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Irreducible components of characteristic varieties

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Abstract

We give a dimension bound on the irreducible components of the characteristic variety of a system of linear partial differential equations defined from a suitable filtration of the Weyl algebra A_n. This generalizes an important consequence of the fact that a characteristic variety defined from the order filtration is involutive. More explicitly, we consider a filtration of A_n induced by any vector $(\mathbf{u},\mathbf{v}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ such that the associated graded algebra is a commutative polynomial ring. Any finitely generated left A_n-module M has a good filtration with respect to $(\mathbf{u},\mathbf{v})$ and this gives rise to a characteristic variety $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ which depends only on $(\mathbf{u},\mathbf{v})$ and M. When $(\mathbf{u},\mathbf{v}) = (\mathbf{0},\mathbf{1})$, the characteristic variety is involutive and this implies that its irreducible components have dimension at least n. In general, the characteristic variety may fail to be involutive, but we are still able to prove that each irreducible component of $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ has dimension at least n. © 2001 Published by Elsevier Science B.V.

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1. Introduction

The geometry of the characteristic variety plays a central role in the study of systems of linear partial differential equations. In algebraic analysis, the characteristic variety is obtained from a filtration of the corresponding $\mathscr{D}$-module. When $\mathscr{D}$ is equipped with the order filtration, Sato et al. [11] and Gabber [6] show that each characteristic variety is involutive with respect to the natural symplectic structure on the cotangent bundle. As a consequence, they deduce the "Strong Fundamental Theorem of Algebraic Analysis", which says: if $\mathscr{D}$ is the sheaf of differential operators on an n-dimensional variety, then each irreducible component of a characteristic variety must have dimension at

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least n. In this paper, we consider left A_n-modules which correspond to systems of linear partial differential equations with polynomial coefficients on $\mathbb{A}^n$. Our goal is to extend this dimension bound to all filtrations of the Weyl algebra A_n for which the associated graded ring is a commutative polynomial ring and to extend this assertion to a larger class of algebras.

We are primarily interested in this larger class of filtrations because of its connection with monomial ideals in a commutative polynomial ring. Specifically, for a generic vectors $(\mathbf{u},\mathbf{v})$, the characteristic variety $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ is given by a square-free monomial ideal or equivalently a simplicial complex. Monomials ideals form an important link between algebraic geometry, combinatorics and commutative algebra. Much of the success of Gröbner bases theory comes from an understanding of monomial ideals. We believe that further exploration of this connection will lead to new insights into A_n-modules. Problems of making effective computations in algebraic analysis, provide a secondary motivation for considering filtrations other than the standard or order filtration. The choice of filtration can significantly effect the complexity of the characteristic variety.

To state our theorems more explicitly, we introduce some notation. We write $x_1,\ldots,x_n$, $y_1,\ldots,y_n$ for the generators of A_n satisfying the relations $x_ix_j - x_jx_i = 0$, $y_iy_j - y_jy_i = 0$ and $y_ix_j - x_jy_i = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Each vector $(\mathbf{u},\mathbf{v}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ induces an increasing filtration on A_n by setting $\deg x_i = \mathbf{u}_i$ and $\deg y_i = \mathbf{v}_i$. We focus those vectors $(\mathbf{u},\mathbf{v})$ for which the associated graded ring $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(A_n)$ is the commutative polynomial ring $S = k[\bar{x}_1,\ldots,\bar{x}_n,\bar{y}_1,\ldots,\bar{y}_n]$. Similarly, we can filter any finitely generated left A_n-module M by assigning degrees to a generating set of M. The associated graded module $\mathrm{gr}\,M$ is a module over the polynomial ring S and, hence, the prime radical of $\mathrm{Ann}_S(\mathrm{gr}\,M)$ defines a variety in $\mathbb{A}^{2n}$. This variety is called the characteristic variety $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ of M. It is independent of the choice of degrees and generators for M, but depends on the filtration of A_n. The main result of this paper is the following:

Theorem 1.1. *Let k be a field of characteristic zero and let M be a finitely generated left A_n-module. If the integer vector $(\mathbf{u},\mathbf{v})$ induces a filtration satisfying $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(A_n) = S$, then every irreducible component of $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ has dimension at least n.*

The conclusion is vacuously satisfied when $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ is empty and this can occur: if $M = A_2/A_2 \cdot I$, where I is the left ideal $\langle y_1 - 1, y_2 - 1\rangle$, then $\mathrm{Ch}_{(\mathbf{2},-\mathbf{1})}(M)$ corresponds to the S-ideal $\langle 1\rangle$ indicating that the characteristic variety is empty. However, if $(\mathbf{u},\mathbf{v})$ is non-negative and $M \neq 0$, then $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ is never empty.

Theorem 1.1 refines Bernstein's inequality [5] which states that there exists an irreducible component of $\mathrm{Ch}_{(\mathbf{1},\mathbf{1})}(M)$ of dimension at least n. On the other hand, Theorem 1.1 follows from the fact that the characteristic variety $\mathrm{Ch}_{(\mathbf{0},\mathbf{1})}(M)$ is involutive with respect to the natural symplectic structure on $\mathbb{A}^{2n}$. The involutivity of $\mathrm{Ch}_{(\mathbf{0},\mathbf{1})}(M)$ was first established by Sato et al. [11] using micro-local analysis; Gabber [6] provided a purely algebraic proof. In our more general case, a different proof is necessary because the characteristic variety $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ is not always involutive under the natural symplectic structure on $\mathbb{A}^{2n}$. For example, the characteristic variety of $A_2/A_2 \cdot I$, where I is the left ideal $\langle y_1^2 - y_2, x_1 y_1 + 2x_2 y_2 \rangle$, with respect to the vector $(1,1,1,3)$ is given by the non-involutive S-ideal $\langle \bar{x}_2, \bar{y}_2 \rangle \cap \langle \bar{y}_1, \bar{y}_2 \rangle$.

The general techniques used in the proof of Theorem 1.1 apply to a larger collection of k-algebras. We develop these methods for a skew polynomial ring R which is an almost centralizing extension of a commutative polynomial ring (see Section 2 for a precise definition). We write GKdim for the Gelfand–Kirillov dimension. The second major result of this paper is the following:

Theorem 1.2. *Assume that $(\mathbf{u},\mathbf{v})$ is an integer vector which induces an increasing filtration on R such that $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(R)$ is a commutative polynomial ring. Let p be a nonnegative integer and let M be a finitely generated R-module. If $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ has an irreducible component of dimension p, then there is a submodule M' of M such that $\mathrm{GKdim}\, M' = p$.*

For the special case in which $\mathrm{GKdim}\, M' = \mathrm{GKdim}\, M$ for every non-zero submodule M' of M (M is called GKdim-pure), this theorem implies that the characteristic variety is equidimensional.

Theorem 1.2 also generalizes known equidimensionality results to a larger class of filtrations. In particular, when R is the enveloping algebra of finite dimensional Lie algebra, it extends Gabber's equidimensionality theorem [7, Théorème 1] beyond the standard filtration. For certain skew polynomial rings, it extends the equidimensionality theorem in Li and van Ostaeyen [8, Corollary III 4.3.6] to non-Zariskian filtrations. For example, the filtration of A_2 induced by the $(\mathbf{2},-\mathbf{1})$ is not Zariskian because the element $1 + y_1$ is not invertible even if y_1 belongs to (-1)-th level of the filtration of A_2. A filtration induced by a vector with negative entries is typically not Zariskian. Our proof of Theorem 1.2 involves studying the growth of filtered modules and Gröbner basis theory and differs significantly from the homological methods used by Björk [3], Gabber [7] and Li and van Oystaeyen [8].

