as $\operatorname{ev}_{\sqrt{z}q}$ on any elements g in R_2, the claim follows. Finally, we observe that the signature of a quadratic form is subadditive. $\square$

3. Bounds for Curves

For real curves with no real points, this section bounds the maximum typical conjugation-invariant length. Despite superficial similarities to Theorem 1 of [BT15], our new inequality is essentially independent and crucial to our proof strategy. Being able to sharpen or generalize the inequality in Theorem 3.3 would ultimately translate into better degree bounds.

The next two lemmas provide topological tools for bounding the maximum typical conjugation-invariant length on certain real subvarieties. The subvariety X is *linearly normal* if the canonical map $H^0\big(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)\big) \to H^0\big(X, \mathscr{O}_X(1)\big)$ is surjective.

Lemma 3.1. *Assume that the real projective subvariety X in $\mathbb{P}^n$ is nondegenerate, linearly normal, and has positive dimension. Let R be the homogeneous coordinate ring of X. For any linear subspace W in R_1, the set of linear functionals $\ell \in R_2^*$ such that the corresponding quadratic form $\varphi_\ell \colon R_1 \to \mathbb{R}$ satisfies the following three properties, is open and nonempty in the Zariski topology.*

(1) *The restriction of quadratic form φ_ℓ to the linear subspace W has rank equal to* $\dim W$.
(2) *For any nonreal point q in X, there exists a complex number z (which may depend on ℓ) such that, for the linear functional $\ell' = z \operatorname{ev}_q + \bar{z} \operatorname{ev}_{\bar{q}}$, the quadratic form φ' corresponding to $\ell + \ell'$ has corank* 1.

(3) *The quadratic form φ' is a smooth point on the differentiable manifold M formed by the quadratic forms having corank* 1*, and the linear map ℓ' is not an element of the tangent space $T_{\varphi'}(M)$.*

Recall that Lemma 2.2 (ii) identifies elements in $T_{\varphi'}(M)$ with appropriate linear maps.

Proof. Fix a linear subspace W in the $(n+1)$-dimensional real vector space R_1.

(1) A quadratic form has maximal rank if and only if its corresponding symmetric matrix is invertible. Hence, the locus of quadratic forms arising from linear functionals on R_2, whose restriction to W has maximal rank, is open in the Zariski topology. We need to show that it is nonempty. Consider the incidence correspondence $\Xi := \big\{(\ell, p) \in R_2^* \times \mathbb{P}(W) \;\big|\; \widehat{p} \in \operatorname{Ker}(\varphi_\ell)\big\}$. The fibre over the point p in $\mathbb{P}(W)$ consists of those linear functionals $\ell\colon R_2 \to \mathbb{R}$ that annihilate the linear subspace $\big(\sum_{j=0}^n (\widehat{p})_j x_j\big) \cdot W$ in R_2. Since X is nondegenerate and irreducible, multiplication by the linear form $\sum_{j=0}^n (\widehat{p})_j x_j$ is injective for any affine representative $\widehat{p}$. Hence, the dimension of the linear subspace $\big(\sum_{j=0}^n (\widehat{p})_j x_j\big) \cdot W$ equals $\dim W$ and the fibre over the point p is a linear subspace of dimension $\dim R_2 - \dim W$. Combining the dimension of the fibres with the dimension of the base, the dimension of Ξ is bounded above by $\dim R_2 - 1$. We deduce that the projection of the incidence correspondence Ξ on first factor R_2^* cannot be surjective. Therefore, a generic quadratic form φ_ℓ has maximal rank when restricted to W.

(2) By making a linear change of variables on the polynomial ring S, we may assume that $x_n(q) \neq 0$ and $x_0, x_1, \dots, x_{n-2}$ form a basis for the linear subspace of forms in S_1 vanishing at the nonreal points q and $\overline{q}$. We may also assume that $[0:0:\cdots:0:\mathrm{i}:1]$ is an affine representative of q. With respect to these coordinates, the symmetric $\big((n+1)\times(n+1)\big)$-matrix associated to $\varphi_{\ell'}$ is

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -u & v \\ 0 & 0 & \cdots & 0 & v & u \end{bmatrix}$$

where $z := \frac{1}{2}u - \mathrm{i}\,v$ for some real numbers u and v. Suppose that $\varphi_\ell\colon S_1 \to \mathbb{R}$ is a quadratic form that has maximal rank when restricted to the linear subspace spanned by $x_0, x_1, \dots, x_{n-2}$. Hence, there exists an invertible $\big((n-1)\times(n-1)\big)$-matrix $\mathbf{A}$, a $\big((n-2)\times 2\big)$-matrix $\mathbf{B}$, and a (2×2)-matrix $\mathbf{C}$ such that the associated symmetric matrix has the block structure

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\mathsf{T} & \mathbf{C} \end{bmatrix}.$$

The Schur complement of the block $\mathbf{A}$ is the (2×2)-matrix

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} := \mathbf{C} - \mathbf{B}^\mathsf{T}\,\mathbf{A}^{-1}\,\mathbf{B}.$$

Set $z := \frac{1}{2}\alpha + \mathrm{i}\,\beta$ for the real numbers α and β. It follows that the analogous Schur complement of the symmetric matrix associated to $\varphi' := \varphi_\ell + \varphi_{\ell'}$ is the (2×2)-matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & \alpha+\gamma \end{bmatrix}$$

so the quadratic form φ' does not have maximal rank. Moreover, φ' has corank 1 if and only if this Schur complement does not have trace zero. The locus of quadratic forms, for which the leading principal $\big((n-1)\times(n-1)\big)$-submatrix is invertible and corresponding Schur complement has nonzero trace, is open in the Zariski topology. We need to show that this locus is nonempty.

To accomplish this, we exhibit a quadratic form having corank 1 such that the leading principal $\big((n-1)\times(n-1)\big)$-submatrix of the associated symmetric matrix is invertible. Since x_n is a general linear form, the Bertini Theorem [Jou83, Théorème 6.3] shows that the hyperplane section $Y := X \cap \mathrm{V}(x_n)$ is reduced and nondegenerate. Moreover, as X is linearly normal, the homogeneous coordinate ring of Y is isomorphic to $R/\langle x_n\rangle$ in degrees at most 2, so any quadratic form on Y lifts to a quadratic form on X; see [Zak99, Lemma 2.9b]. When X has dimension greater than 1, the scheme Y is irreducible and no product of linear forms vanishes on Y. Part (1) implies that there exists a quadratic form on Y whose restriction to any linear subspace has maximal rank. Hence, the leading principal $\big((n-1)\times(n-1)\big)$-submatrix of the associated symmetric matrix is invertible, and this quadratic form on Y lifts to quadratic form on X with the required properties. When X has dimension 1, the hyperplane section Y is a reduced nondegenerate set of points in linearly general position; see [ACGH85, p. 109]. Choose a minimal conjugation-invariant generating set $\{p_1, p_2, \dots, p_r, q_1, \overline{q_1}, q_2, \overline{q_2}, \dots, q_c, \overline{q_c}\}$ in Y that spans $\mathbb{P}^{n-1}$ containing r real points and c pairs of conjugate complex points. The minimality implies that $r+c$ is either n or $n+1$. For any general real numbers $a_1, a_2, \dots, a_r$ and any general complex numbers $z_1, z_2, \dots, z_c$, the quadratic form corresponding to the linear function

$$a_1 \operatorname{ev}_{p_1} + a_2 \operatorname{ev}_{p_2} + \cdots + a_r \operatorname{ev}_{p_r} + (z_1 \operatorname{ev}_{q_1} + \overline{z_1} \operatorname{ev}_{\overline{q_1}}) + (z_2 \operatorname{ev}_{q_2} + \overline{z_2} \operatorname{ev}_{\overline{q_2}}) + \cdots + (z_c \operatorname{ev}_{q_c} + \overline{z}_c \operatorname{ev}_{\overline{q_c}}),$$

has maximal rank when restricted to the linear subspaces spanned by $x_0, x_1, \dots, x_{n-1}$ and $x_0, x_1, \dots, x_{n-2}$. When $r+c = n$, the general linear combination gives a quadratic form of rank n. The restriction to the linear subspace spanned by $x_0, x_1, \dots, x_{n-2}$ is the quadratic form corresponding to the evaluations at the projections of the points to the corresponding hyperplane $\mathrm{V}(x_{n-1})$ in $\mathbb{P}^{n-1}$. The projections are still in linearly general position and hence the general linear combination still has full rank. The case $r+c = n+1$ is analogous. It follows that this quadratic form on Y lifts to quadratic form on X with the required properties.

(3) Part (2) establishes that the quadratic form φ' has corank 1, so Lemma 2.2 (i) demonstrates that this quadratic form determines a point on the differentiable manifold M. Since the irreducibility of X ensures that no nonzero linear form is a zerodivisor on R, Lemma 2.2 (ii) shows that the point φ' is nonsingular. Moreover, the tangent space $T_{\varphi'}(M)$ consists of those linear maps $\psi \colon S_2 \to \mathbb{R}$ such that $\psi(g^2) = 0$ for all polynomials in the kernel of φ'. From the chosen coordinates in part (2), we see that the linear map $\ell' \colon S_2 \to \mathbb{R}$ does not lie in the tangent space $T_{\varphi'}(M)$ if and only if the real number α, which is defined to be the $(1,1)$-entry in the Schur complement of the leading principal $\big((n-1)\times(n-1)\big)$-submatrix, is nonzero. The locus of quadratic forms, for which the leading principal $\big((n-1)\times(n-1)\big)$-submatrix is invertible and the $(1,1)$-entry in the corresponding Schur complement is nonzero, is open in the Zariski topology. It is nonempty because there exists a real number u such that, for the linear functional $\ell'' := u(\operatorname{ev}_q + \operatorname{ev}_{\overline{q}})$, the quadratic form $(\varphi_\ell + \varphi_{\ell''}) + (\varphi_{\ell'} - \varphi_{\ell''})$ satisfies all three properties. □

The second lemma brings the Euclidean topology into play.

Lemma 3.2. *Assume that the real projective subvariety X in $\mathbb{P}^n$ is irreducible, linearly normal, and has positive dimension. Let R be the homogeneous coordinate ring of X. Fix a nonreal point q in X and let $U \subset R_2^*$ be a nonempty Euclidean open set.*

(i) *There exist a linear functional ℓ in U and a complex number z such that, for the linear functional $\ell_q := z\,\mathrm{ev}_q + \bar{z}\,\mathrm{ev}_{\overline{q}}$, the quadratic form corresponding to $\ell + \ell_q$ has corank* 1.

(ii) *There exists a Euclidean open set U' containing the linear functional ℓ from part (i) and a differentiable function $\lambda\colon U' \to \mathbb{R}$ such that U' is a dense subset of U and, for all ℓ' in U', the quadratic form corresponding to $\ell' + \lambda(\ell')\,\ell_q$ has corank* 1. *Moreover, the locus of points in $\mathbb{P}^n(\mathbb{R})$ determined by the kernels of quadratic forms corresponding to elements in U' contains a Euclidean open subset.*

Proof. For part (i), Lemma 3.1 shows that the locus U'' of linear functionals $\ell\colon R_2 \to \mathbb{R}$, such that there exists a complex number z and the linear functional $\ell' := z\,\mathrm{ev}_q + \bar{z}\,\mathrm{ev}_{\overline{q}}$ for which the quadratic form φ' corresponding to $\ell + \ell'$ has corank 1 and the linear functional ℓ' is not an element in the tangent space $T_{\varphi'}(M)$, is nonempty and Zariski open. It follows that U'' has a nontrivial intersection with any nonempty Euclidean open set. Hence, we have $U'' \cap U \neq \varnothing$. Any quadratic form ℓ in $U'' \cap U$, together with the associated complex number z, proves part (i).

For part (ii), the determinant of the symmetric matrix associated to $\varphi_\ell + t\,\varphi_{\ell'}$ is a polynomial in $\mathbb{R}[t]$ having a simple root at $t = 1$, because the quadratic form $\varphi' := \varphi_\ell + \varphi_{\ell'}$ has corank 1. By the Implicit Function Theorem, it follows that there exists a Euclidean open subset U' with $\varphi_\ell \in U' \subseteq U$ and a differentiable function $\lambda\colon U' \to \mathbb{R}$ such that, for all φ in U', the quadratic form $\varphi + \lambda(\varphi)\,\varphi_{\ell'}$ has corank 1. Since the linear function ℓ' is not an element of the tangent space $T_{\varphi'}(M)$, the differential of this map is surjective at φ_ℓ. Hence, this differential is an open map is some neighbourhood of the quadratic form φ_ℓ. The locus where this fails is determined by the algebraic condition that the differential does not have full rank and is therefore lower-dimensional in U'. $\square$

To bound the maximum typical conjugation-invariant length, we restrict our attention to curves with no real points.

Theorem 3.3. *Assume that the real projective subvariety X is nondegenerate, geometrically irreducible, and has dimension* 1. *Let R be the homogeneous coordinate ring of X. When X is linearly normal and has no real points, we have the inequality* $\mathrm{mclen}(X) \leqslant 1 + \lceil (\dim R_2 - \dim R_1)/2 \rceil$.

Proof. Let $\ell\colon R_2 \to \mathbb{R}$ be a linear functional. We first reduce to points by identifying a suitable hyperplane section of the curve X. Pick a nonreal point q in X. By Lemma 3.2, there is a nonempty Euclidean open subset $U' \subset R_2^*$ containing ℓ in its closure and a differentiable function $\lambda\colon U' \to \mathbb{R}$ such that for all ℓ' and ℓ'' in U', the quadratic form corresponding to $\ell'' + \lambda(\ell'')\,\ell'$ has corank 1. Moreover, the locus of points in $\mathbb{P}^n(\mathbb{R})$ determined by the kernels of quadratic forms corresponding to elements in U' contains a Euclidean open subset. Consider the nonempty Euclidean open set $U'' \subseteq U'$ obtained by intersecting U' with the nonempty Zariski open set of linear functionals such that a generator h in S_1 of the kernel of the corresponding quadratic form is a nonzero divisor on R and $X \cap \mathrm{V}(h)$ is a reduced nondegenerate set of points in linearly general position. The Bertini Theorem [Jou83, Théorème 6.3] shows that a general hyperplane section of X is reduced and nondegenerate, and the General Position Theorem [ACGH85, p. 109] shows that the points are in linearly general position. Hence, the linear functional ℓ is in the closure of U''. It suffices to show that, for any ℓ'' in U'', we have the desired inequality for $\mathrm{clen}(\ell'')$.

To produce the desired inequality, fix a general hyperplane h and consider the hyperplane section $Y := X \cap \mathrm{V}(h)$ of X. As X is linearly normal, the homogeneous coordinate ring T of Y is isomorphic to $R/\langle h \rangle$ in degrees at most 2; see [Zak99, Lemma 2.9b]. Hence, the ring T is reduced in degree

2, so every linear functional $\ell \colon T_2 \otimes \mathbb{C} \to \mathbb{C}$ is a linear combination of point evaluations; compare with Lemma 2.3. Since Y consists of points in linearly general position and contains no real points, the $\mathbb{C}$-vector space $T_2 \otimes_{\mathbb{R}} \mathbb{C}$ has a generating set that is invariant under complex conjugation: $T_2 \otimes_{\mathbb{R}} \mathbb{C}$ is spanned by $\{\mathrm{ev}_{q_i}, \mathrm{ev}_{\overline{q_i}} \mid 1 \leqslant i \leqslant r\}$. It follows that the $\mathbb{R}$-vector space T_2 is spanned by the set $\{\mathrm{ev}_{q_i} + \mathrm{ev}_{\overline{q}_i} \mid 1 \leqslant i \leqslant r\}$. Hence, the linear functional ℓ is a linear combination of $\lceil (\dim T_2)/2 \rceil$ conjugate pairs. For all $\ell'' \in U'$, there exists $\ell' = z\,\mathrm{ev}_q + \overline{z}\,\mathrm{ev}_{\overline{q}}$ such that the quadratic form corresponding to $\ell'' + \lambda(\ell'')\ell'$ has corank 1, so we obtain

$$\mathrm{clen}(\ell'') \leqslant 1 + \mathrm{clen}\big(\ell'' + \lambda(\ell'')\ell'\big) \leqslant 1 + \big\lceil (\dim T_2)/2 \big\rceil = 1 + \big\lceil (\dim R_2 - \dim R_1)/2 \big\rceil \,.$$

Lastly, we conclude that $\mathrm{mclen}(X) \leqslant 1 + \big\lceil (\dim R_2 - \dim R_1)/2 \big\rceil$ because any $\ell \in R_2^*$ is the limit of linear functionals of conjugation-invariant length at most this bound. □

4. Nonnegative Multipliers on Surfaces

This section presents the major technical result in the paper: we exhibit cohomological conditions on a real surface that lead to certificates for nonnegativity. Assume that the real projective variety X is *totally real*, meaning the set of real points in X is Zariski dense in the set of complex points or, equivalently, the variety has a nonsingular real point. A divisor D on X is a Cartier divisor that is locally defined by a rational function with *real* coefficients. The divisor D determines the invertible sheaf or line bundle $\mathscr{O}_X(D)$ on X.

A global section f in $H^0\big(X, \mathscr{O}_X(2D)\big)$ is *nonnegative* if its sign at every real point in X is not negative. The sign is well-defined at a real point because f is locally defined by a rational function with real coefficients, the ratio of any two local representatives is the square of an invertible section of $\mathscr{O}_X$ evaluated at the real point, and the square of any real number is nonnegative. Similarly, a global section f in $H^0\big(X, \mathscr{O}_X(2D)\big)$ is a *sum of squares* if there exists global sections $h_1, h_2, \dots, h_r$ in $H^0\big(X, \mathscr{O}_X(D)\big)$ such that $f = h_1^2 + h_2^2 + \cdots + h_r^2$. Both of these properties make sense for line bundles associated to an even divisor, i.e., the square of a line bundle on X: $\mathscr{O}_X(2D) = \mathscr{O}_X(D) \otimes \mathscr{O}_X(D)$.

Building on these concepts, we introduce the following terminology.

Definition 4.1. A divisor E *supports multipliers* for a divisor D if, for any nonnegative global section f in $H^0\big(X, \mathscr{O}_X(2D)\big)$, there exists a nonzero global section g in $H^0\big(X, \mathscr{O}_X(2E)\big)$ such that the product $g\,f$ in $H^0\big(X, \mathscr{O}_X(2D+2E)\big)$ is a sum of squares.

The next result gives effective conditions for a divisor on a real surface X to support multipliers for another divisor. For any integer i and any divisor D on X, set $h^i(X,D) := \dim H^i\big(X, \mathscr{O}_X(D)\big)$.

Theorem 4.2. *Assume that X is a totally-real geometrically-integral projective surface. Let D and E be divisors on X with D free (the line bundle $\mathscr{O}_X(D)$ is globally generated), $D+E$ very ample, and $H^0\big(X, \mathscr{O}_X(E-D)\big) = H^1\big(X, \mathscr{O}_X(D+E)\big) = H^1\big(X, \mathscr{O}_X(2E)\big) = 0$. The inequality*

$$h^0(X, D+E) > 1 + \left\lceil \frac{h^0(X, 2D+2E) - h^0(X, 2E) - h^0(X, D+E) - h^1(X, E-D)}{2} \right\rceil$$

implies that the divisor E supports multipliers for the divisor D.

Proof by contrapositive. Suppose that the divisor E does not support multipliers for the divisor D. We first identify a special witness for this failure to support multipliers. By definition, there exists a nonnegative global section f in $H^0\big(X, \mathscr{O}_X(2D)\big)$ such that, for any nonzero global section g in $H^0\big(X, \mathscr{O}_X(2E)\big)$, the product $g\,f$ in $H^0\big(X, \mathscr{O}_X(2D+2E)\big)$ is not a sum of squares. Since X

is totally real and geometrically integral, the cone Σ_{2D+2E} formed by the sums of squares in $H^0\big(X,\mathscr{O}_X(2D+2E)\big)$ is pointed (closed in the Euclidean topology and contains no lines); see [BSV19, Proposition 2.5]. We deduce that the linear space $f\cdot H^0\big(X,\mathscr{O}_X(2E)\big)$ and the cone Σ_{2D+2E} are well-separated: there is a linear functional that is positive on the nonzero elements in the linear space and negative on the nonzero elements in the cone. Continuity implies that this property also holds for any global section f' sufficiently close to f in the Euclidean norm. Since the base locus of the divisor D is empty, there exist global sections $h_1,h_2,\dots,h_k$ in $H^0\big(X,\mathscr{O}_X(D)\big)$ that have no common zero on X. Hence, for any sufficiently small positive real number ε, the global section $f' := f+\varepsilon(h_1^2+h_2^2+\cdots+h_k^2)$ is positive and $f'\cdot H^0\big(X,\mathscr{O}_X(2E)\big)\cap\Sigma_{2D+2E}=\{0\}$. It follows that the set of positive global sections f' in $H^0\big(X,\mathscr{O}_X(2D)\big)$ such that the linear space $f'\cdot H^0\big(X,\mathscr{O}_X(2E)\big)$ and the cone Σ_{2D+2E} are well-separated has nonempty interior in the Euclidean topology on $H^0\big(X,\mathscr{O}_X(2D)\big)$; compare with [BSV19, Theorem 3.1]. As a consequence, this nonempty Euclidean open set must intersect the Zariski open set of global sections f'' in $H^0\big(X,\mathscr{O}_X(2D)\big)$ for which the curve Y on X defined by the vanishing of f'' is reduced and geometrically integral. Our assumption that D is free, combined with the Bertini Theorem [Jou83, Théorème 6.3], ensure that this Zariski open set is nonempty. Therefore, we may assume that the global section f in $H^0\big(X,\mathscr{O}_X(2D)\big)$ is nonnegative and that the associated curve $Y=\mathrm{V}(f)$ on X is geometrically integral and contains no real points.

We now show that the curve Y in X is equipped with some special linear functionals. Consider the section ring $\widehat{R} := \bigoplus_{n\in\mathbb{N}} H^0\big(X,\mathscr{O}_X(nD+nE)\big)$ of the surface X. As an algebraic counterpart to Y, let T be the quotient ring of $\widehat{R}$ by the ideal generated by the linear subspace $f\cdot H^0\big(X,\mathscr{O}_X(2E)\big)$ in $\widehat{R}_2 = H^0\big(X,\mathscr{O}_X(2D+2E)\big)$. Since $f\cdot H^0\big(X,\mathscr{O}_X(2E)\big)\cap\Sigma_{2D+2E}=\{0\}$, the image of the cone Σ_{2D+2E} in T_2 is pointed. Hence, there exists a Euclidean open subset of linear functionals from T_2 to $\mathbb{R}$ that are positive on the nonzero squares of elements from $T_1=\widehat{R}_1=H^0\big(X,\mathscr{O}_X(D+E)\big)$. As a second variant of Y, let $\widehat{Y}$ be the image of Y in $\mathbb{P}\big(H^0\big(X,\mathscr{O}_X(D+E)\big)^*\big)$ under the morphism determined by the complete linear series of the line bundle $\mathscr{O}_Y(D+E)$. By hypothesis, the divisor $D+E$ is very ample on X, so its restriction to the subvariety Y is also very ample. It follows that $\widehat{Y}$ is isomorphic to Y and it contains no real points. In addition, the projective curve $\widehat{Y}$ is linearly normal and its homogeneous coordinate ring $\widehat{B}$ is the subalgebra of the section ring $\widehat{T} := \bigoplus_{n\in\mathbb{N}} H^0\big(Y,\mathscr{O}_Y(nD+nE)\big)$ generated by $H^0\big(Y,\mathscr{O}_X(D+E)\big)$. Tensoring the short exact sequence

$$0 \longrightarrow \mathscr{O}_X(-2D) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_Y \longrightarrow 0$$

of coherent sheaves with the line bundle $\mathscr{O}_X(D+E)$ and the line bundle $\mathscr{O}_X(2D+2E)$, we obtain the following two short exact sequences in cohomology

$$\begin{array}{l} 0 \longrightarrow H^0\big(X,\mathscr{O}_X(D+E)\big) \longrightarrow H^0\big(X,\mathscr{O}_Y(D+E)\big) \longrightarrow H^1\big(X,\mathscr{O}_X(E-D)\big) \longrightarrow 0 \\ 0 \longrightarrow H^0\big(X,\mathscr{O}_X(2E)\big) \longrightarrow H^0\big(X,\mathscr{O}_X(2D+2E)\big) \longrightarrow H^0\big(X,\mathscr{O}_Y(2D+2E)\big) \longrightarrow 0\,, \end{array}$$

because $H^0\big(X,\mathscr{O}_X(E-D)\big)=H^1\big(X,\mathscr{O}_X(D+E)\big)=H^1\big(X,\mathscr{O}_X(2E)\big)=0$. We deduce that

$$\begin{aligned} \dim\widehat{T}_1 &= h^0(Y,D+E)=h^0(X,D+E)+h^1(X,E-D)\geqslant \dim T_1\,, \\ \dim\widehat{T}_2 &= h^0(Y,2D+2E)=h^0(X,2D+2E)-h^0(X,2E)=\dim T_2\,, \end{aligned}$$

and $T_2\cong\widehat{T}_2$. This canonical isomorphism implies that there exists a Euclidean open subset of linear functionals from $\widehat{T}_2$ to $\mathbb{R}$ that are positive on nonzero squares of the elements from $T_1\subseteq\widehat{T}_1$. From the inclusion $\widehat{B} \subseteq \widehat{T}$, we conclude that there exists a nonempty Euclidean open set U of linear functionals from $\widehat{B}_2$ to $\mathbb{R}$ that are positive on nonzero squares of the elements from $T_1 \subseteq \widehat{T}_1 = \widehat{B}_1$.

To complete the proof, we produce the appropriate inequality. Applied to the projective curve $\widehat{Y}$, Theorem 3.3 shows that there exists a linear functional $\ell \in U$ such that

$$\operatorname{clen}(\ell) \leqslant \operatorname{mclen}(X) \leqslant 1 + \left\lceil (\dim \widehat{B}_2 - \dim \widehat{B}_1)/2 \right\rceil \leqslant 1 + \left\lceil (\dim \widehat{T}_2 - \dim \widehat{T}_1)/2 \right\rceil .$$

By construction, the linear functional $\ell \colon \widehat{B}_2 \to \mathbb{R}$ is positive on the nonzero squares of elements from $T_1 \subseteq \widehat{B}_1$, so the associated quadratic form φ_ℓ is positive definite on T_1. Since the dimension of T_1 is bounded above by the number $\operatorname{clen}(\ell)$ of positive eigenvalues of the form, we have the inequality

$$h^0(X, D+E) = \dim T_1 \leqslant 1 + \left\lceil (\dim \widehat{T}_2 - \dim \widehat{T}_1)/2 \right\rceil . \qquad \square$$

When the divisors D and E are sufficiently positive, the inequality in Theorem 4.2 can be rephrased in terms of Euler characteristics. For any integer m, the *Euler characteristic* the divisor mD on X is the integer $\chi(mD) := \sum_i (-1)^i h^i(X, mD)$.

Corollary 4.3. *Assume that X is a totally-real geometrically-integral projective surface. Let D and E be divisors on X with D free, $D+E$ very ample and $h^0(X, E-D) = 0$. When $h^i(X, mE) = 0$ and $h^i(X, mD+mE) = 0$ for any positive integers i and m, the inequality*

$$\chi(2E) + h^1(X, E-D) > \chi(-D-E)$$

implies that the divisor E supports multipliers for the divisor D. When X is nonsingular and K_X is its canonical divisor, this inequality is equivalent to $h^0(X, 2E) + h^1(X, E-D) > h^0(X, K_X + D + E)$.