We now describe the contents of this paper. In the next section, we gather global notation and preliminary results. In Section 3, we connect irreducible components of the characteristic variety to the Gelfand–Kirillov dimension of submodules. To guarantee that the Gelfand–Kirillov dimension is well behaved, we restrict our attention to finite dimensional filtrations throughout this section. Section 4 reviews the basics of Gröbner basis theory and constructs a combinatorial object, called the Gröbner fan. This generalizes the Gröbner fan of Mora and Robbiano [10] in the case of commutative polynomial rings and Assi et al. [1] in the case of the Weyl algebra. In the last section, we use the use the Gröbner fan to extend our results for finite dimensional filtrations to all filtrations in the polynomial region and prove our main theorems.

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2. Preliminaries

Throughout this paper k denotes a field. Let B be the commutative polynomial ring $k[x_1,\dots,x_m]$. We concentrate on a k-algebra R which is generated by $x_1,\dots,x_m,\ y_1,\dots,y_n$ subject only to the relations:

(R1) $y_i x_j - x_j y_i = Q^1_{i,j}(x)$,

(R2) $y_i y_j - y_j y_i = Q^2_{i,j}(x,y) = Q^{2,0}_{i,j}(x) + \sum_{\ell=1}^{n} Q^{2,\ell}_{i,j}(x) y_\ell$,

where $Q^1_{i,j}(x) \in B$ and $Q^{2,\ell}_{i,j}(x) \in B$ for all $1 \leqslant \ell \leqslant n$. The skew polynomial ring R is called an almost centralizing extension of B. The set of standard monomial $\{x^{\mathbf{a}} y^{\mathbf{b}} = x_1^{\mathbf{a}_1} \cdots x_m^{\mathbf{a}_m} y_1^{\mathbf{b}_1} \cdots y_n^{\mathbf{b}_n}\colon (\mathbf{a},\mathbf{b}) \in \mathbb{N}^m \times \mathbb{N}^n\}$ forms a k-basis for R. In particular, each element $f \in R$ has a unique standard expression of the form $\sum \kappa_{\mathbf{a},\mathbf{b}} x^{\mathbf{a}} y^{\mathbf{b}}$. For more information about almost centralizing extension, see Subsections 8.6.6–8.6.7 in McConnell and Robson [9].

Example 2.1. If $\mathfrak{g}$ is a finite dimensional Lie algebra over k then any crossed product $B * U(\mathfrak{g})$ is an almost centralizing extension of B. Notably, the commutative polynomial ring, the Weyl algebra A_n and the enveloping algebra $U(\mathfrak{g})$ all have this form.

A vector $(\mathbf{u},\mathbf{v}) \in \mathbb{Z}^m \times \mathbb{Z}^n$ induces an increasing filtration of R: for $i \in \mathbb{Z}$, set $F_i R := k\{x^{\mathbf{a}} y^{\mathbf{b}}\colon \mathbf{u}\cdot\mathbf{a} + \mathbf{v}\cdot\mathbf{b} \leqslant i, \mathbf{a} \in \mathbb{N}^m, \mathbf{b} \in \mathbb{N}^n\}$. This clearly gives an increasing sequence of vectors spaces satisfying the conditions $1 \in F_0 R$ and $\bigcup_{i\in\mathbb{Z}} F_i R = R$. When $F_i R \cdot F_j R \subseteq F_{i+j} R$, the associated graded ring is $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(R) = \bigoplus_{i\in\mathbb{Z}} F_i R / F_{i-1} R$. An element f belonging to the vector space $F_i R - F_{i-1} R$ is said to have degree i and we write $\deg_{(\mathbf{u}\,\mathbf{v})} f = i$.

The polynomial region associated to R, denoted $\mathrm{PR}(R)$, is set of all vectors $(\mathbf{u},\mathbf{v})$ such that $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(R)$ is the commutative polynomial ring generated by the initial forms $\bar{x}_i$ and $\bar{y}_i$ of x_i and y_i, respectively. We denote this commutative polynomial ring by S. Since S is noetherian ring, R is both a left and a right noetherian ring. We will focus exclusively on vectors $(\mathbf{u},\mathbf{v})$ belonging to the polynomial region. The next proposition provides a more explicit interpretation of $\mathrm{PR}(R)$.

Proposition 2.2. *The polynomial region* $\mathrm{PR}(R)$ *is the open convex polyhedral cone given by the intersection of the following open half-spaces*:

$$(\mathrm{I}) \quad \begin{cases} \deg_{(\mathbf{u},\mathbf{v})} x_i y_j > \deg_{(\mathbf{u},\mathbf{v})} Q^1_{i,j} & \textit{for all } 1 \leqslant j \leqslant n \textit{ and } 1 \leqslant i \leqslant m, \\ \deg_{(\mathbf{u},\mathbf{v})} y_i y_j > \deg_{(\mathbf{u},\mathbf{v})} Q^2_{i,j} & \textit{for all } 1 \leqslant i,j \leqslant n. \end{cases}$$

Proof. Let $f_1,\dots,f_j$ be elements of the k-vector space generated by the elements $x_1,\dots,x_m,\ y_1,\dots,y_n$. Now, if σ is a permutation of $\{1,\dots,j\}$, we claim that $f_1 f_2 \cdots f_j \in f_{\sigma(1)} f_{\sigma(2)} \cdots f_{\sigma(j)} + F_{\ell-1} R$, where ℓ is the degree of the left-hand side. Since the elements $x_1,\dots,x_m$ commute and $Q^1_{i,j}(x)$ and $Q^2_{i,j}(x,y)$ are at most linear in the variables $y_1,\dots,y_n$, it suffices to prove this when σ is a transposition. By linearity, the assertion is equivalent to the conditions:

$$y_i x_j - x_j y_i = Q^1_{i,j}(x) \in F_{u_i+u_j-1}R,$$

$$y_i y_j - y_j y_i = Q^2_{i,j}(x,y) \in F_{v_i+v_j-1}R.$$

We conclude that $F_iR \cdot F_jR \subseteq F_{i+j}R$ and that $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(R)$ is a commutative k-algebra generated by $\bar{x}_1,\dots,\bar{x}_m,\ \bar{y}_1,\dots,\bar{y}_n$.

To see that $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(R) = S$, it suffices to see that there are no k-linear relations between the monomials in $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(R)$. A relation among the monomials in $\mathrm{gr}_{(\mathbf{u},\mathbf{v})}(R)$, would yield a relation among the standard monomials in R. However, the standard monomials form a k-basis for R which completes the proof. □

Remark 2.3. If $p = \max\{\deg_{(\mathbf{1},\mathbf{0})} Q^\ell_{i,j}$: for all $i,j,\ell\} + 1$, then the positive vector $(\mathbf{1}, p\mathbf{1})$ belongs to the polynomial region $\mathrm{PR}(R)$.

Remark 2.4. The set of all real vectors satisfying the inequalities (I) is denoted $\mathrm{PR}(R) \otimes \mathbb{R}$—it will play an important role in Section 4.