Proof. For the first part, it suffices to prove that the inequality in Theorem 4.2 follows from the first inequality. Because X is a surface, the Riemann–Roch Theorem [Bea96, Theorem I.12] shows that $\chi(mE)$ and $\chi(mD+mE)$ are quadratic polynomials in m with half-integer coefficients whose constant terms equal 1. Setting $\chi(E) = 1 + b_1 m + b_2 m^2$ and $\chi(mD+mE) = 1 + a_1 m + a_2 m^2$ for some coefficients b_1, b_2, a_1, a_2 in $\mathbb{Z}\left[\frac{1}{2}\right]$, the inequality $\chi(2E) + h^1(X, E-D) > \chi(-D-E)$ becomes

$$\begin{aligned} & 1 + 2b_1 + 4b_2 + h^1(X, E-D) > 1 - a_1 + a_2 \\ \Leftrightarrow \qquad & a_1 + a_2 > \frac{2 + a_1 + a_2 - h^1(X, E-D)}{2} + a_2 - (1 + b_1 + 2b_2) . \end{aligned}$$

Since $\chi(D+E) = 1 + a_1 + a_2$ is an integer, the left side of this last inequality is an integer and the right side is either an integer or half-integer. For any integer k, the inequality $\frac{k}{2} \geqslant \left\lceil \frac{k}{2} - \frac{1}{2} \right\rceil$ gives

$$\begin{aligned} \chi(D+E) - 1 = a_1 + a_2 &> \left\lceil \frac{1 + a_1 + a_2 - h^1(X, E-D)}{2} + a_2 - (1 + b_1 + 2b_2) \right\rceil \\ &= \left\lceil \frac{\chi(2D+2E) - \chi(2E) - \chi(D+E) - h^1(E-D)}{2} \right\rceil . \end{aligned}$$

For any positive integers i and m, the hypothesis that $h^i(X, mE) = 0$ and $h^i(X, mD+mE) = 0$ implies that $\chi(X, mE) = h^0(X, mE)$ and $\chi(mD+mE) = h^0(mD+mE)$. Hence, we obtain the desired inequality

$$h^0(X, D+E) > 1 + \left\lceil \frac{h^0(X, 2D+2E) - h^0(X, 2E) - h^0(X, D+E) - h^1(X, E-D)}{2} \right\rceil .$$

For the second part, the surface X is nonsingular. Serre Duality [Bea96, Theorem I.11] shows that $\chi(-D-E) = \chi(K_X + D + E)$. Since the divisor $D+E$ is very ample (and thereby ample), the Kodaira Vanishing Theorem [Laz04, Theorem 4.2.1] asserts that $h^i(X, K_X + D + E) = 0$ for all positive integers i. We conclude that $\chi(K_X + D + E) = h^0(X, K_X + D + E)$. $\square$

In some situations, Theorem 4.2 can also be used inductively. A variety is *strongly totally-real* if the real points of every nonempty Zariski open set are dense in the Euclidean topology on X. Equivalently, every real point of X is a *central point*: it lies in the Euclidean closure of the nonsingular real points; see [BCR98, §7.6]. For instance, this holds whenever the nonsingular real points of X are dense in the Euclidean topology, so it holds for nonsingular surfaces. On any strongly totally-real surface, a multiplier g in $H^0\big(X, \mathscr{O}_X(2E)\big)$ constructed via Theorem 4.2 is necessarily a nonnegative global section, which allows for the repeated applications of Theorem 4.2. To formalize this idea, we generalize Definition 4.1.

Definition 4.4. Let D and E be divisors on X. For any positive integer t, a sequence $(D_0, D_1, \dots, D_t)$ of divisors on X is a *t-step transfer* from D to E if $D = D_0$, $E = D_t$, and the divisor D_i supports multipliers for the divisor D_{i-1} for all $1 \leqslant i \leqslant t$.

Given a t-step transfer from D to E, the next corollary shows that the problem of deciding whether a global section in $H^0\big(X, \mathscr{O}_X(2D)\big)$ is nonnegative reduces to solving t semidefinite programming problems and deciding whether a global section in $H^0\big(X, \mathscr{O}_X(2E)\big)$ is nonnegative.

Corollary 4.5. *Assume that X is strongly-totally-real geometrically-integral projective surface. Let $(D_0, D_1, \dots, D_t)$ be a t-step transfer on X from a divisor D to a divisor E. A global section f_0 in $H^0\big(X, \mathscr{O}_X(2D)\big)$ is nonnegative if and only if, for any integer i satisfying $1 \leqslant i \leqslant t$, there exists a nonnegative global section f_i in $H^0\big(X, \mathscr{O}_X(2D_i)\big)$ and a sums-of-squares h_i in $H^0\big(X, \mathscr{O}_X(2D_{i-1} + 2D_i)\big)$ such that $f_{i-1} f_i = h_i$. Moreover, if f_t is a sum of squares, then both the multiplier $f_t \cdot f_{t-1} \cdots f_1$ in $H^0\big(X, \mathscr{O}_X(\sum_{i=1}^t D_i)\big)$ and the product $f_t \cdot f_{t-1} \cdots f_1 \cdot f_0$ in $H^0\big(X, \mathscr{O}_X(\sum_{i=0}^t D_i)\big)$ are sums of squares.*

Proof. By definition, the divisor D_i supports multipliers on the divisor D_{i-1} for any integer i satisfying $1 \leqslant i \leqslant t$. Hence, for any nonnegative global section f_{i-1} in $H^0\big(X, \mathscr{O}_X(2D_{i-1})\big)$, there exists a global section f_i in $H^0\big(X, \mathscr{O}_X(2D_i)\big)$ and a sum-of-squares h_i in $H^0\big(X, \mathscr{O}_X(2D_{i-1} + 2D_i)\big)$ such that $f_{i-1} f_i = h_i$. Since X is strongly totally-real, this equality establishes that f_i is also nonnegative. As this equality holds for all $1 \leqslant i \leqslant t$, the nonnegativity of f_0 implies that nonnegativity of f_t. Conversely, when f_t is a nonnegative global section and the equality $f_{i-1} f_i = h_i$ holds for all $1 \leqslant i \leqslant t$, we successively deduce that $f_{t-1}, f_{t-2}, \dots, f_1, f_0$ are all nonnegative.

Now assume that f_t in $H^0\big(X, \mathscr{O}_X(D_t)\big)$ is a sum of squares. To show the last claim, we assume that $t \geqslant 2$ and even. The odd case follows similarly. Since products of sums of squares are again sums of squares, we see that both the multiplier $f_t f_{t-1} f_{t-2} \cdots f_1 f_0 = (f_t f_{t-1}) \cdots (f_4 f_3)(f_2 f_1)$ and the product $f_t f_{t-1} f_{t-2} \cdots f_1 f_0 = f_t (f_{t-1} f_{t-2}) \cdots (f_3 f_2)(f_1 f_0)$ are sums of squares. $\square$

We end this section by clarifying the difference between the nonnegativity of global sections and the nonnegativity of homogeneous forms. For a free divisor D on X, consider the morphism $\nu \colon X \to \mathbb{P}\big(H^0\big(X, \mathscr{O}_X(D)\big)^*\big)$ determined by the complete linear series of the line bundle $\mathscr{O}_X(D)$. Let $X' := \nu(X)$ be the image subvariety and let R' be its homogeneous coordinate ring. The pullback under ν of a homogeneous element in R'_2 that is nonnegative on X' is a nonnegative global section in $H^0\big(X, \mathscr{O}_X(2D)\big)$. However, there could be nonnegative global sections in $H^0\big(X, \mathscr{O}_X(2D)\big)$ that do not correspond to nonnegative elements in R'_2. Indeed, the definitions of nonnegativity on X and X' may not coincide because the target X' could have real points that are not images under ν of the real points in X. To avoid this disparity, we offer the following definition.

Definition 4.6. A morphism $\mu\colon X \to X'$ between real varieties is *strongly dominant* if the set of real points in X' with the Euclidean topology have the image of the real points in X as a dense subset. A free divisor D on X is *strongly dominant* if the morphism from X to its image subvariety under the associated complete linear series is strongly dominant.

By design, the nonnegativity of a homogeneous element in R'_2 is equivalent to the nonnegativity of a global section in $H^0\big(X, \mathscr{O}_X(2D)\big)$ whenever the free divisor D is strongly dominant. For the sake of completeness, we exhibit a free divisor on a smooth surface that is not strongly dominant.

Example 4.7. Consider the surface $Q^{3,1}$ in $\mathbb{P}^3$ defined by $x_0^2 - x_1^2 - x_2^2 - x_3^2$ in $\mathbb{R}[x_0, x_1, x_2, x_3]$. In the affine open subset defined by $x_0 \neq 0$, the real points in $Q^{3,1}$ form a sphere S^2. Choose real line L disjoint from this sphere. It follows that the line L intersects the subvariety $Q^{3,1}$ in a pair of complex conjugate points q and $\overline{q}$. Let $Q^{3,1}(0,2)$ to be the blow-up of $Q^{3,1}$ at these two points. The real points of $Q^{3,1}(0,2)$ are in bijection with the real points of $Q^{3,1}$ under the canonical morphism $\pi\colon Q^{3,1}(0,2) \to Q^{3,1}$. Write E_1 and E_2 for the exceptional divisors on $Q^{3,1}(0,2)$ over the points q and $\overline{q}$ respectively, and set H to be the pullback to $Q^{3,1}(0,2)$ of the hyperplane class in $\mathbb{P}^3$ to $Q^{3,1}$. The complete linear system of the real divisor $H - E_1 - E_2$ corresponds to a pencil of hyperplanes in $\mathbb{P}^3$ containing the line L. The corresponding morphism $\nu\colon Q^{3,1}(0,2) \to \mathbb{P}^1$ maps the real points in $Q^{3,1}(0,2)$ to the closed interval consisting of those hyperplanes which intersect the sphere. Since this interval is not dense in the Euclidean topology on the real points in $\mathbb{P}^1$, we conclude that the divisor $H - E_1 - E_2$ is not strongly dominant. $\diamond$

5. Applications to Toric Surfaces

In this section, we apply Theorem 4.2 to the construction of nonnegativity certificates on toric surfaces.

A binary Laurent polynomial is an expression $f = \sum_{(a,b)\in S} c_{(a,b)} x^a y^b$ where $S \subseteq \mathbb{Z}^2$ is a given finite set of exponents and $c_{(a,b)}$ are real numbers. If $C \subseteq \mathbb{R}^2$ is any subset, we say that f has *monomial support on* C if $S \subseteq C$ and that f is nonnegative if $f(\alpha, \beta) \geqslant 0$ for every nonzero real numbers α, β (i.e. for $(\alpha, \beta) \in (\mathbb{R}^*)^2$ in the real points of the 2-dimensional algebraic torus). In this section, we study the problem of certifying the nonnegativity of binary Laurent polynomials with monomial support on $2P$ where P is a given lattice polygon.

The main result of this section is Theorem 5.1 which provides a combinatorial criterion on another lattice polygon Q so that $2Q$ supports multipliers for all nonnegative Laurent polynomials f with monomial support on $2P$. To state our main result we introduce the following terminology: If $A \subseteq \mathbb{R}^2$ then the number of *reduced connected components of* A is one less than the number of connected components of A and a *lattice translate of* A is a set of the form $A + m$ for $m \in \mathbb{Z}^2$. We write $\#A$ for the number of lattice points contained in A and A° for the interior of A.

Theorem 5.1. *Assume that P and Q are convex lattice polygons such that no lattice translate of P is contained in Q. Let h be the total number of reduced connected components of the set differences $P \setminus Q'$ as Q' ranges over all lattice translates of Q. The inequality*

$$\#(2Q) + h > \#\big((P+Q)^\circ\big)$$

implies that Q supports multipliers for P (i.e. for every nonnegative Laurent polynomial f with monomial support in $2P$ there exists a Laurent polynomial g with monomial support in $2Q$ such that fg is a sum of squares of Laurent polynomials with monomial support in $2(P+Q)$).

If X is a toric variety and D is a torus-invariant Weil divisor on X then the action of the torus T on X defines a grading of the cohomology groups of D. At the level of global sections this is well-known allowing us to identify the global sections of $\mathcal{O}_X(D)$ with the lattice points of the polytope $P_D \subseteq M_{\mathbb{R}}$ corresponding to D. What is less well-known is that a similar "visualization" of higher cohomology groups is also possible thanks to a theorem of Altmann, Buczynski, Kastner and Winz [ABKW20, Theorem III.6] whose proof was greatly simplified by Altmann and Ploog in [AP20, Main Theorem] which gives us a topological interpretation of the graded components $H^i(X,D)(m)$ for $m \in M$ and any torus invariant Cartier divisor D.

To precisely describe their result we need to introduce more specific notation. Assume M is the lattice of characters of a torus T and let N be the lattice of one-parameter subgroups of T. Let $\langle \cdot,\cdot \rangle : M \times N \to \mathbb{Z}$ denote the natural pairing between them. Recall that a toric variety is specified by a rational polyhedral fan F in N. Let $u_1, u_2, \dots, u_n \in N$ be the first lattice points in each ray of F and recall [Ful93] that there is a correspondence between such rays and the torus invariant divisors D_i on X. If $D = \sum a_j D_j$ define the polytope $P_D \subseteq M \otimes_{\mathbb{Z}} \mathbb{R}$ corresponding to D as

$$P_D := \{m \in M \otimes \mathbb{R} \mid \langle m, u_i \rangle \geqslant -a_i \text{for } i = 1,2,\dots,n\}.$$

Recall that the global sections of $\mathcal{O}_X(D)$ are in correspondence with the lattice points $P_D \cap M$. There are easy combinatorial criteria for determining when D is a Cartier divisor [CLS11, Theorem 4.2.8] and when D is nef [CLS11, Theorem 6.3.12 and Proposition 6.1.1]. Furthermore every torus-invariant Cartier divisor D is linearly equivalent to a difference of T-invariant nef divisors $D = D^+ - D^-$ and any such decomposition can be used to compute the cohomology groups of D via the following.

Theorem 5.2 ([AP20, Main Theorem])**.** *If X is a projective toric variety and $D = D^+ - D^-$ is a difference of nef divisors with corresponding polytopes $\Delta^+, \Delta^- \subseteq M_{\mathbb{R}}$ then there is an isomorphism of vector spaces*

$$H^i\big(X, \mathcal{O}_X[D]\big)(m) \cong \widetilde{H}^{i-1}\big(\Delta^- \setminus (\Delta^+ - m)\big)$$

where $\widetilde{H^i}$ denotes the reduced singular cohomology groups of a topological space and $\Delta^+ - m$ means the translate by $-m$ of Δ^+ in $M_{\mathbb{R}}$.

Proof of Theorem 5.1. Let X be the normal toric variety defined by the normal fan of the lattice polygon $P+Q$. Via the usual correspondence [CLS11, Proposition 6.1.10], the polygons P and Q define torus-invariant Weil divisors D_P and D_Q on X whose corresponding sheaves have spaces of global sections spanned by the lattice points of P and Q respectively. Restricting sections to the points of the torus $T \subseteq X$ we obtain a correspondence between Laurent polynomials supported in P (resp. Q) and global sections of $H^0(X, D_P)$ (resp. $H^0(X, D_Q)$).

Furthermore the divisors D_P and D_Q are Cartier divisors with Cartier data given by the vertices of P and Q [CLS11, Theorem 4.2.8]. Since the vertices of P and Q are global sections of the corresponding line bundles we conclude that D_P and D_Q are basepoint-free and therefore nef divisors on X [CLS11, Theorem 6.3.12 and Proposition 6.1.1]. Applying Theorem 5.2 we obtain:

1. $h^i(X, mD) = 0$ for any positive integer i, any positive integer m, and any divisor D in $\{D_Q, D_P, D_P + D_Q\}$. This follows from writing $D = D - 0$.

2. The quantity $h^1(X, D_Q - D_P)$ equals the total sum of the dimensions of the reduced singular cohomology groups $\widetilde{H}^0(P \setminus (Q-m))$ as m ranges over $\mathbb{Z}^2$. So $h^1(X, D_Q - D_P)$ agrees with the quantity h defined in the statement of the Theorem.

By Corollary 4.3 and Theorem 4.2, we conclude that a sufficient condition for D_Q to support multipliers for D_P is given by the inequality

$$\phi_Q(2) + h > \phi_{P+Q}(-1)$$

where $\phi_E(m) = \chi(\mathscr{O}_X[mD_E])$ coincides with the Ehrhart polynomial of $E \in \{Q, P+Q\}$. By Ehrhart reciprocity (or toric Serre duality) the right-hand side equals the number of interior lattice points of $P+Q$, proving the claim. □

Any multiplier g constructed via Theorem 5.1 is necessarily nonnegative. Iterated application of Theorem 5.1 can therefore lead to rational sum-of-square certificates of nonnegativity provided the multiplier polygons Q are chosen judiciously. This follows from [BSV16, Theorem 1.1] applied to the case of surfaces: on totally-real non-degenerate surfaces of minimal degree, every nonnegative quadric is a sum of squares. Surfaces of minimal degree are classified (by Bertini) and happen to be toric, corresponding to the $2\Delta = \operatorname{conv}\{(0,0),(2,0),(0,2)\}$ and Lawrence prisms of dimension 2. A lattice polygon S is a Lawrence prism with heights h_1, h_2 if it is lattice congruent to $\operatorname{conv}(0, e_1, h_1e_2, e_1 + h_2e_2)$ for some nonnegative integers h_1, h_2.

Example 5.3. Let P be a square in $\mathbb{Z}^2$ with side length 2. Hence, the forms with support in $2P$ correspond to bihomogeneous forms in two sets of variables (x_1, y_1) and (x_2, y_2) which have degree 4 with respect to each pair (x_i, y_i). For the polytope Q, we choose a square in $\mathbb{Z}^2$ with side length 1. As illustrated in Figure 5.4, we have $h = 0$, $\#(2Q) = 9$ and $\#(P+Q)^\circ = 4$. Thus, Theorem 5.1 establishes that Q supports multipliers for P. Since Q is a Lawrence prism every nonnegative multiplier g is a sum of squares. ◇

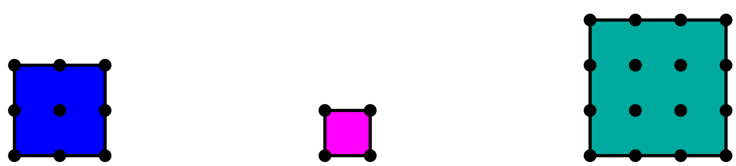

FIGURE 5.4. The lattice polygons $P = 2Q$, Q, and $P+Q$

Let $X \subset \mathbb{P}^3$ be the toric surface corresponding to embedding $\mathbb{P}^1 \times \mathbb{P}^1$ via the line bundle $\mathscr{O}(1) \times \mathscr{O}(1)$, and let R denote the graded coordinate ring of X. Example 5.3 shows that degree 4 nonnegative forms on X have a sum of squares multiplier of degree 2. We now generalize this to multiplier degree bounds for nonnegative forms of degree $2d$ on X. This is done by iteratively transferring nonnegativity to simpler polygons, until we can descend to a variety of minimal degree. This is a warm-up to our improvement of Hilbert's bounds for ternary forms, but polygonal geometry of rectangles is slightly simpler than that of triangles.

Example 5.5. Let $P = [0,d]^2$ be the square in $\mathbb{Z}^2$ with side length d, which corresponds to degree d forms on $X = \mathbb{P}^1 \times \mathbb{P}^1$. We let Q be the square with side length $d-1$. Then we have $h = 0$, $\#(2Q) = (2d-1)^2$ and $\#(P+Q)^\circ = (2d-2)^2$. Therefore, we transfer nonnegativity from $2P$ to $2Q$: given a nonnegative form g_0 with support in $2P$ we can find a nonnegative form g_1 with support in $2Q$ such that g_0g_1 is a sum of squares. We can now apply this result to g_1 and continue. We produce a sequence of nonnegative multipliers $g_1,\dots,g_{d-1}$ with $g_i \in R_{2d-2i}$ such that g_ig_{i+1} is a sum of squares for $i=0,\dots,d-2$. Note that g_{d-1} is a sum of squares in R_2. It follows that for any nonnegative form f of degree $2d$ on X there exists a sum of squares multiplier g with $\deg g = d(d-1)$ such that fg is a sum of squares, see Corollary 4.5.

We can improve on this bound by utilizing rectangles instead of squares. Concretely, we can now lower the degree from $2d$ to $2d-6$ in two steps, while in the previous strategy we went from $2d$ to $2d-4$. So let P be again the square in $\mathbb{Z}^2$ with side length d, and take $Q_1 = [0,d-1]\times[0,d-2]$ to be the $(d-1)\times(d-2)$ rectangle. Then $h=0$, $\#(2Q_1) = (2d-1)(2d-3)$ and $\#(P+Q_1)^\circ = (2d-2)(2d-3)$. Therefore we can transfer nonnegativity from the $d\times d$ square to the $(d-1)\times(d-2)$ rectangle. Next, let Q_2 be the $(d-3)\times(d-3)$ square. Then we have $h=0$, $\#(2Q_2)=(d-5)^2$ and $\#(Q_1+Q_2)^\circ = (d-5)\times(d-6)$. Therefore we can transfer nonnegativity from the $(d-1)\times(d-2)$ rectangle to the $(d-3)\times(d-3)$ square. So in two steps, we went from $P=[0,d]^2$ to $[0,d-3]^2$ improving the degree bounds faster than in the first strategy which only used squares.

This allows us to improve degree bounds. For instance when $d\equiv 1 \mod 3$, we have that a nonnegative form f of degree $2d$ on $\mathbb{P}^1\times\mathbb{P}^1$ has a sum of squares multiplier g with $\deg g = d(d-1)-\frac{1}{3}d(d-1) = \frac{2}{3}d(d-1)$ such that fg is a sum of squares, see Corollary 4.5.

We do not make any claims on optimality of these bounds, especially for high degree d. It is possible to use polygons that are different from rectangles, and they may lead to tighter bounds. $\diamond$

We now use the freedom of choosing multipliers with special support to give an improvement to Hilbert's rational sum-of-squares certificates for ternary forms. We first explain Hilbert's method. Let Δ be the right triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. We call the polygon $d\cdot\Delta$ the Veronese triangle of degree d.

Example 5.6 (Hilbert's bound for ternary forms)**.** [Hil93] shows that for any nonnegative ternary form f of degree $2d$, there exists a nonnegative form g_1 of degree $2d-4$ such that g_1f is a sum of squares. We can derive this result from Theorem 5.1 by setting $P=d\cdot\Delta$ to be the Veronese triangle of degree d and $Q=(d-2)\cdot\Delta$ to be the Veronese triangle of degree $d-2$. We have $h=0$, $\#(2Q)=\binom{2d-2}{2}$ and $\#(P+Q)^\circ=\binom{2d-3}{2}$, and therefore we can transfer nonnegativity from P to Q.

We can now apply the result to g_1 and produce a multiplier g_2 of degree $2d-8$ such that g_2g_1 is a sum of squares. Applying this result iteratively we eventually produce a nonnegative multiplier g_k of degree either 2 or 4, such that g_kg_{k-1} is a sum of squares. We observe that g_k must be a sum of squares, since nonnegative ternary quartics and quadrics are sums of squares by Hilbert's earlier result [Hil88]. Therefore we see that a nonnegative form of degree $2d$ has a sum of squares multiplier g such that fg is a sum of squares and $\deg g = \frac{1}{2}d(d-2)$ when d is even, and $\deg g = \frac{1}{2}(d-1)^2$ when d is odd, see Corollary 4.5. $\diamond$

Remark 5.7. We cannot drop degree by more than 4 in Hilbert's method using our inequality: if P is the Veronese triangle of degree d and Q is the Veronese triangle of degree $d-3$, then the numbers come out to be $h=0$, $\#(2Q)=\binom{2d-4}{2}$ and $\#(P+Q)^\circ=\binom{2d-4}{2}$. Thus we obtain equality, instead of strict inequality in Theorem 5.1. For the cases of ternary sextics and octics, i.e. $d=3,4$ we know that we cannot transfer nonnegativity of degree $2d$ to degree $2d-6$ [BSV19], and therefore we see that the bound of Theorem 5.1 is tight in these cases.

Even though we cannot immediately go from degree d Veronese triangle to degree $d-3$ Veronese triangle, we note that the inequality from degree $d-2$ Veronese triangle has some slack in it (see

Example 5.6). Therefore, we may choose a smaller polytope for Q than the $d-2$ Veronese triangle, and this allows us to arrive at a variety of minimal degree faster, similar to Example 5.5.

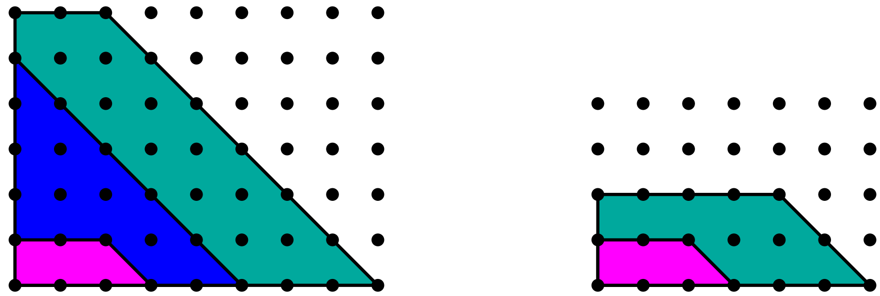

FIGURE 5.8. Left: P (blue), Q (magenta) and $P+Q$ (emerald). Right: Q (magenta) and $2Q$ (emerald).

Example 5.9 (Improving Hilbert's bound for degree 10)**.** Let P be the Veronese triangle of degree 5, and Q be the Lawrence prism with heights 3 and 2. Then, as can be seen in Figure 5.8, $\#(2Q) = 18$ and $\#(P+Q)^\circ = 20$ and furthermore $h = 3$ since there are exactly three lattice translates of Q which disconnect P. Since all nonnegative Laurent polynomials with support in $2Q$ are sums of squares by [BSV19], it follows that any nonnegative ternary form f of degree 10 has a sum of squares multiplier g of degree 6 such that gf is a sum of squares. This improves Hilbert's bound from 1893, which was the best known bound.

Applying Hilbert's 1893 result iteratively to a ternary form f of degree 10 leads to a sum-of-squares multiplier g of degree 8. Using the flexibility in choosing polygons that are not Veronese triangles, our method shows that the multiplier g can be taken to be a ternary sextic, whose monomial support lies in twice a Lawrence prism so that g is already a sum of squares. Using our method, we do not need a second iteration step. $\diamond$

Example 5.10 (Improving Hilbert's bound for general degrees)**.** Given two nonnegative integers d and m, let $T(d,m)$ be the lattice trapezoid defined by inequalities $x \geqslant 0$, $0 \leqslant y \leqslant d-m$, and $x+y \leqslant d$. The trapezoid $T(d,m)$ corresponds to forms of degree d vanishing to order m at a torus-invariant point of $\mathbb{P}^2$. We can think of $T(d,m)$ as the Veronese triangle of degree d with a cut off corner.

Let $P = T(d_1,m_1)$ and $Q = T(d_2,m_2)$ (see Figure 5.8, which shows $T(8,2)$ in emerald on the left). Then

$$\#(2Q) = \binom{2d_2+2}{2} - \binom{2m_2+1}{2} \qquad \text{and} \qquad \#(P+Q)^\circ = \binom{d_1+d_2-1}{2} - \binom{m_1+m_2}{2}.$$

We take $d_2 \leqslant d_1$.

As we observed in Example 5.6, Hilbert's proof took $m_1 = m_2 = 0$ and $d_2 = d_1 - 2$, and choosing $m_1 = m_2 = 0$ and $d_2 = d_1 - 3$ is not possible, since the strict inequality required to apply Theorem 5.1 is an equality. However, we can decrease the degree by 3, if we first "bite off" a corner of the Veronese triangle.