Example 2.5. All vectors in $\mathbb{Z}^m \times \mathbb{Z}^n$ belong to the polynomial region for S and $\mathrm{PR}(A_n) = \{(\mathbf{u},\mathbf{v})$: $u_i + v_i > 0$ for all $1 \leqslant i \leqslant n\}$. If the Lie algebra $\mathfrak{sl}_2$ has the standard basis y_1, y_2, y_3 such that $y_2y_3 - y_3y_2 = 2y_3$, $y_2y_1 - y_1y_2 = -2y_1$ and $y_1y_3 - y_3y_1 = y_2$, then $\mathrm{PR}(U(\mathfrak{sl}_2))$ is the open cone defined by the inequalities $v_1 + v_3 > v_2$ and $v_2 > 0$.

The associated Rees ring of R with respect to $(\mathbf{u},\mathbf{v})$ is $\tilde{R} = \bigoplus_{i\in\mathbb{Z}} F_iR$; the k-algebra structure on R makes it into a graded k-algebra. For $f \in F_iR$, we write the homogeneous element represented by f in $\tilde{R}_i$ as $(\tilde{f})_i$. The central non-zero-divisor represented by the identity 1 of R in $\tilde{R}_1$ is denoted by x_0. More concretely, the Rees ring $\tilde{R}$ is generated by $x_0,\dots,x_m,\ y_1,\dots,y_n$ subject to the relations:

($\tilde{\mathrm{R}}$1) $y_i x_j - x_j y_i = x_0^{\mathbf{u}_j+\mathbf{v}_i-q^1_{i,j}} Q^1_{i,j}(x)$,

($\tilde{\mathrm{R}}$2) $y_i y_j - y_j y_i = x_0^{\mathbf{v}_i+\mathbf{v}_j-q^2_{i,j}} Q^2_{i,j}(x,y)$,

where $Q^1_{0,j}(x) = 0$ and $q^\ell_{i,j} = \deg_{(\mathbf{u},\mathbf{v})} Q^\ell_{i,j}$ for $\ell = 1,2$. We stress that $\tilde{R}$ is an almost centralizing extension of $B[x_0]$ and the relations ($\tilde{\mathrm{R}}$1) and ($\tilde{\mathrm{R}}$2) are homogeneous with respect to $(\mathbf{u},\mathbf{v})$. The condition that $(\mathbf{u},\mathbf{v})$ belongs to the polynomial region $\mathrm{PR}(R)$ insures that x_0 has a nonnegative exponent in relations ($\tilde{\mathrm{R}}$1) and ($\tilde{\mathrm{R}}$2). The homogenization map from R to $\tilde{R}$ is defined by sending $f = \sum \kappa_{\mathbf{a},\mathbf{b}} x^{\mathbf{a}} y^{\mathbf{b}}$ in R to $\tilde{f} = \sum \kappa_{\mathbf{a},\mathbf{b}} x_0^{i-\mathbf{u}\cdot\mathbf{a}-\mathbf{v}\cdot\mathbf{b}} x^{\mathbf{a}} y^{\mathbf{b}}$ where $\deg_{(\mathbf{u},\mathbf{v})}(f) = i$. In the other direction, the substitution $x_0 = 1$ gives a k-algebra homomorphism from $\tilde{R}$ to R. More details on Rees rings can be found in Section I.4 of Li [8].

All modules considered in this paper will be finitely generated left modules. By a filtered R-module, we mean an R-module with an increasing sequence of vector spaces F_iM satisfying the conditions $F_iR \cdot F_jM \subseteq F_{i+j}M$, and $\bigcup_{i\in\mathbb{Z}} F_iM = M$. The associated Rees module is the graded k-module $\tilde{M} = \bigoplus_{i\in\mathbb{Z}} F_iM$—the R-module structure on M

makes $\tilde{M}$ into a graded $\tilde{R}$-algebra. The associated graded module is $\operatorname{gr} M = \bigoplus_{i\in\mathbb{Z}} F_iM/F_{i-1}M$. It follows that $\operatorname{gr} M$ is a graded $\operatorname{gr}_{(\mathbf{u},\mathbf{v})}(R)$-module. A good filtration is a filtration of an R-module M for which there are $z_1,\dots,z_p \in M$ and $w_1,\dots,w_p \in \mathbb{Z}$ such that $F_iM = \sum_{j=1}^{p} F_{i-w_j}Rz_j$. Every finitely generated R-module M has a good filtration and any module with a good filtration is necessarily finitely generated over R. For a good filtration of M, we define the characteristic ideal $I(M)$ to be the prime radical of $\operatorname{Ann}_S(\operatorname{gr} M)$. Since any two good filtrations are equivalent, the characteristic ideal $I(M)$ is independent of the choice of good filtration; however $I(M)$ does depend on $(\mathbf{u},\mathbf{v})$. The characteristic variety of M is the reduced scheme $\operatorname{Ch}_{(\mathbf{u},\mathbf{v})}(M) = \operatorname{Spec}(\operatorname{gr}_{(\mathbf{u},\mathbf{v})}(R)/I(M))$.

3. Finite dimensional filtrations

Under the assumption that R has a finite dimensional filtration, we are able to relate the dimension of the irreducible components of $\operatorname{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ to the Gelfand–Kirillov dimension of submodules of M. We accomplish this by using the Rees module $\tilde{M}$ to link submodules of M and graded submodules of $\operatorname{gr}(M)$. We begin with a brief discussion of Gelfand–Kirillov dimension.

We define the Gelfand–Kirillov dimension only for R-modules with a given good finite dimensional filtration. For a function $\phi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$, set $\gamma(\phi) = \inf\{d: f(i) \leqslant i^d$ for $i \gg 0\}$. The Gelfand–Kirillov dimension of a filtered R-module M is defined to be $\operatorname{GKdim} M = \gamma(\dim_k F_iM)$. Subsection 8.6.18 in McConnell and Robson [9] implies that $\operatorname{GKdim} M$ is independent of the choice of the good finite dimensional filtration. However, changing the filtration of R may significantly alter the filtration of M and the module $\operatorname{gr} M$. Regardless, Theorem 5.2 shows that $\operatorname{GKdim} M$ is also independent of the finite dimensional filtration of R.

Now, if $(\mathbf{u},\mathbf{v}) \in \operatorname{PR}(R)$ is not positive, then $\dim_k F_iR$ is infinite for $i \gg 0$. Thus, R has a finite dimensional filtration if and only if $(\mathbf{u},\mathbf{v})$ is positive. With this additional hypothesis, we can give a useful description of the function $i \mapsto \dim_k F_iM$. Recall that a function $\theta : \mathbb{Z} \to \mathbb{C}$ is called a quasi-polynomial if there exists a positive integer p and polynomials Q_j for $0 \leqslant j \leqslant p-1$ such that, for all $i \in \mathbb{Z}$, we have $\theta(i) = Q_j(i)$ where $i = rp + j$ with $0 \leqslant j \leqslant p-1$. The degree of a quasi-polynomial is the maximum of degree of the polynomials Q_j. The next proposition generalizes the almost commutative results in Section 8.4 of McConnell and Robson [9].

Proposition 3.1. *Assume the vector* $(\mathbf{u},\mathbf{v}) \in \operatorname{PR}(R)$ *is positive. If* M *is a non-zero* R*-module with a good finite dimensional filtration such that* $\operatorname{gr} M$ *has Krull dimension* d, *then one has the following*:

(1) *There are positive integers* $c_0,\dots,c_d$ *and* $Q(t) \in \mathbb{Z}[t,t^{-1}]$ *such that*

$$\sum_{i\geqslant 0} (\dim_k F_iM)\cdot t^i = \frac{Q(t)}{\prod_{j=0}^{d}(1-t^{c_j})}, \quad \textit{with } Q(1) > 0.$$

(2) *The function* $i \mapsto \dim_k F_iM$ *is a quasi-polynomial of degree* d.