This leads to the following procedure: take one step to "bite" off a corner of the degree d Veronese triangle $d\Delta$ as much as possible. At the start we have $m_1 = 0$ and $d_1 = d_2 = d$, so that $\#(P+Q)^\circ = \binom{2d-1}{2} - \binom{m_2}{2}$ and $\#(2Q) = \binom{2d+2}{2} - \binom{2m_2+1}{2}$. We can make $m_2 \approx 2\sqrt{d}$. For the next step we have $d_1 = d$ and $m_1 \approx 2\sqrt{d}$. Then we can take $d_2 = d_1 - 3$ and $m_2 = m_1 - 3$:

$$\#(2Q) - \#(P+Q)^\circ = \binom{2m_1-4}{2} - \binom{2m_1-5}{2} > 0\,.$$

From this inequality we see that we can continue decreasing the degree by 3 until the "bitten off" corner (i.e. m_1) falls below 3. Then we repeat.

In this process we take roughly $d/3$ steps and, in each step (except roughly $\sqrt{d}$ "biting off" steps), we decrease the degree by 3. Therefore, the total degree of the multiplier is bounded by $\frac{d^2}{6} + \textit{lower order terms}$, which is an asymptotic improvement over Hilbert's bound, which has a leading order of $\frac{d^2}{4}$. $\diamond$

6. Applications to del Pezzo Surfaces

We now characterize nonnegative global sections for all even divisors on totally-real del Pezzo surfaces having degree at least 3. Beyond their prominence in the theory of algebraic surfaces, del Pezzo surfaces are interesting within real algebraic geometry because they admit several distinct real structures. More significantly, these surfaces are a successor to varieties of minimal degree (surfaces X in $\mathbb{P}^n$ such that $\deg(X) = 1 + \operatorname{codim}(X)$); see [Dol12, §8.1]. Indeed, the linearly normal nonsingular surfaces of almost minimal degree (surfaces X in $\mathbb{P}^n$ such that $\deg(X) = 2 + \operatorname{codim}(X)$) are del Pezzo surfaces of degree at least 3 embedded via their anticanonical linear series; see [Dol12, §8.3]. From this perspective, our characterization extends the degree bounds for sum-of-squares multipliers on surfaces of minimal degree in [BSV19, Theorem 1.2].

The theorem in this section encapsulates the major insights by showing that certificates of nonnegativity for global sections of a divisor $2D$ may always be obtained from analogous certificates for simpler divisors.

Theorem 6.1. *Let X be a totally-real del Pezzo surface having degree at least 3 and canonical divisor K_X. For any nonzero real effective divisor D on X, there exists a finite sequence $D_0, D_1, \dots, D_k$ of effective divisors on X with $D_0 = D$ such that $-K_X \cdot D_i < -K_X \cdot D_{i-1}$, D_i supports multipliers for D_{i-1} for all $1 \leqslant i \leqslant k$, and D_k is either zero or a positive multiple of a conic bundle. In particular, the length k of the sequence is bounded above by $-K_X \cdot D$.*

Before delving into the proof, we recount some features of del Pezzo surfaces. A *del Pezzo surface* is a nonsingular geometrically-irreducible surface X whose anticanonical divisor $-K_X$ is ample; see [Dol12, Definition 8.1.2]. Its *degree* is the self-intersection number $d := K_X \cdot K_X$, which satisfies $1 \leqslant d \leqslant 9$; see [Dol12, Proposition 8.1.6]. Over $\mathbb{C}$, del Pezzo surfaces form a single sequence with one exception. Other than $\mathbb{P}^1 \times \mathbb{P}^1$ which has degree 8, a del Pezzo surface is a blow-up of $\mathbb{P}^2$ at $9-d$ general points: no three lie on a line, no six line on a conic, and no eight lie on a singular cubic with one of the points at the singularity; see [Dol12, Proposition 8.1.18]. Hence, the del Pezzo surfaces having degree d greater than 5 are all normal toric varieties. However, there are infinitely many nonisomorphic del Pezzo surfaces of each degree less than 5; see [Dol12, Sections 8.5–8.8].

Over the complex numbers, the birational geometry of del Pezzo surfaces is comparatively simple. The Picard group of a del Pezzo surface is a free abelian group of rank $10-d$. For the special case $\mathbb{P}^1 \times \mathbb{P}^1$, the Picard group is generated by the divisor classes of the two rulings. Otherwise, $X = \operatorname{Bl}_{p_1,p_2,\dots,p_{9-d}}(\mathbb{P}^2)$ and the Picard group $\operatorname{Pic}(X)$ is generated by the pullback H of the hyperplane class on $\mathbb{P}^2$ (under the canonical morphism $\pi\colon X \to \mathbb{P}^2$) and the classes of the exceptional divisors

$E_1, E_2, \dots, E_{9-d}$, which are the preimages of the points $p_1, p_2, \dots, p_{9-d}$. In particular, we have $K_X = -3H + E_1 + E_2 + \cdots + E_{9-d}$. Moreover, the intersection product is determined by $H \cdot H = 1$, $H \cdot E_i = 0$ for all $1 \leqslant i \leqslant 9 - r$, and $E_i \cdot E_j = -\delta_{i,j}$ for all $1 \leqslant i \leqslant j \leqslant 9 - r$. A (-1)-*curve* on a surface X is a divisor class C satisfying $C \cdot C = -1$ and $K_X \cdot C = -1$ where K_X is the canonical divisor on X. When $1 \leqslant d \leqslant 7$, the cone of curves on a del Pezzo surface of degree d is the closed cone in the real vector space $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by the classes of (-1)-curves; see [Dol12, Theorem 8.2.19]. Moreover, the finitely many (-1)-curves are explicitly enumerated in [Dol12, Proposition 8.2.15].

Over the real numbers, the classification of del Pezzo surfaces is more involved because a complex del Pezzo surface may have more than one real structure. The two del Pezzo surfaces of degree 8 in the subsequent example begin to reveal some of the intricacies; see [Rus02, Proposition 1.2].

Example 6.2. Consider the totally-real surfaces $Q^{2,2}$ and $Q^{3,1}$ in $\mathbb{P}^3$ defined by $x_0^2 + x_1^2 - x_2^2 - x_3^2$ and $x_0^2 + x_1^2 + x_2^3 - x_3^2$ respectively. Over $\mathbb{C}$, these subvarieties are isomorphic (in fact, projectively equivalent) because their defining quadratic polynomials have the same rank. No such isomorphism exists over $\mathbb{R}$, because the quadratic polynomials have different signatures. Topologically, the set $Q^{2,2}(\mathbb{R})$ of real points is the torus $S^1 \times S^1$ and $Q^{3,1}(\mathbb{R})$ is the sphere S^2 which are not homeomorphic. Geometrically, the real variety $Q^{2,2}$ is ruled by real lines, whereas conjugation on $\mathbb{P}^3$ exchanges the complex lines of the two rulings through each real point of $Q^{3,1}$. $\diamond$

More generally, a real structure on a complex variety is a choice of an antiholomorphic involution. The real points are, by definition, the subset of points fixed by the involution. For any real scheme X, its complexification $X_{\mathbb{C}} := X \times_{\operatorname{Spec}(\mathbb{R})} \operatorname{Spec}(\mathbb{C})$ has a canonical antiholomorphic involution induced by complex conjugation on $\mathbb{C}$. The study of equivalence classes of real structures on a complex projective variety $X_{\mathbb{C}}$ is equivalent to the study of isomorphism classes of real projective varieties X whose complexification is isomorphic over $\mathbb{C}$ to $X_{\mathbb{C}}$; see [Rus02, §1]. The ensuing example, constructed via an antiholomorphic involution, manifests a real del Pezzo surface of degree 4 whose real points form a disconnected topological space; see [Rus02, Example 2] for further details.

Example 6.3. Let Γ be a nonsingular real plane cubic curve having two real components. Choose a general real point p_1 on Γ. The intersection of Γ with its polar curve with respect to p_1 has degree 6. Since p_1 is a general point on Γ, this intersection contains of four distinct real points p_2, p_3, p_4, p_5 on Γ (in addition to p_1) such that the tangent to Γ at p_i passes through p_1 for all $2 \leqslant i \leqslant 5$; see [Dol12, Theorem 1.1.1]. Let $\mathbb{D}_{\mathbb{C}} := \operatorname{Bl}_{p_1, p_2, \dots, p_5}(\mathbb{P}^2)$ be the associated complex del Pezzo surface of degree 4. The de Jonquières birational involution of $\mathbb{P}^2$ is uniquely determined by the property that its restriction to a general line L passing through p_1 coincides with the involution of $\mathbb{P}^1$ that interchanges the residual intersection points of L with Γ and fixes p_1; see [Dol12, §7.3.6]. This birational involution lifts to an antiholomorphic involution $\tau \colon \mathbb{D}_{\mathbb{C}} \to \mathbb{D}_{\mathbb{C}}$ that sends E_1 to $2H - E_1 - E_2 - E_3 - E_4 - E_5$ and sends E_i to $H - E_1 - E_i$ for all $2 \leqslant i \leqslant 5$. Hence, every (-1)-curve C on $\mathbb{D}$ satisfies $C \cdot \tau(C) = 1$. Setting $\mathbb{D}$ to be the totally-real del Pezzo surface corresponding to this antiholomorphic involution on $\mathbb{D}_{\mathbb{C}}$, we see that its set of real points is $S^2 \sqcup S^2$. $\diamond$

To catalogue the relevant real structures on complex del Pezzo surfaces, we collect some notation. For nonnegative integers a and b, let $Q^{a,b}$ be the real subvariety in $\mathbb{P}^{a+b-1}$ defined by the quadratic polynomial $x_0^2 + x_1^2 + \cdots + x_{a-1}^2 - x_a^2 - x_{a+1}^2 - \cdots - x_{a+b-1}^2$ in $\mathbb{R}[x_0, x_1, \dots, x_{a+b-1}]$. For any real surface X, let $X(a, 2b)$ be the real surface obtained from X by blowing-up a distinct real points and b pairs of conjugate nonreal points. With these definitions, we have $Q^{2,2} \cong Q^{2,1} \times Q^{2,1}$, $\mathbb{P}^2(2,0) \cong$

$Q^{2,2}(1,0)$, and $\mathbb{P}^2(0,2) \cong Q^{3,1}(1,0)$. Table 6.4 lists the 24 totally-real del Pezzo surfaces of degree at least 3; see [Rus02, Proposition 1.2 and Corollaries 2.4, 3.2–3.3] or [Kol01, Proposition 86] for a complete classification including those containing no real points. In Table 6.4, the column heading "$\rho(X_{\mathbb{R}})$" stands for the rank of the real Picard group of X and the column heading "# real (-1)'s" is an abbreviation for the number of real (-1)-curves on X.

TABLE 6.4. Totally-real del Pezzo surfaces of degree at least 3

Degree	X	$\rho(X_{\mathbb{R}})$	# real (-1)'s
9	$\mathbb{P}^2$	1	0
8	$\mathbb{P}^2(1,0)$	2	1
8	$Q^{2,2}$	2	0
8	$Q^{3,1}$	1	0
7	$\mathbb{P}^2(2,0)$	3	3
7	$\mathbb{P}^2(0,2)$	2	1
6	$\mathbb{P}^2(3,0)$	4	6
6	$\mathbb{P}^2(1,2)$	3	6
6	$Q^{3,1}(0,2)$	2	0
6	$Q^{2,2}(0,2)$	3	0
5	$\mathbb{P}^2(4,0)$	5	10
5	$\mathbb{P}^2(2,2)$	4	4

Degree	X	$\rho(X_{\mathbb{R}})$	# real (-1)'s
5	$\mathbb{P}^2(0,4)$	3	2
4	$\mathbb{P}^2(5,0)$	6	16
4	$\mathbb{P}^2(3,2)$	5	8
4	$\mathbb{P}^2(1,4)$	4	4
4	$Q^{3,1}(0,4)$	3	0
4	$Q^{2,2}(0,4)$	4	0
4	$\mathbb{D}$	2	0
3	$\mathbb{P}^2(6,0)$	7	27
3	$\mathbb{P}^2(4,2)$	6	15
3	$\mathbb{P}^2(2,4)$	5	7
3	$\mathbb{P}^2(0,6)$	4	3
3	$\mathbb{D}(1,0)$	3	3

Over the real numbers, the birational geometry of surfaces is more complicated: the real Picard group may have smaller rank and there are more possibilities for the extremal rays in the cone of curves. Conic bundles provide one new kind of extremal ray. On a del Pezzo surface X, a *conic bundle* is a divisor class B such that $B \cdot B = 0$ and $-K_X \cdot B = 2$. By the Riemann–Roch Theorem, the complete linear series of B defines a surjective morphism $\pi_B \colon X \to \mathbb{P}^1$ such that every fibre is isomorphic to a plane conic. As [Kol01, Theorem 29] establishes that a conic bundle can be an extremal ray only when the rank of the real Picard group is 2, the following example shows that inventorying the minimal conic bundles is relatively straightforward.

Example 6.5. From Table 6.4, we see that there are 5 totally-real del Pezzo surfaces with real Picard rank equal to 2. Since any 2-dimensional closed convex cone has two extremal rays, there are just two divisors classes on each surface to analyze.

- Suppose that $X = \mathbb{P}^2(1,0)$. Let H denote the pullback of the hyperplane class on $\mathbb{P}^2$ and let E_1 be the exceptional divisor over the distinguished real point in $\mathbb{P}^2$. The extremal rays on X are the real (-1)-curve E_1 and the real conic bundle $B := H - E_1$. Moreover, [Kol01, Theorem 29] implies that $\pi_B\big(X(\mathbb{R})\big) = \mathbb{P}^1(\mathbb{R})$.
- Suppose that $X = Q^{2,2} \cong Q^{2,1} \times Q^{2,1}$. The extremal rays on X are given by the divisor classes of the two real rulings which are real conic bundles. In either case, we have $\pi_B\big(X(\mathbb{R})\big) = \mathbb{P}^1(\mathbb{R})$.
- Suppose that $X = \mathbb{P}^2(0,2) \cong Q^{2,2}(1,0)$. One extremal ray contracts the real (-1)-curve and the other contracts the disjoint pair of conjugate exceptional curves, so there is no conic bundle. As

overkill, [Kol01, Theorem 29] also proves that a conic bundle can only be an extremal ray on a del Pezzo surface having even degree.

- Suppose that $X = Q^{3,1}(0,2)$. Let L_1 and L_2 be the pullback of the two rulings on $Q^{3,1}$ and let E_1 and E_2 be the exceptional divisors over the distinguished pair of conjugate nonreal points. One extremal ray contracts the disjoint pair of conjugate exceptional curves. The second is the real conic bundle $B := L_1 + L_2 - E_1 - E_2$. [Kol01, Theorem 29] confirms that $\pi_B \colon X \to \mathbb{P}^1$ has two singular fibres, so the image $\pi_B\big(X(\mathbb{R})\big)$ is a closed interval in $\mathbb{P}^1(\mathbb{R})$ whose endpoints correspond to the singular fibres.
- Suppose that $X = \mathbb{D}$. Let H denote the pullback of the hyperplane class on $\mathbb{P}^2$ and let $E_1, E_2, \dots, E_5$ be the exceptional divisors over the special real points in $\mathbb{P}^2$. The extremal rays on X are the two real conic bundles $H - E_1$ and $2H - E_2 - E_3 - E_4 - E_5$. In both cases, [Kol01, Theorem 29] establishes that $\pi_B \colon X \to \mathbb{P}^1$ has four singular fibres, so the image $\pi_B\big(X(\mathbb{R})\big)$ consists of 2 disjoint closed intervals in $\mathbb{P}^1(\mathbb{R})$ whose endpoints correspond to the singular fibres. $\diamond$

Remark 6.6. Applying the minimal model program for real algebraic surface [Kol01, Theorem 30], we see that every totally-real del Pezzo surface X of degree at least 3 is obtained from $\mathbb{P}^2$, $Q^{2,2}$, $Q^{3,1}$, or $\mathbb{D}$ from a sequence of blow-ups at either a real point or a pair of conjugate nonreal points. The birational map associated to either type of blow-up is strongly dominant over $\mathbb{R}$. It follows that any real conic bundle on X is the pullback of a minimal conic bundle appearing in Example 6.5.

Global sections of a real conic bundle may require modified certificates of nonnegativity.

Remark 6.7. Let B be a real conic bundle on a totally-real del Pezzo surface X and let $\pi_B \colon X \to \mathbb{P}^1$ be the associated surjective morphism. For any positive integer c, consider f in $H^0\big(X, \mathscr{O}_X(2cB)\big)$. The global section f is the pullback of a unique homogeneous polynomial g in $\mathbb{R}[x_0, x_1]$ of degree $2c$. Moreover, f is nonnegative if and only if g is nonnegative on $\pi_B\big(X(\mathbb{R})\big)$. When $\pi_B\big(X(\mathbb{R})\big) = \mathbb{P}^1(\mathbb{R})$, every nonnegative g can be expressed as a sum of squares in $\mathbb{R}[x_0, x_1]$. For the remaining cases, choose real coordinates on $\mathbb{P}^1$ such that $[1:0]$ is not in $\pi_B\big(X(\mathbb{R})\big)$. Under the map $[x_0 : x_1] \mapsto x_0/x_1$, the image $\pi_B\big(X(\mathbb{R})\big)$ corresponds to the closed interval $[a_0, a_1]$ or the disjoint union $[a_0, a_1] \sqcup [a_2, a_3]$. In the first case, every nonnegative g can be expressed as $h_0 + h_1(a_0 x_1 - x_0)(x_0 - a_1 x_1)$ where h_0 and h_1 are sums of squares in $\mathbb{R}[x_0, x_1]$. In second case, every nonnegative g can be expressed as $h_0 + h_1(x_0 - a_1 x_1)(x_0 - a_2 x_1) + h_2(a_0 x_1 - x_0)(x_0 - a_4 x_1)$ where h_0, h_1, and h_2 are sums of squares.

Given this background on totally-real del Pezzo surfaces having degree at least 3, we now present four lemmas about divisors needed for our proof of Theorem 6.1. A divisor D on a surface X is *nef* if $D \cdot C \geqslant 0$ for any effective divisor C.

Lemma 6.8. *Let X be a totally-real del Pezzo surface of degree at least* 3. *Assume that D is a nonzero effective divisor on X. When the divisor D is not nef, there exists a divisor E on X, which is either a real (-1)-curve or the sum of a disjoint pair of conjugate complex (-1)-curves, such that $H^0\big(X, \mathscr{O}_X(D)\big) = H^0\big(X, \mathscr{O}_X(D - E)\big)$. When the divisor D is nef but not ample, then either*

(i) *there exists a real (-1)-curve C such that $D \cdot C = 0$,*
(ii) *there is a pair $(C, \overline{C})$ of disjoint conjugate (-1)-curves such that $D \cdot C = D \cdot \overline{C} = 0$, or*
(iii) *D is a positive multiple of a conic bundle.*

In cases (i) *and* (ii)*, there exists a real birational morphism $\pi \colon X \to X'$ from X to a totally-real del Pezzo surface X' of larger degree than X and a nef divisor N on X' such that $D = \pi^*(N)$.*

Proof. Since the divisor D is not nef, there exists a (-1)-curve C such that $D \cdot C < 0$. Assuming that $C \neq \overline{C}$ and $C \cdot \overline{C} \neq 0$, the divisor $B := C + \overline{C}$ would be a real conic bundle satisfying $D \cdot B < 0$, which contradicts the hypothesis that D is effective because every conic bundle is nef. It follows that either C is real or $(C, \overline{C})$ is a disjoint pair of conjugate (-1)-curves. Let E be the real divisor defined by either C or $C + \overline{C}$. The long exact sequence in cohomology associated to the short exact sequence

$$0 \longrightarrow \mathscr{O}_X(D-E) \longrightarrow \mathscr{O}_X(D) \longrightarrow \mathscr{O}_E(D) \longrightarrow 0$$

yields the desired equality of global sections.

Suppose that D is nef but not ample. When $d \leqslant 7$, there exists a (-1)-curve C such that $D \cdot C = 0$. Assuming that $C \neq \overline{C}$ and $C \cdot \overline{C} \neq 0$, the divisor $B := C + \overline{C}$ is a real conic bundle satisfying $D \cdot B = 0$ which implies that D is a positive multiple of B. When either C is real or $(C, \overline{C})$ is a disjoint pair of conjugate (-1)-curves, let E be the real divisor defined by either C or $C + \overline{C}$. The target of the real birational morphism $\pi \colon X \to X'$ that contracts E is a totally-real del Pezzo surface of higher degree. Since $D \cdot E = 0$, we see that D is a pullback of a nef divisor on X'. Lastly, when $8 \leqslant d \leqslant 9$, the hypothesis that D is nef but not ample implies that $d = 8$ and there are two options: D is either a positive multiple of $H - E_1$ on $X = \mathbb{P}^1(1,0)$ or a positive multiple of a real ruling on $X = Q^{2,2}$. Thus, Example 6.5 implies that the D is a positive multiple of a conic bundle in these cases. □

Every ample divisor D on a complex del Pezzo surface having degree at least 3 can be written as $D = A + N$ where A is the *minimal ample divisor* defined in Table 6.9 and N is some nef divisor. When X is a real, the minimal ample divisor is real, so the nef divisor N is also real.

TABLE 6.9. Minimal ample divisor on complex del Pezzo surfaces

Degree	X	A
9	$\mathbb{P}^2$	$H = -\frac{1}{3}K_X$
8	$\mathrm{Bl}_{p_1}(\mathbb{P}^2)$	$2H - E_1$
8	$\mathbb{P}^1 \times \mathbb{P}^1$	$L_1 + L_2 = -\frac{1}{2}K_X$
$d \leqslant 7$	$\mathrm{Bl}_{p_1, p_2, \dots, p_{9-d}}(\mathbb{P}^2)$	$-K_X$

Lemma 6.10. *Let X be a totally-real del Pezzo surface of degree d at least 3 and let A denote the minimal ample divisor on X. There exists a real effective divisor C and nef divisors N and M such that $A = C + N$, $-K_X = C + M$ and a general section of M is a smooth rational curve. When $d \leqslant 7$ or $X = \mathbb{P}^2(1,0)$, the divisor C can be chosen to be a real (-1)-curve or a real conic bundle.*

Proof. Suppose that $8 \leqslant d \leqslant 9$. When $X = \mathbb{P}^2$, setting $C := H$, $N := 0$, and $M := 2H$ implies that $A = C + N$, $-K_X = C + M$, and a general section of M is smooth rational curve. When $X = \mathbb{P}^2(1,0)$, setting $C := H - E_1$, $N := H$, and $M := 2H$ ensures that $A = C + N$, $-K_X = C + M$ and the genus formula [Bea96, I.15] shows that a section of M has genus zero. When X is $Q^{2,2}$ or $Q^{3,1}$, let L_1 and L_2 be the divisors classes of the two rulings. Setting $C := A$, $N := 0$, and $M := L_1 + L_2$, we see that $A = C + N$, $-K_X = C + M$, and the genus formula again shows that a general section of M is a smooth rational curve.

Suppose that $3 \leqslant d \leqslant 7$. When the surface X contains a real (-1)-curve C, set $N := -K_X - C$ and $M := N$. For any (-1)-curve C', we have $N \cdot C' = 1 - C \cdot C'$. Since $d \geqslant 3$, any two (-1)-curves on X intersect in a most one point; see [Dol12, Proposition 8.2.15]. It follows that N is nef. Combining the definition of a (-1)-curve and the genus formula, we deduce that general section of M is a smooth rational curve. When the surface X does not contain a real (-1)-curve, Table 6.4 shows that $d \geqslant 4$. Moreover, Example 6.5 and Remark 6.6 also establish that X has a real conic bundle B. Set $C := B$, $N := -K_X - B$, and $M := N$ so that $A = C+N$ and $-K_X = C+M$. The divisor class N is nef because no del Pezzo surface of degree at least 4 contains a triangle which implies that $B \cdot C' \leqslant 1$ for any (-1)-curve C'; see [Dol12, §8.4.1, §8.4.2, §8.5.1, §8.6.3]. Combining the definition of a conic bundle and the genus formula, we see that general section of M is a smooth rational curve. $\square$

Lemma 6.11. *Every nef divisor N on a del Pezzo surface X is effective. Moreover, for any positive integer i and any nonnegative integer m, we have $h^i(X, mN) = 0$.*

Proof. Since $-K_X$ is ample, the divisor class $N - \frac{1}{m}K_X$ is ample for any positive integer m; see [Laz04, Corollary 1.4.10]. Hence, the Nakai Criterion [Laz04, Theorem 1.2.23] establishes that $(mN - K_X)^2 > 0$ for any nonnegative integer m, which implies that $mN - K_X$ is big and nef; see [Laz04, Theorem 2.2.16]. The Kawamata–Viehweg Vanishing Theorem [Laz04, Theorem 4.3.1] demonstrates that $h^i\big(X, (mN - K_X) + K_X\big) = 0$ for any positive integer i. The effectiveness of mN then follows from the Riemann–Roch Theorem. $\square$

Lemma 6.12. *For any ample divisor D on a totally-real del Pezzo surface of degree at least 3, there exists a nonzero effective divisor C such that $E := D - C$ is effective and supports multipliers for D.*

Proof. Let X be a totally-real del Pezzo surface of degree d where $d \geqslant 3$. Lemma 6.10 proves that there exists a nonzero effective divisor C on X and nef divisors N and M on X such that $A = C+N$, $-K_X = C+M$, and a general section of M is a smooth rational curve. We claim that the divisor $E := D - C$ is effective and supports multipliers for D.

Suppose that $E = 0$. Since $M \neq 0$, it follows that $d \geqslant 8$ and there are four cases: the pair (X, D) is $(\mathbb{P}^2, H)$, $(\mathbb{P}^2, 2H)$, $\big(\mathrm{Bl}_{p_1}(\mathbb{P}^2), 2H - E_1\big)$ or $(\mathbb{P}^1 \times \mathbb{P}^1, L_1 + L_2)$. In all of these cases, the nonnegative global sections of $\mathscr{O}_X(2D)$ coincide with the sums of squares because the surface X embedded the very ample line bundle $\mathscr{O}_X(D)$ is a variety of minimal degree; see [BSV16, Theorem 1.1].

Suppose that $E \neq 0$. To prove the claim, it is enough to verify the hypotheses of Corollary 4.3. Since the divisor D is ample, there exists a unique nef divisor N' such that $D = A + N'$. It follows that $E = D - C = N + N'$ is nef and Lemma 6.11 shows that E is effective and $h^i(X, mE) = 0$ for any positive integers i and m. The divisor $E - D = -C$ has negative intersection with the ample divisor $-K_X$ which gives $h^0(X, E-D) = 0$. The assumption that $d \geqslant 3$ ensures that the minimal ample divisor A on X is very ample, so the divisor $D = A + N'$ is free and the divisor $D + E = A + N + 2N'$ is very ample. Hence, Lemma 6.11 also shows that $h^i(X, mD + mE) = 0$ for any positive integers i and m. All that remains is to confirm the inequality $h^0(X, 2E) + h^1(X, E-D) > h^0(X, K_X + D + E)$. The choice of C ensures that $K_X + D + E = 2E - M$. Consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_X(2E - M) \longrightarrow \mathscr{O}_X(2E) \longrightarrow \mathscr{O}_M(2E|_M) \longrightarrow 0\,.$$

As $D + E = A + N + 2N'$ is big and nef, the Kawamata–Viehweg Vanishing Theorem demonstrates that $h^1(X, 2E - M) = h^1(X, K_X + D + E) = 0$, so we obtain the exact sequence

$$0 \longrightarrow H^0\big(X, \mathscr{O}_X(2E - M)\big) \longrightarrow H^0\big(X, \mathscr{O}_X(2E)\big) \longrightarrow H^0\big(M, \mathscr{O}_M(2E|_M)\big) \longrightarrow 0\,.$$

The divisor $2E = 2(N + N')$ being nef establishes that $2E \cdot M \geqslant 0$. It follows that $2E|_M$ is a divisor of nonnegative degree on the smooth rational curve M, so $h^0\big(M, \mathscr{O}_M(2E|_M)\big) > 0$. We conclude that $h^0(X, 2E) + h^1(X, E-D) \geqslant h^0(X, 2E) > h^0(X, 2E - M) = h^0(X, K_X + D + E)$. $\square$

We now prove the main result of this section.