Proof. By assumption, S is positively graded commutative k-algebra, so Proposition 4.4.1 in Bruns and Herzog [2] implies that there are positive integers $c_1,\dots,c_d$ and $Q(t)\in\mathbb{Z}[t,t^{-1}]$ such that

$$\sum_{i\geqslant 0}(\dim_k(\operatorname{gr} M)_i)t^i=\frac{Q(t)}{\prod_{j=1}^{d}(1-t^{c_j})}\quad\text{and}\quad Q(1)>0.$$

Hence, we have

$$\sum_{i\geqslant 0}(\dim_k F_iM)t^i=\left(\frac{Q(t)}{\prod_{j=1}^{d}(1-t^{c_j})}\right)\left(\frac{1}{1-t}\right),$$

which proves part (1). Part (2) follows immediately from part (1) by applying Proposition 4.4.1 in Stanley [13]. $\square$

The second part of this proposition clearly implies the following:

Corollary 3.2. *If the vector* $(\mathbf{u},\mathbf{v})\in\mathrm{PR}(R)$ *is positive and* N *is a finitely generated graded* S*-module, then the Gelfand–Kirillov dimension and Krull dimension of* N *are equal.*

We next provide morphisms linking the submodules of M, $\tilde{M}$ and $\operatorname{gr}(M)$. We always assume that the filtration on a submodule is the unique filtration induced by the module containing it. Because R is left noetherian, good filtrations induce good filtrations on submodules.

Proposition 3.3. *Let* x_0 *be the canonical central regular element of degree* 1 *in Rees ring* $\tilde{R}$. *If* M *is a filtered* R*-module and* $\tilde{M}$ *is the associated Rees module, one has the following*:

(1) *There exists a surjective homomorphism* $\pi_1:\tilde{M}\to M$ *such that* $\operatorname{Ker}\pi_1=(1-x_0)\cdot M$. *Moreover, for all submodules* $M'\subseteq M$, *one has* $\pi_1(\widetilde{M'})=M'$.
(2) *There exists a surjective graded homomorphism* $\pi_0:\tilde{M}\to\operatorname{gr}(M)$ *such that* $\operatorname{Ker}\pi_0=x_0M$. *Furthermore,* π_0 *maps graded submodules of* $\tilde{M}$ *to graded submodules of* $\operatorname{gr}(M)$ *and every graded submodule of* $\operatorname{gr}(M)$ *arises in this manner.*

Proof. (1) Every element $\tilde{z}\in\tilde{M}$ can be written uniquely as a finite sum of homogeneous components; $\tilde{z}=\sum_{j=0}^{p}(\tilde{z})_{\ell_j}$ where $\ell_0<\cdots<\ell_p$. Let $\pi_1:\tilde{M}\to M$ be defined by $\pi_1(\tilde{z})=\sum_{j=0}^{p}(z)_{\ell_j}$ where $(z)_{\ell_j}\in F_{\ell_j}M$. The definition of the $\tilde{R}$-module structure on $\tilde{M}$ insures that π_1 is a k-module homomorphism and the image is an R-module. It is clearly surjective. Now, if $\sum_{j=0}^{p}(z)_{\ell_j}=0$ then $\sum_{j=0}^{p}(\tilde{z})_{\ell_j}x_0^{\ell_p-\ell_j}=0$. Hence, the element

$$\tilde{z}=\sum_{j=0}^{p}(\tilde{z})_{\ell_j}-\sum_{j=0}^{p}(\tilde{z})_{\ell_j}x_0^{\ell_p-\ell_j}=\sum_{j=0}^{p}((\tilde{z})_{\ell_j}-(\tilde{z})_{\ell_j}x_0^{\ell_p-\ell_j})$$

belongs to $(1-x_0)\tilde{M}$ and we have $\operatorname{Ker}\pi_1\subseteq(1-x_0)\tilde{M}$. It is obvious from the definition of π_1 that we have $(1-x_0)\tilde{M}\subseteq\operatorname{Ker}\pi_1$ and $\pi_1(\widetilde{M'})=M'$.

(2) For all $i \in \mathbb{Z}$, we have isomorphisms $\tilde{M}_i/x_0\tilde{M}_{i-1} \cong F_iM/F_{i-1}M = \mathrm{gr}(M)_i$. Combining these maps gives the required isomorphism $\tilde{M}/x_0\tilde{M} \cong \mathrm{gr}(M)$. Moreover, we have

$$\pi_0(\tilde{f}\tilde{z}) = fz + (x_0M) = (f + x_0R)(z + x_0M) = \pi_0(\tilde{f})\pi_0(\tilde{z})$$

and, thus, π_0 takes $\tilde{R}$-modules to $\mathrm{gr}(R)$-modules. Finally, for a graded submodule N of $\mathrm{gr}(M)$, consider the $\tilde{R}$-submodule L of $\tilde{M}$ generated by the set $\pi_0^{-1}(N)$. To demonstrate that $\pi_0(L) = N$, it suffices to show $\pi_0(L) \subseteq N$. Every element of L can be written in the form $\sum_{j=0}^{p} \tilde{f}_j\tilde{z}_j$ for some $\tilde{f}_j \in \tilde{R}$ and $\tilde{z}_j \in \pi_0^{-1}(N)$. Applying π_0, we obtain

$$\pi_0\left(\sum_{j=0}^{p} \tilde{f}_j\tilde{z}_j\right) = \sum_{j=0}^{p} \pi_0(\tilde{f}_j)\pi_0(\tilde{z}_j) = \sum_{j=0}^{p} f_jz_j,$$

where $f_j \in \mathrm{gr}(A)$ and $z_j \in N$. Therefore, we have $\pi_0(L) \subseteq N$ which completes the proof. $\square$

Before studying the Gelfand–Kirillov dimension of graded submodules of $\tilde{M}$, we record a useful lemma; see Proposition 2.16 in Björk [4].

Lemma 3.4 (Björk [4]). *Let M be a filtered R-module. If L be a graded submodule of $\tilde{M}$, then the graded module $(\pi_1(L))^{\sim}$ contains L and the quotient $(\pi_1(L))^{\sim}/L$ is an x_0-torsion module.*

Proposition 3.5. *If $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$ is positive, M is an R-module with a good finite dimensional filtration and L is a graded submodule of $\tilde{M}$, then one has the following:*
(1) $\mathrm{GKdim}\, L = \mathrm{GKdim}(\pi_1(L))^{\sim}$;
(2) $1 + \mathrm{GKdim}\, M = \mathrm{GKdim}\, \tilde{M}$;
(3) $1 + \mathrm{GKdim}\, \pi_1(L) = \mathrm{GKdim}\, L$.

Proof. (1) Let $\phi(i) = \dim_k L_i$, $L' = (\pi_1(L))^{\sim}$ and $\phi'(i) = \dim_k L'_i$. By Lemma 3.4, L_i is a subvector space of L'_i which implies $\phi(i) \leqslant \phi'(i)$ and $\mathrm{GKdim}\, L \leqslant \mathrm{GKdim}\, L'$. On the other hand, Lemma 3.4 also states that the quotient L'/L is an x_0-torsion module. Since L' is a finitely generated module, there exists an integer ℓ such that $x_0^{\ell}L'_i \subseteq L_{i+\ell}$. Thus, we have $\phi'(i) \leqslant \phi(i+\ell)$ which implies $\mathrm{GKdim}\, L' \leqslant \mathrm{GKdim}\, L$. Combining the two inequality yields the first part.

(2) The definition of GKdim implies that $\mathrm{GKdim}\, M = \gamma(\dim_k F_iM)$ and $\mathrm{GKdim}\, \tilde{M} = \gamma(\sum_{j=0}^{i} \dim_k F_jM)$ However, a monotonically increasing function $\phi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ and the function $\psi(i) = \sum_{j=0}^{i} \phi(j)$ are related by the equation $\gamma(\psi) = \gamma(\phi) + 1$ and this proves the second assertion.

(3) Applying part (2) gives $1 + \mathrm{GKdim}\, \pi_1(L) = \mathrm{GKdim}(\pi_1(L))^{\sim}$ and combining this part (1) yields the third assertion. $\square$

We have the analogous result for submodules of $\tilde{M}$ and $\mathrm{gr}\, M$.

Proposition 3.6. *Let* $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$ *be a positive vector and let* M *be an* R*-module with a good filtration. If* L *is a graded* $\tilde{R}$*-submodule of* $\tilde{M}$ *then* $1 + \mathrm{GKdim}\,\pi_0(L) = \mathrm{GKdim}\,L$.

Proof. Observe that $\dim_k F_i(\pi_0(L)) = \sum_{j=0}^{i} \dim_k L_j/x_0 L_{j-1} = \dim_k L_i$ and $\dim_k F_i L = \sum_{j=0}^{i} \dim_k L_j$. Since x_0 is a non-zero-divisor of degree 1 on L, we have $\dim_k L_i \leqslant \dim_k L_{i+1}$. Thus, applying the formula $\gamma(\psi) = \gamma(\phi) + 1$, relating an increasing function $\phi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ and $\psi(i) = \sum_{j=0}^{i} \phi(j)$, completes the proof. □

We now prove the main result in this section.

Theorem 3.7. *Let* p *be a non-negative integer, let* $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$ *be a positive vector and let* M *be a filtered* R*-module. If* $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ *has an irreducible component of dimension* p *then there exists a submodule* M' *of* M *such that* $\mathrm{GKdim}\, M' = p$.

Proof. By definition, irreducible components of $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ correspond to minimal primes in the support of $\mathrm{gr}\, M$. Hence, if there exists an irreducible component of $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ of dimension p, then we have a minimal prime $\mathfrak{p}$ in the support of $\mathrm{gr}\, M$ of dimension p. Each minimal prime $\mathfrak{p}$ in the support of $\mathrm{gr}\, M$ corresponds to a graded submodule of $\mathrm{gr}\, M$ of the form $(S/\mathfrak{p})(j)$ for some $j \in \mathbb{Z}$ and Corollary 3.2 implies the Krull dimension of $(S/\mathfrak{p})(j)$ is equal to its Gelfand–Kirillov dimension. Thus, we have a graded submodule of $\mathrm{gr}\, M$ with Gelfand–Kirillov dimension p. We complete this proof by showing that the following three conditions are equivalent:

(a) there exists a submodule M' of M with $\mathrm{GKdim}\, M' = p$;
(b) there exists a graded submodule L of $\tilde{M}$ with $\mathrm{GKdim}\, L = p + 1$;
(c) there exists a graded submodule N of $\mathrm{gr}(M)$ with $\mathrm{GKdim}\, N = p$.

Indeed, we have

(a) $\Rightarrow$ (b): By Proposition 3.5.2, the graded submodule $\tilde{M}'$ of $\tilde{M}$ has Gelfand–Kirillov dimension $p + 1$.
(b) $\Rightarrow$ (a): Follows immediately from Proposition 3.5.3.
(b) $\Rightarrow$ (c): Follows immediately from Proposition 3.6.
(c) $\Rightarrow$ (b): By Proposition 3.3 there exists a graded submodule L of $\tilde{M}$ such that $\pi_0(L) = N$. □

Remark 3.8. Since $\mathrm{GKdim}\, M = \mathrm{GKdim}\,\mathrm{gr}\, M$ for any R-module M, we see that the Gelfand–Kirillov dimension of M is an upper bound on the Krull dimension of each irreducible component of $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$.

Remark 3.9. We have not used the fact that R is an almost centralizing extension of B, so Theorem 3.7 holds for a module M with a good finite dimensional filtration over a filtered k-algebra.

4. Gröbner fan

To study characteristic ideals and the natural adjacency relations among them, we describe the Gröbner fan of an R-ideal. In particular, we generalize Mora and Robbiano's [10] construction for commutative polynomial rings and Assi et al. [1] work on the Weyl algebra. Our setting has the advantage that the commutative polynomial ring, Weyl algebra and homogenized Weyl algebra are all done at once, shortening the treatment in Saito et al. [12].

For the reader's convenience, we recall some "Gröbner basics". For $f=\sum \kappa_{\mathbf{a},\mathbf{b}}x^{\mathbf{a}}y^{\mathbf{b}}\in R$, the initial form of f with respect to vector $(\mathbf{u},\mathbf{v})\in\mathbb{R}^m\otimes\mathbb{R}^n$ is the element $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(f)=\sum_{\mathbf{u}\cdot\mathbf{a}+\mathbf{v}\cdot\mathbf{b}=\ell}\kappa_{\mathbf{a},\mathbf{b}}\bar{x}^{\mathbf{a}}\bar{y}^{\mathbf{b}}$ in S, where $\ell=\max\{\mathbf{u}\cdot\mathbf{a}+\mathbf{v}\cdot\mathbf{b}\colon \kappa_{\mathbf{a},\mathbf{b}}\neq 0\}$.

Proposition 4.1. *If I is an R-ideal and $(\mathbf{u},\mathbf{v})\in\mathrm{PR}(R)\otimes\mathbb{R}$ then $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)=k\cdot\{\mathrm{in}_{(\mathbf{u},\mathbf{v})}(f)\colon f\in I\}$ is an S-ideal. Moreover, if $(\mathbf{u},\mathbf{v})$ belongs to $\mathrm{PR}(R)$ then $\mathrm{gr}\,I$ is isomorphic to $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$.*

Proof. Since $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ is closed under addition, it suffices to show that it is closed under left multiplication by $\bar{x}_1,\dots,\bar{x}_m,\ \bar{y}_1,\dots,\bar{y}_n$. For an element $f=\sum\kappa_{\mathbf{a},\mathbf{b}}x^{\mathbf{a}}y^{\mathbf{b}}\in I$, we have

$$\bar{x}_i\cdot\mathrm{in}_{(\mathbf{u},\mathbf{v})}(f)=\mathrm{in}_{(\mathbf{u},\mathbf{v})}\left(\sum\kappa_{\mathbf{a},\mathbf{b}}x^{\mathbf{a}+\mathbf{e}_i}y^{\mathbf{b}}\right)=\mathrm{in}_{(\mathbf{u},\mathbf{v})}(x_i\cdot f),$$

where $\mathbf{e}_i$ is the ith standard basis vector. Similarly, we obtain

$$\bar{y}_i\mathrm{in}_{(\mathbf{u},\mathbf{v})}(f)=\mathrm{in}_{(\mathbf{u},\mathbf{v})}\left(\sum\kappa_{\mathbf{a},\mathbf{b}}x^{\mathbf{a}}y^{\mathbf{b}+\mathbf{e}_i}+Q(x,y)\right)=\mathrm{in}_{(\mathbf{u},\mathbf{v})}(y_if),$$

because $(\mathbf{u},\mathbf{v})\in\mathrm{PR}(R)\otimes\mathbb{R}$ implies that $Q(x,y)\in R$ has a smaller initial form than $\sum\kappa_{\mathbf{a},\mathbf{b}}x^{\mathbf{a}}y^{\mathbf{b}+\mathbf{e}_i}$.