Proof of Theorem 6.1. Suppose that D is not nef. Lemma 6.8 shows that there exists a divisor E on X, which is either a real (-1)-curve or the sum of a disjoint pair of conjugate (-1)-curves, such that $H^0\big(X,\mathscr{O}_X(D)\big) = H^0\big(X,\mathscr{O}_X(D-E)\big)$. Iterating this step a finite number of times, we reach a nef divisor D'. By construction, we have $-K_X \cdot D' < -K_X \cdot D$ and D' supports multipliers for D.

For some nonnegative integer j, we may assume that there exists divisors $D_0, D_1, \dots, D_j$ with $D_0 = D$ such that $-K_X \cdot D_{i-1} < -K_X \cdot D_i$ and D_i supports multipliers for D_{i-1} for all $1 \leqslant i \leqslant j$, and D_j is nef. If $D_j = 0$ or D_j is a multiple of a conic bundle, then we are done. If $D_j \neq 0$ and not ample, Lemma 6.8 shows that there exists a sequence of birational morphisms, contracting a real (-1)-curve or a conjugate pair of (-1)-curves at each step, such that the composition $\pi \colon X \to X'$ is a strongly dominant morphism onto a real del Pezzo surface of degree greater than $d := \deg(X)$ and D_j is the pullback under π of an ample divisor D_j' on X'. Since π is strongly dominant, the nonnegative global sections of $\mathscr{O}_X(D_j)$ coincide with the nonnegative global sections of $\mathscr{O}_{X'}(D_j')$. Hence, we may work on X' or, equivalently, assume that D_j is ample. If D_j is ample, the Lemma 6.12 demonstrates that there exists a nonzero effective divisor C such that $D_{j+1} := D_j - C$ is effective and supports multipliers for D_j. Since $-K_X$ is ample and C is effective, we must have $-K_X \cdot D_{j+1} < -K_X \cdot D_j$. This process must terminate after at most $-K_X \cdot D$ steps. □

From the algorithm outlined in the proof of Theorem 6.1, we see that nonnegativity certificates on totally-real del Pezzo surfaces of degree at least 3 can be computed via an explicit sequence of semidefinite programs. The next example shows that these semidefinite programs depend on the real structure on X.

Example 6.13. Let X be a real cubic surface in $\mathbb{P}^3$ and let f be a nonnegative global section in $H^0\big(X,\mathscr{O}_X(-2K_X)\big)$. Equivalently, f is a homogeneous quartic polynomial which is nonnegative on $X(\mathbb{R})$. To highlight the importance of the real structure, we consider on two cases: $\mathbb{P}^2(6,0)$ and $\mathbb{D}(1,0)$. In both of these cases, the surface contains at least one real (-1)-curve C, so the divisor $-K_X - C$ supports multipliers for $-K_X$. In other words, there exists a nonnegative global section g in $H^0\big(X,\mathscr{O}_X(-2K_X-2C)\big)$ and a sum-of-squares h in $H^0\big(X,\mathscr{O}_X(-4K_X-2C)\big)$ such that $fg = h$. Moreover, $B := -K_X - C$ is a real conic bundle. When $X = \mathbb{P}^2(6,0)$, the contraction associated to B is surjective on real points, so g is a sum-of-squares of global sections in $H^0\big(X,\mathscr{O}_X(B)\big)$ and $f = h/g$. When $X = \mathbb{D}(1,0)$, the contraction associated to B sends the real points $X(\mathbb{R})$ to the disjoint union $[a_1,a_2] \sqcup [a_3,a_4]$. As in Remark 6.7, the nonnegative global section g can be expressed as $g_0 + c_1(x_0 - a_1x_1)(x_0 - a_2x_1) + c_2(a_0x_1 - x_0)(x_0 - a_4x_1)$ where g_0 is a sum-of-squares in $H^0\big(X,\mathscr{O}_X(B)\big)$ and c_1, c_2 are nonnegative real numbers, so we obtain

$$f = \frac{h}{g_0 + c_1(x_0 - a_1x_1)(x_0 - a_2x_1) + c_2(a_0x_1 - x_0)(x_0 - a_4x_1)}.$$

Solely in terms of degree bounds, a nonnegative quartic form on a cubic surface admits a quadratic nonnegative multiplier. In the first case (but not the second), the multiplier is a sum of squares. ⋄

Remark 6.14. Combining Theorem 6.1 and Remark 6.7, we see that there are three kinds of multiplier certificates on del Pezzo surfaces having degree at least 3. Table 6.15 summarizes the relationship between surface type and nonnegativity certificates.

Although modified certificates are necessary to characterize nonnegativity for arbitrary divisors on a del Pezzo surface, we demonstrate that sums of squares may suffice for a specific divisor.

TABLE 6.15. Three kinds of multiplier certificates

Surface type	Nonnegativity certificate type
Admits a birational morphism to $\mathbb{D}$	Modified SOS multiplier with 2 intervals
Admits a birational morphism to $Q^{3,1}(0,2)$	Modified SOS multiplier with 1 interval
Otherwise	SOS-multipliers

Example 6.16. The surface $Q^{3,1}(0,2)$ embedded via its anticanonical bundle is a subvariety X in $\mathbb{P}^6$ of degree 6. A nonnegative quadratic form f on X is a nonnegative global section of $H^0\big(X, \mathscr{O}_X(-2K_X)\big)$. The divisor $B := L_1 + L_2 - E_1 - E_2$ be the unique real conic bundle on X. Following the algorithm outlined in the proof of Theorem 6.1, the divisor $D := -K_X - B = L_1 + L_2$ supports multipliers for the divisor $-K_X$ and the divisor 0 supports multipliers for D. Hence, there exists an equation of the form $fg = h$ where g and h are sums of squares. More precisely, g is a sum of squares of linear forms vanishing on B, h is a sum of squares of quadratic forms vanishing on B, and f admits a quadratic sum-of-squares multiplier. ⋄

7. ASYMPTOTIC MULTIPLIERS BOUNDS

This section gives asymptotic bounds for the degree of multipliers on certain embedded surfaces. We provide quadratic upper bounds on the growth rate rather than exact bounds, because we have more control over the transfer steps than the base case of the induction. Nevertheless, our novel results constitute the first multiplier bounds for nonrational surfaces beyond the elementary recursive degree estimates that apply to all real varieties; compare with [LPR20, Theorem 1.5.7]. In hindsight, our methods handle a totally-real smooth surface X with a very ample divisor A such that $-K_X \cdot A > 0$. Despite apparently aligning with algebraic surfaces of Kodaira dimension $-\infty$, it is unclear whether this is an artifact of our techniques or reflects some deeper aspect of nonnegativity certificates.

As in the minimal model program for algebraic surfaces, our approach exploits the (-1)-curves on a surface. We start by showing how a single (-1)-curve can produce 1-step transfers.

Lemma 7.1. *Assume that X is a totally-real smooth surface with a very ample divisor A. Let $\pi \colon Z := \mathrm{Bl}_p(X) \to X$ be the blow-up of X at a real point p, let $E := \pi^{-1}(p)$ be the exceptional divisor, and set $H := \pi^*(A)$. Fix a positive integer m and choose a nonnegative integer ℓ such that the divisor $\ell H - K_Z$ is big and nef.*

(i) *For any positive integer k and any integer d satisfying $d \geqslant 2m + k + \ell$, the inequality*
$$2d(-K_Z \cdot H) - (2m+k)(k+1) - \chi(\mathscr{O}_Z) > 0$$
implies that the divisor $dH - (m+k)E$ supports multipliers on the divisor $dH - mE$.

(ii) *For any integer d satisfying $d \geqslant 2m + \ell$, the inequality*
$$(1-2d)\big(H \cdot (H + K_Z)\big) + m(m-1) - \chi(\mathscr{O}_Z) > 0$$
implies that the divisor $(d-1)H$ supports multipliers on the divisor $dH - mE$.

Before proving the lemma, we list a few rudimentary properties of the surface Z. Firstly, we have $K_Z = \pi^*(K_X) + E$. Secondly, the divisor $2H - E$ on Z is very ample; for example see [BS96, Theorem 2.1]. Thirdly, for any sufficiently large integer ℓ, the Nakai Criterion implies that the divisor $\ell H - K_Z$ on Z is big and nef, because $(\ell H - K_Z)^2 = \ell^2 A^2 - 2\ell A \cdot K_X + (K_X)^2 - 1$ and $A^2 > 0$.

Proof. To establish part (i), it suffices to verify the hypotheses of Corollary 4.3. Set $D_0 := dH - mE$ and $D_1 := dH - (m+k)E$. The inequality $d \geqslant 2m+k+\ell$ means that there exists a nonnegative integer j such that $d = 2m+j+k+\ell$. As the sum of an very ample divisor and a free divisor, both $D_0 = m(2H-E)+(j+k+\ell)H$ and $D_0+D_1 = (2m+k)(2H-E)+2(j+\ell)H$ are very ample and thereby free; see [Har77, §II.7, Exercise 7.5d]. Since $k > 0$ and the divisor E is effective, the equality $D_1 - D_0 = -kE$ shows that $h^0(Z, D_1 - D_0) = 0$. The divisor $\ell' H - K_Z$ being big and nef, for all integers $\ell' \geqslant \ell$, guarantees that, for any positive integer c, the divisors

$$
\begin{aligned}
cD_0 - K_Z &= cm(2H-E) + \big(c(j+k+\ell)H - K_Z\big)\,, \\
cD_0 + cD_1 - K_Z &= (2m+k)(2H-E) + \big(2c(j+\ell)H - K_Z\big)
\end{aligned}
$$

are also big and nef. Hence, the Kawamata–Viehweg Vanishing Theorem gives $h^i(Z, cD_0) = 0$ and $h^i\big(Z, cD_0 + cD_1\big) = 0$ for any positive integers i and c. All that remains is confirm the inequality $\chi(2D_1) + h^1(Z, D_1 - D_0) > \chi(-D_0 - D_1)$. Applying the Riemann–Roch Formula, we deduce that

$$
\begin{aligned}
&\chi(2D_1) + h^1(Z, D_1 - D_0) - \chi(-D_0 - D_1) \\
&= \chi(2D_1) - \chi(D_1 - D_0) + h^2(X, D_1 - D_0) - \chi(-D_0 - D_1) \\
&\geqslant \tfrac{1}{2}\big((2D_1)^2 - 2D_1 \cdot K_Z - (D_1 - D_0)^2 + (D_1 - D_0)\cdot K_Z - (D_0 + D_1)^2 - (D_0 + D_1)\cdot K_Z\big) - \chi(\mathscr{O}_Z) \\
&= D_1^2 - D_1 \cdot K_Z - D_0^2 - D_0 \cdot K_Z - \chi(\mathscr{O}_Z) \\
&= d^2H^2 - (m+k)^2 - dH\cdot K_Z - m - d^2H^2 + m^2 - dH \cdot K_Z - (m+k) - \chi(\mathscr{O}_Z) \\
&= 2d(-K_Z \cdot H) - (2m+k)(k+1) - \chi(\mathscr{O}_Z),
\end{aligned}
$$

which is positive by assumption.

For part (ii), it again suffices to verify the hypotheses of Corollary 4.3. Set $D_0 := dH - mE$ and $D_1 := (d-1)H$. The inequality $d \geqslant 2m + \ell$ means that there exists a nonnegative integer j such that $d = 2m + j + \ell$. As the sum of an very ample and a free divisor, both $D_0 = m(2H-E) + (j+\ell)H$ and $D_0 + D_1$ are very ample and free. Since $H \cdot (D_1 - D_0) = H \cdot (mE - H) = -H^2 < 0$, the divisor $D_1 - D_0$ is not effective and $h^0(Z, D_1 - D_0) = 0$. The divisor $\ell' H - K_Z$ being big and nef, for all integers $\ell' \geqslant \ell$, also guarantees that, for any positive integer c, the divisors

$$
\begin{aligned}
cD_0 - K_Z &= cm(2H-E) + \big(c(j+\ell)H - K_Z\big)\,, \\
cD_0 + cD_1 - K_Z &= cm(2H-E) + \big(c(2m+2j+\ell-2)H - K_Z\big)
\end{aligned}
$$

are big and nef. Hence, the Kawamata–Viehweg Vanishing Theorem gives $h^i(Z, cD_0) = 0$ and $h^i\big(Z, cD_0 + cD_1\big) = 0$ for any positive integers i and c. As in part (i), it remains to confirm that $\chi(2D_1) + h^1(Z, D_1 - D_0) > \chi(-D_0 - D_1)$. Applying the Riemann–Roch Formula, we deduce that

$$
\begin{aligned}
\chi(2D_1) + h^1(Z, D_1 - D_0) - \chi(-D_0 - D_1) &\geqslant D_1^2 - D_1 \cdot K_Z - D_0^2 - D_0 \cdot K_Z - \chi(\mathscr{O}_Z) \\
&= (d-1)^2 H^2 - (d-1)H \cdot K_Z - d^2H^2 + m^2 - dH \cdot K_Z - m - \chi(\mathscr{O}_Z) \\
&= (1-2d)\big(H \cdot (H + K_Z)\big) + m(m-1) - \chi(\mathscr{O}_Z)
\end{aligned}
$$

which is again positive by assumption. □

Remark 7.2. The hypotheses in Lemma 7.1 constrain the underlying surface X. To have the inequality $2d(-K_Z \cdot H) - (2m+k)(k+1) - \chi(\mathscr{O}_Z) > 0$ hold for some choice of positive integers m and k and any sufficiently large integer d requires $-K_Z \cdot H > 0$. As $-K_Z \cdot H = -K_X \cdot A$ and A is a very ample divisor on X, it follows that no multiple of K_X can be effective, so $h^0(X, cK_X) = 0$ for any positive integer c. Hence, the Enriques characterization [Bea96, Theorem VI.17] implies that X must be a *ruled surface*: birationally equivalently to a product $C \times \mathbb{P}^1$ for some nonsingular curve C. Furthermore, the condition $-K_X \cdot A > 0$ is the same as $A^2 > A \cdot (A + K_X)$. Assuming that the surface X is embedded into projective space via the complete linear series associated to A, the adjunction formula [Bea96, Remarks I.16] implies that the genus $\mathrm{g}(X, A)$ of a general hyperplane section is $\frac{1}{2}\big(A \cdot (A + K_X)\big) + 1$. Hence, the degree of this embedded surface is greater than $2\,\mathrm{g}(X, A) - 2$.