When $(\mathbf{u},\mathbf{v})\in\mathrm{PR}(R)$, $\{\mathrm{in}_{(\mathbf{u},\mathbf{v})}(f)\colon f\in I$ and $\deg_{(\mathbf{u},\mathbf{v})}(f)=i\}$ is a complete set of representatives for the cosets of $(\mathrm{gr}\,I)_i$. Hence, there exists a bijective set map between $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ and $\mathrm{gr}\,I$. One easily verifies that the S-module structure of $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ and $\mathrm{gr}\,I$ agree under this correspondence. □

The S-ideal $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ is called the initial ideal of the R-ideal I with respect to $(\mathbf{u},\mathbf{v})$. A finite subset $\mathscr{G}$ is a Gröbner basis of I with respect to $(\mathbf{u},\mathbf{v})$ if I is generated by $\mathscr{G}$ and $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ is generated by the initial forms $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(\mathscr{G})=\{\mathrm{in}_{(\mathbf{u},\mathbf{v})}(g)\colon g\in\mathscr{G}\}$.

Now, there is a second type of Gröbner basis arising from certain orderings on R. A total ordering $\prec$ on the standard monomials in R is called a multiplicative order when the following three conditions hold:

(M1) $x^{\mathbf{a}}\prec x_iy_j$ for all monomials $x^{\mathbf{a}}$ appearing in $Q^1_{i,j}(x)$;

(M2) $x^{\mathbf{a}}y_\ell\prec y_iy_j$ for all monomials $x^{\mathbf{a}}y_\ell$ appearing in $Q^2_{i,j}(x,y)$;

(M3) $x^{\mathbf{a}}y^{\mathbf{b}}\prec x^{\mathbf{a}'}y^{\mathbf{b}'}\Rightarrow x^{\mathbf{a}+\mathbf{c}}y^{\mathbf{b}+\mathbf{d}}\prec x^{\mathbf{a}'+\mathbf{c}}y^{\mathbf{b}'+\mathbf{d}}$ for all $(\mathbf{c},\mathbf{d})\in\mathbb{N}^m\times\mathbb{N}^n$.

A multiplicative order $\prec$ is called a term order if $1 = x^0 y^0$ is the smallest element of $\prec$. A multiplicative order which is not a term order has infinite strictly decreasing chains but a term order does not.

Remark 4.2. Conditions (M1) and (M2) correspond directly to the relations (R1) and (R2) in the definition of R. Without these assumptions the order would not be compatible with multiplication.

Fix a multiplicative order $\prec$. The initial monomial $\mathrm{in}_{\prec}(f)$ of $f \in R$ is the monomial $\bar{x}^{\mathbf{a}} \bar{y}^{\mathbf{b}} \in S$ such that $x^{\mathbf{a}} y^{\mathbf{b}}$ is the $\prec$-largest monomial appearing in the standard expansion of f in R. For an R-ideal I, the initial ideal $\mathrm{in}_{\prec}(I)$ is the monomial ideal in S generated by $\{\mathrm{in}_{\prec}(f)\colon f \in I\}$. A finite subset $\mathscr{G}$ is a Gröbner basis of I with respect to $\prec$ if I is generated by $\mathscr{G}$ and $\mathrm{in}_{\prec}(I)$ is generated by $\mathrm{in}_{\prec}(\mathscr{G}) = \{\mathrm{in}_{\prec}(g)\colon g \in \mathscr{G}\}$. A Gröbner basis is reduced if, for any two distinct elements g, $g' \in \mathscr{G}$, the exponent vector of $\mathrm{in}_{\prec}(g)$ is componentwise larger than any exponent vector appearing in the standard expression of g' in R.

The next two propositions relate the two different notions of Gröbner basis in R. For a vector $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R) \otimes \mathbb{R}$ and a term order $\prec$, the multiplicative monomial order $\prec_{(\mathbf{u},\mathbf{v})}$ is defined as follows:

$$x^{\mathbf{a}'} y^{\mathbf{b}'} \prec_{(\mathbf{u},\mathbf{v})} x^{\mathbf{a}} y^{\mathbf{b}} \Leftrightarrow \begin{cases} \mathbf{u} \cdot (\mathbf{a} - \mathbf{a}') + \mathbf{v} \cdot (\mathbf{b} - \mathbf{b}') > 0 \\ \text{or} \quad \begin{pmatrix} \mathbf{u} \cdot (\mathbf{a} - \mathbf{a}') + \mathbf{v} \cdot (\mathbf{b} - \mathbf{b}') = 0 \\ \text{and } x^{\mathbf{a}'} y^{\mathbf{b}'} \prec x^{\mathbf{a}} y^{\mathbf{b}} \end{pmatrix} \end{cases}.$$

Note that $\prec_{(\mathbf{u},\mathbf{v})}$ is a term order if and only if $(\mathbf{u}, \mathbf{v})$ is non-negative.

Proposition 4.3. *Let I be any R-ideal, $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(R) \otimes \mathbb{R}$ and let $\prec$ be any term order. If $\mathscr{G}$ a Gröbner basis for I with respect to $\prec_{(\mathbf{u},\mathbf{v})}$, then one has*:

(1) *The set $\mathscr{G}$ is a Gröbner basis for I with respect to $(\mathbf{u}, \mathbf{v})$;*
(2) *The set $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(\mathscr{G}) = \{\mathrm{in}_{(\mathbf{u},\mathbf{v})}(g)\colon g \in \mathscr{G}\}$ is a Gröbner basis for $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ with respect to $\prec$;*
(3) *If $\mathscr{G}$ is the reduced Gröbner basis for I with respect to $\prec_{(\mathbf{u},\mathbf{v})}$, then $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(\mathscr{G})$ is also reduced.*

Proof. Parts (1) and (2) are analogous to Theorem 1.1.6 in Saito et al. [12]. Part (3) follows from the fact that the exponent vectors appearing in $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(g)$ for $g \in \mathscr{G}$ form a subset of the exponent vectors appearing the standard expression of g in R. □

Although generally there are infinitely many different term orders, this does not lead to an infinite number of distinct initial ideals.

Theorem 4.4. *An R-ideal I has only finitely many distinct initial ideals $\mathrm{in}_{\prec}(I)$ where $\prec$ is a term order.*

Proof. See Theorem 1.2 in Sturmfels [14]. $\square$

A multiplicative order $\prec$ on R lifts to a multiplicative order $<$ on $\tilde{R}$ by the following convention:

$$x_0^{a_0'}x^{\mathbf{a}'}y^{\mathbf{b}'} < x_0^{a_0}x^{\mathbf{a}}y^{\mathbf{b}} \Leftrightarrow \begin{cases} a_0' - a_0 > 0 \\ \text{or} \begin{pmatrix} a_0' - a_0 = 0 \\ \text{and } x^{\mathbf{a}'}y^{\mathbf{b}'} \prec x^{\mathbf{a}}y^{\mathbf{b}} \end{pmatrix}. \end{cases}$$

Note that $\prec$ is a term order if and only if $<$ is a term order.

Proposition 4.5. *Let $\prec$ be a multiplicative order on R and let $\tilde{I}$ be the homogenization of an R-ideal I with respect to $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$. If $\tilde{G}$ is a Gröbner basis for $\tilde{I}$ with respect to $<$ then its dehomogenization $\mathscr{G}$ is a Gröbner basis for I with respect to $\prec$.*

Proof. Since $\tilde{I}|_{x_0=1} = I$, the set $\mathscr{G}$ generates I if and only if $\tilde{\mathscr{G}}$ generates $\tilde{I}$. Thus, it suffices to study the initial ideals. Clearly, $h \in \mathrm{in}_{\prec}(I)$ implies $h \in \mathrm{in}_{<}(\tilde{I})$. Since $\tilde{\mathscr{G}}$ is a Gröbner basis, $h = \mathrm{in}_{<}(\tilde{f}\tilde{g})$ for some $\tilde{f} \in \tilde{R}$ and $\tilde{g} \in \tilde{\mathscr{G}}$. Dehomogenizing, we obtain $h = \mathrm{in}_{\prec}(\tilde{f}|_{x_0=1}\tilde{g}|_{x_0=1})$ which implies $h \in \mathrm{in}_{\prec}(\mathscr{G})$ as required. $\square$

Fix an R-ideal I. Two degree vectors $(\mathbf{u},\mathbf{v})$ and $(\mathbf{u}',\mathbf{v}')$ in $\mathrm{PR}(R) \otimes \mathbb{R}$ are equivalent with respect to I if $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I) = \mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(I)$. We denote the equivalence class of vectors $(\mathbf{u},\mathbf{v})$ with respect to I by $C_I[(\mathbf{u},\mathbf{v})]$. The Gröbner region $\mathrm{GR}(I)$ is the set of all $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R) \otimes \mathbb{R}$ such that $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I) = \mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(I)$ for some positive vector $(\mathbf{u}',\mathbf{v}') \in \mathrm{PR}(R)$.

Proposition 4.6. *Suppose that R is a graded k-algebra with respect to a positive vector $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$. If I is a homogeneous R-ideal, then we have $\mathrm{GR}(I) = \mathrm{PR}(R) \otimes \mathbb{R}$.*

Proof. See Proposition 1.12 in Sturmfels [14]. $\square$

Notice that, for any R-ideal I and a positive vector $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$, the Rees ring $\tilde{R}$ and $\tilde{I}$ satisfy the hypothesis of the above proposition.

A finite subset $\mathscr{U}$ of I is called a universal Gröbner basis if it is simultaneously a Gröbner basis of I with respect to all $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$. This definition is different than the one found Sturmfels [14]; Sturmfels' considers only vectors $(\mathbf{u},\mathbf{v})$ in the Gröbner region $\mathrm{GR}(I)$. Proposition 4.6 shows that these two different notions of a universal Gröbner basis agree for homogeneous ideals in a graded ring.

Corollary 4.7. *Every R-ideal I has a finite universal Gröbner basis.*

Proof. Consider homogenization $\tilde{I}$ of I with respect to a positive vector in $\mathrm{PR}(R)$. By Theorem 4.4 there exists only finitely many distinct reduced Gröbner basis for $\tilde{I}$ with respect to term orders—let $\tilde{\mathscr{G}}$ be their union. Choose $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$ and fix a term order $\prec$ on R. Let $<_{(\mathbf{u},\mathbf{v})}$ denote the multiplicative order on $\tilde{R}$ obtained from $\prec_{(\mathbf{u},\mathbf{v})}$. By construction, $\tilde{\mathscr{G}}$ is a Gröbner basis with respect to $<_{(\mathbf{u},\mathbf{v})}$ when $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$ is positive. Applying Proposition 4.6, it follows that $\tilde{\mathscr{G}}$ is a Gröbner basis with respect to $<_{(\mathbf{u},\mathbf{v})}$ for all $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$. If $\mathscr{G}$ is the dehomogenization of $\tilde{\mathscr{G}}$, then Proposition 4.5 implies that $\mathscr{G}$ is a Gröbner basis of I for $\prec_{(\mathbf{u},\mathbf{v})}$ where $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R)$. Finally, Proposition 4.3 shows that $\mathscr{G}$ is a universal Gröbner basis for I. $\square$

The next proposition shows that Gröbner bases with respect to vectors $(\mathbf{u},\mathbf{v})$ generalize those with respect to term orders.

Proposition 4.8. *Let I be an R-ideal. For any term order $\prec$ there exists a positive vector $(\mathbf{u},\mathbf{v}) \in \mathrm{PR}(R) \otimes \mathbb{R}$ such that $\mathrm{in}_{\prec}(I) = \mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$.*

Proof. See Proposition 2.1.5 in Saito et al. [12]. $\square$

We prove a key tool in the construction of the Gröbner fan.

Proposition 4.9. *Let I be an R-ideal, $(\mathbf{u}',\mathbf{v}') \in \mathrm{PR}(R) \otimes \mathbb{R}$ and let $(\mathbf{u},\mathbf{v})$ belong to $\mathrm{PR}(S) \otimes \mathbb{R} = \mathbb{R}^m \times \mathbb{R}^n$. If $\varepsilon > 0$ is sufficiently small, then one has $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(\mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(I)) = \mathrm{in}_{(\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v})}(I)$.*

Proof. Let $\prec$ be any term order and let $\prec_\varepsilon$ be the multiplicative order defined as follows:

$$x^{\mathbf{a}'}y^{\mathbf{b}'} \prec_\varepsilon x^{\mathbf{a}}y^{\mathbf{b}} \Leftrightarrow \begin{cases} (\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v}) \cdot (\mathbf{a}-\mathbf{a}',\mathbf{b}-\mathbf{b}') > 0 \\ \text{or} \begin{pmatrix} (\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v}) \cdot (\mathbf{a}-\mathbf{a}',\mathbf{b}-\mathbf{b}') = 0 \\ \text{and } (\mathbf{u}',\mathbf{v}') \cdot (\mathbf{a}-\mathbf{a}',\mathbf{b}-\mathbf{b}') > 0 \end{pmatrix} \\ \text{or} \begin{pmatrix} (\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v}) \cdot (\mathbf{a}-\mathbf{a}',\mathbf{b}-\mathbf{b}') = 0 \\ (\mathbf{u}',\mathbf{v}') \cdot (\mathbf{a}-\mathbf{a}',\mathbf{b}-\mathbf{b}') = 0, \\ \text{and } x^{\mathbf{a}'}y^{\mathbf{b}'} \prec x^{\mathbf{a}}y^{\mathbf{b}}. \end{pmatrix} \end{cases}.$$

Fix a universal Gröbner basis $\mathscr{U}$ for I and choose ε small enough so that the following assertions hold:

(1) $(\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v}) \in \mathrm{PR}(R)$,

(2) for all elements g in $\mathscr{U}$, the standard form of g breaks into four pieces $g(x,y) = g_0(x,y)+g_1(x,y)+g_2(x,y)+g_3(x,y)$ such that $\mathrm{in}_{\prec_\varepsilon}(g) = g_0(\bar{x},\bar{y})$, $\mathrm{in}_{(\mathbf{u}'+\varepsilon\mathbf{u},\mathbf{v}'+\varepsilon\mathbf{v})}(g) = g_0(\bar{x},\bar{y}) + g_1(\bar{x},\bar{y})$, and $\mathrm{in}_{(\mathbf{u}',\mathbf{v}')}(g) = g_0(\bar{x},\bar{y}) + g_1(\bar{x},\bar{y}) + g_2(\bar{x},\bar{y})$.