The next theorem comes from repeated use of the 1-step transfers arising from (-1)-curves.

Theorem 7.3. *Assume that X is a totally-real smooth surface with a very ample divisor A satisfying $-K_X \cdot A > 0$. Let $\pi \colon Z := \mathrm{Bl}_p(X) \to X$ be the blow-up of X at a real point p and set $H := \pi^*(A)$. Fix s to be the smallest positive integer such that $s(-K_X \cdot A) > A \cdot (A + K_X)$ and choose a positive integer t such that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{t+1} > 2(1 + \sqrt{s})$. For all sufficiently large integers d, there exists an $(t+1)$-step transfer on Z from dH to $(d-1)H$.*

The harmonic series being divergent affirms the existence of the positive integer t.

Proof. Let $E := \pi^{-1}(p)$ be the exceptional divisor on Z. We claim that, for any sufficiently large integer d, there exist positive integers $m_0, m_1, \dots, m_t$ such that the divisors

$$D_0 := dH\,, \quad D_1 := dH - m_1 E\,, \quad D_2 := dH - m_2 E\,, \quad \dots, \quad D_t := dH - m_t E\,, \quad D_{t+1} := (d-1)H$$

form an $(t+1)$-step transfer from dH to $(d-1)H$. Consider the function $\Lambda \colon \mathbb{N} \to \mathbb{Z}$ defined by $\Lambda(d) = 2d(-K_Z \cdot H) - \chi(\mathscr{O}_Z)$. Since t depends only on s (and not d), this function enjoys the following three properties. First, for any integer j satisfying $1 \leqslant j \leqslant t$, there exists a positive integer k_j such that, for any sufficiently large integer d, we have

$$\frac{1}{2(j+1)}\sqrt{\Lambda(d)} \leqslant k_j \leqslant \frac{1}{2j}\sqrt{\Lambda(d)} - 1\,.$$

Second, for any positive integer ℓ such that the divisor $\ell H - K_Z$ on Z is big and nef, and any sufficiently large integer d, we have

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{t+1}\right)\sqrt{\Lambda(d)} < d - \ell\,.$$

Third, for any sufficiently large integer d, we have $d > -H \cdot (H + K_Z)$ and $d > -(s+1)\chi(\mathscr{O}_Z)$. Assume that the integer d is large enough that both of these properties hold. Set $m_0 := 0$ and, for all $1 \leqslant j \leqslant t$, set $m_j := \sum_{i=1}^{j} k_j$. For any $1 \leqslant j \leqslant t$, the two properties give

$$2m_j \leqslant 2\sum_{i=1}^{j}\left(\frac{1}{2j}\sqrt{\Lambda(d)} - 1\right) \leqslant \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{t+1}\right)\sqrt{\Lambda(d)} < d - \ell\,,$$

so we obtain $d > 2m_j + \ell > 2m_{j-1} + k_j + \ell$. Since $k_j + 1 \leqslant \frac{1}{2j}\sqrt{\Lambda(d)}$, we also have

$$\begin{aligned}
(2m_{j-1} + k_j)(k_j + 1) &\leqslant 2m_{j-1}(k_j + 1) + (k_j + 1)^2 \\
&\leqslant \left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j-1}\right)\sqrt{\Lambda(d)}\right]\left[\frac{\sqrt{\Lambda(d)}}{2j}\right] + \left[\frac{\sqrt{\Lambda(d)}}{2j}\right]^2 \\
&< \left(\frac{j-1}{2j} + \frac{1}{4j^2}\right)\Lambda(d) < \Lambda(d) = 2d(-K_Z \cdot H) - \chi(\mathscr{O}_Z)\,.
\end{aligned}$$

Thus, Lemma 7.1.i demonstrates that, for all $1 \leqslant j \leqslant t$, the divisor D_j supports multipliers for the divisor D_{j-1}. For the last transfer, the choice of t and the first property give

$$m_t - 1 \geqslant \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{t+1}\right)\sqrt{\Lambda(d)} - 1 \geqslant (1+\sqrt{s})\sqrt{\Lambda(d)} - 1 > \sqrt{s\,\Lambda(d)}\,,$$

so we obtain $m_t(m_t - 1) > (m_t - 1)^2 > s\,\Lambda(d) = 2ds(-K_Z \cdot H) - s\chi(\mathscr{O}_Z)$. Combining the inequality $s(-K_Z \cdot H) = s(-K_X \cdot A) > A \cdot (A + K_X) = H \cdot (H + K_Z)$ with the third property, it follows that

$$\begin{aligned}(1-2d)\big(H\cdot(H+K_Z)\big) &+ m_t(m_t-1) - \chi(\mathscr{O}_Z)\\ &> (1-2d)\big(H\cdot(H+K_Z)\big) + 2ds(-K_Z\cdot H) - (s+1)\chi(\mathscr{O}_Z)\\ &= H\cdot(H+K_Z) + 2d\big(s(-K_Z\cdot H) - H\cdot(H+K_Z)\big) - (s+1)\chi(\mathscr{O}_Z) > 0\,.\end{aligned}$$

Therefore, Lemma 7.1.ii proves that the divisor D_{t+1} supports multipliers for the divisor D_t. □

Remark 7.4. Both Example 5.10 and the proof of Theorem 7.3 use the same inherent strategy. A more detailed understanding of $\mathrm{Bl}_p(\mathbb{P}^2)$ in the first case is the only substantial difference.

From this theorem, we extract a quadratic upper bound on the growth rate of the degree of multipliers on embedded ruled surfaces.

Corollary 7.5. *Let X be a totally-real smooth surface with a very ample divisor A satisfying $-K_X \cdot A > 0$. For any positive integer d and any nonnegative global section f in $H^0\big(X, \mathscr{O}_X(2dA)\big)$, there exists a nonzero sum-of-squares global section g in $H^0\big(X, \mathscr{O}_X(2eA)\big)$ such that $e = O(d^2)$ and the product fg is a sum of squares.*

Proof. Let $\pi\colon Z := \mathrm{Bl}_p(X) \to X$ be the blow-up of X at a real point p and set $H := \pi^*(A)$. By repeated applications of Theorem 7.3, there exists an integer d_0 such that, for any $d > d_0$, there exists a $(t+1)(d-d_0)$-step transfer $D_0, D_1, \dots, D_{(t+1)(d-d_0)}$ on $Z := \mathrm{Bl}_p(X)$ from the divisor dH to the divisor d_0H. Set $r := (t+2)(d-d_0)$. In particular, for any nonnegative global section f_0 in $H^0\big(X, \mathscr{O}_X(2dA)\big) = H^0\big(Z, \mathscr{O}_X(2dH)\big)$, there exists nonnegative global sections f_i in $H^0\big(Z, \mathscr{O}_Z(2D_i)\big)$ and sums of squares $g_{i-1,i}$ in $H^0\big(Z, \mathscr{O}_Z(2D_{i-1} + 2D_i)\big)$ such that $f_{i-1}f_i = g_{i-1,i}$ for all $1 \leqslant i \leqslant r$. Furthermore, from the proof of Theorem 7.3, we see that the first $(t+1)$ global sections f_i correspond to forms of degree $2d$ on X and the next $(t+1)$ global sections f_i correspond to forms of degree $2(d-1)$ on X. Continuing this pattern, the degrees of the corresponding forms weakly decrease until the very last form f_r has degree $2d_0$. Invoking [LPR20, Theorem 1.3.2], the nonnegativity of f_r on X implies that there exists sums of squares g' and g'' such that $g' f_{(t+1)(d-d_0)} = g''$ where the degrees of g' and g'' are bounded above by a constant e_0 which depends only on d_0. If r is odd, then we have $f_0(g_{1,2}g_{3,4}\cdots g_{r-2,r-1}g'') = f_0f_1\cdots f_rg' = g_{0,1}g_{2,3}\cdots g_{r-1,r}g''$. If r is odd, then we have $f_0(g_{1,2}g_{3,4}\cdots g_{r-1,r}g') = f_0f_1\cdots f_rg' = g_{0,1}g_{2,3}\cdots g_{r-2,r-1}g''$. Therefore, we deduce that f_0 has of sum-of-square multiplier of degree $O(d^2)$. □

Making stronger assumptions on the underlying surface allows for a more streamlined conclusion.

Proposition 7.6. *Assume that X is a nondegenerate nonrational totally-real smooth surface with a very ample divisor A such that its sectional genus $\mathrm{g}(X,A)$ equals 1. Let $\pi\colon Z := \mathrm{Bl}_p(X) \to X$ be the blow-up of X at a real point p and set $H := \pi^*(A)$. For any integer d satisfying $d \geqslant 5$, there exists a 2-step transfer on Z from dH to $(d-1)H$.*

Proof. The genus formula is $2\,\mathrm{g}(X,A)-2 = A\cdot(A+K_X)$, so the assumption that $\mathrm{g}(X,A)=1$ is equivalent to $0 = A\cdot(A+K_X) = H\cdot(H+K_Z)$. Combined with the nondegeneracy of X, we deduce that $-K_Z\cdot H \geqslant 2$. As in Remark 7.2, it follows that Z is a ruled surface birational to $C\times\mathbb{P}^1$ for some nonsingular curve C. The Euler characteristic of a surface being a birational invariant [Bea96, Proposition III.20] implies that $\chi(\mathscr{O}_Z) = \chi(\mathscr{O}_{C\times\mathbb{P}^1}) = 1-\mathrm{g}(C)$. Hence, the hypothesis that the surface X is not rational gives $-\chi(\mathscr{O}_Z)\geqslant 0$.

Let $E := \pi^{-1}(p)$ be the exceptional divisor on Z. When $d\geqslant 5$, we claim that the divisors $D_0 := dH$, $D_1 := dH-2E$, and $D_2 := (d-1)H$ form a 2-step transfer on Z. Since the divisor $H-K_Z$ is big and nef on Z and $d\geqslant 5$, we see that $2d(-K_Z\cdot H)-2(2+1)-\chi(\mathscr{O}_Z)\geqslant 4(5)-6>0$. Thus, Lemma 7.1.i establishes that the divisor $D_1 = dH-2E$ supports multipliers on the divisor $D_0 = dH$. Similarly, as $d\geqslant 2$ and $(1-2d)\big(H\cdot(H+K_Z)\big)+2(2-1)-\chi(\mathscr{O}_Z)\geqslant 2>0$, Lemma 7.1.ii establishes that the divisor $D_2 = (d-1)H$ supports multipliers on the divisor $D_1 = dH-2E$. $\square$

We finish with a couple of examples. Consider the ruled surface $X = C\times\mathbb{P}^1$ and let $\pi_1\colon X\to C$ be the canonical projection onto the smooth curve C. Choose a fibre F of π_1 and a section C (by a slight abuse of notation). The Picard group of X is generated by the class of C and pullbacks of elements from the Picard group of the curve C, and the Néon–Severi group of X is generated by C and F; see [Har77, Proposition V.2.3] or [Bea96, Proposition III.18]. Moreover, we have $C\cdot F = 1$, $C^2 = F^2 = 0$, and $K_X \equiv -2C+\omega F$ where ω is the canonical divisor on C; see [Har77, Lemma V.2.10].

Example 7.7. For a totally-real elliptic curve C, let X be the ruled surface $C\times\mathbb{P}^1$. Choose a point p on C and consider a divisor $A\equiv C+(3p)F$. This divisor is very ample and coincides with the Segre embedding of X into $\mathbb{P}^5$ as a surface of degree 6. The canonical divisor is $K_X = -2C$, so $A\cdot(A+K_X) = 0$. It follows that the sectional genus $\mathrm{g}(X,A)$ is 1. Therefore, Proposition 7.6 establishes that, for all $d\geqslant 5$, there is a 2-step transfer from dA to $(d-1)A$. ◇

Example 7.8. Let C be a totally-real nonsingular curve of genus g that is not hyperelliptic and let X be the ruled surface $X = C\times\mathbb{P}^1$. For an integer $m>0$, consider the divisor $A := m(C+\omega F)$. The divisor A is very ample because it is a multiple of $C+\omega F$ which is the pullback of hyperplane class under the Segre embedding of the closed immersion $\kappa\times\mathrm{id}\colon C\times\mathbb{P}^1\to\mathbb{P}^{g-1}\times\mathbb{P}^1$ where κ is the canonical embedding of C. Since $-K_X\cdot A = m\deg(\omega)>0$, Theorem 7.3 produces asymptotic transfer results for such surfaces. From the equality $A\cdot(A+K_X) = (2m^2-m)\deg(\omega)$, we see that the ration $(A\cdot(A+K_X)/(-K_X\cdot A) = 2m-1$ can be made arbitrarily large. We conclude that the number of transfer steps required to pass from dA to $(d-1)A$ via Theorem 7.3 is not uniformly bounded on all surfaces satisfying $-K_X\cdot A>0$. ◇

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