In particular, we have

(†) $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(g_0(x,y) + g_1(x,y) + g_2(x,y)) = g_0(\bar{x},\bar{y}) + g_1(\bar{x},\bar{y})$.

Since $\mathscr{U}$ is a Gröbner basis with respect to $\prec_\varepsilon$, Proposition 4.3 provides two additional Gröbner bases in the polynomial ring S:

(i) The initial forms $g_0(\bar{x}, \bar{y}) + g_1(\bar{x}, \bar{y})$ for $g \in \mathscr{U}$ are a Gröbner basis for the initial ideal $\mathrm{in}_{(\mathbf{u}'+\varepsilon\mathbf{u}, \mathbf{v}'+\varepsilon\mathbf{v})}(I)$ with respect to $\prec_\varepsilon$.
(ii) The initial forms $g_0(\bar{x}, \bar{y}) + g_1(\bar{x}, \bar{y}) + g_2(\bar{x}, \bar{y})$ for $g \in \mathscr{U}$ are a Gröbner basis for the initial ideal $\mathrm{in}_{(\mathbf{u}', \mathbf{v}')}(I)$ with respect to $\prec_\varepsilon$.

Now, the definition of $\prec_\varepsilon$, statement (ii) and Proposition 4.3.2 imply that the polynomials $g_0(\bar{x}, \bar{y}) + g_1(\bar{x}, \bar{y}) + g_2(\bar{x}, \bar{y})$ are a Gröbner basis for the ideal $\mathrm{in}_{(\mathbf{u}', \mathbf{v}')}(I)$ with respect the vector $(\mathbf{u}, \mathbf{v})$. Moreover, (†) indicates that the polynomials $g_0(\bar{x}, \bar{y}) + g_1(\bar{x}, \bar{y})$ generate the ideal $\mathrm{in}_{(\mathbf{u}, \mathbf{v})}(\mathrm{in}_{(\mathbf{u}', \mathbf{v}')}(I))$ and therefore statement (ii) completes the proof. □

We are now in a position to give a description of the equivalence classes $C_I[(\mathbf{u}, \mathbf{v})]$.

Proposition 4.10. *Let I be an R-ideal, let $\prec$ be a term order and let $(\mathbf{u}, \mathbf{v})$ belong to* $\mathrm{GR}(I)$. *If $\mathscr{G}$ is the reduced Gröbner basis of I with respect to $\prec_{(\mathbf{u}, \mathbf{v})}$, then one has*

$$C_I[(\mathbf{u}, \mathbf{v})] = \{(\mathbf{u}', \mathbf{v}') \in \mathrm{GR}(I):\ \mathrm{in}_{(\mathbf{u}, \mathbf{v})}(g) = \mathrm{in}_{(\mathbf{u}', \mathbf{v}')}(g)\ \forall g \in \mathscr{G}\}$$

and, hence, each equivalence class $C_I[(\mathbf{u}, \mathbf{v})]$ is a relatively open rational convex polyhedral cone.

Proof. See Proposition 2.3 in Sturmfels [14]. □

We end with the main result of this section.

Theorem 4.11. *For I an R-ideal, the finite set*

$$\mathrm{GF}(I) := \{\overline{C_I[(\mathbf{u}, \mathbf{v})]}: \textit{for all } (\mathbf{u}, \mathbf{v}) \in \mathrm{GR}(I)\}$$

forms a fan, called the Gröbner fan of I.

Proof. Given the above lemmas and propositions, the proof is now identical to Proposition 2.4 in Sturmfels [14]. □

5. Bounds on the irreducible components

This section contains the proofs of the main results of this paper. We start by stating an elementary lemma from commutative algebra—see Lemma 2.2.2 in Saito et al. [12].

Lemma 5.1. *If J is any ideal in S and $(\mathbf{u}, \mathbf{v}) \in \mathrm{PR}(S) = \mathbb{R}^m \times \mathbb{R}^n$ then one has the following*:
(1) $\mathrm{Kdim}\,\mathrm{in}_{(\mathbf{u}, \mathbf{v})}(J) \leqslant \mathrm{Kdim}\,J$.
(2) *If $(\mathbf{u}, \mathbf{v})$ is positive, then* $\mathrm{Kdim}\,\mathrm{in}_{(\mathbf{u}, \mathbf{v})}(J) = \mathrm{Kdim}\,J$.

Making use of the Gröbner fan, we next study effect that varying the filtration of R has on the Gelfand–Kirillov dimension of an R-module. Recall that M has a good finite dimensional filtration if and only if the vector $(\mathbf{u}, \mathbf{v})$ is positive.

Proposition 5.2. *If M is a finitely generated R-module, then the Gelfand–Kirillov dimension of M is independent of the positive vector $(\mathbf{u},\mathbf{v})\in \mathrm{PR}(R)$.*

Proof. Subsection 8.6.5 in McConnell and Robson [9] states $\operatorname{GKdim} M = \operatorname{GKdim} \operatorname{gr} M$ when M has a good finite dimensional filtration. We also know that the Gelfand–Kirillov dimension and Krull dimension of $\operatorname{gr}(M)$ are equal by Corollary 3.2. Since the Krull dimension of a finitely generated module is the Krull dimension of its support, it suffices to consider ideals. In particular, one reduces to proving that, for an R-ideal I, the initial ideal $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(I)$ is independent of the choice of positive vector $(\mathbf{u},\mathbf{v})\in \mathrm{PR}(R)$.

We prove this statement by constructing a homotopy between two initial ideals. Let $(\mathbf{u}_1,\mathbf{v}_1)$ and $(\mathbf{u}_2,\mathbf{v}_2)$ be two positive vectors in $\mathrm{PR}(R)$; we claim that $\operatorname{Kdim} \mathrm{in}_{(\mathbf{u}_1,\mathbf{v}_1)}(I) = \operatorname{Kdim} \mathrm{in}_{(\mathbf{u}_2,\mathbf{v}_2)}(I)$. Proposition 4.10 implies that each equivalence class $C_I[(\mathbf{u}_2,\mathbf{v}_2)]$ is a convex cone, so $\mathrm{in}_{(\mathbf{u}_2,\mathbf{v}_2)}(I) = \mathrm{in}_{(r\mathbf{u}_2,r\mathbf{v}_2)}(I)$ for any $0<r\in\mathbb{R}$. By replacing $(\mathbf{u}_2,\mathbf{v}_2)$ with a scalar multiple, we may guarantee that $(\mathbf{u}_2-\mathbf{u}_1,\mathbf{v}_2-\mathbf{v}_1)$ is a positive vector. It follows that $(1-r)(\mathbf{u}_1,\mathbf{v}_1)+r(\mathbf{u}_2,\mathbf{v}_2)$ is a positive vector and belongs to $\mathrm{GR}(I)$ for all $r\in[0,1]$. Define J_r to be the ideal $\mathrm{in}_{(1-r)\cdot(\mathbf{u}_1,\mathbf{v}_1)+r\cdot(\mathbf{u}_2,\mathbf{v}_2)}(I)$. Since the line segment from $(\mathbf{u}_1,\mathbf{v}_1)$ to $(\mathbf{u}_2,\mathbf{v}_2)$ intersects finitely many distinct walls of the Gröbner fan, there are real numbers $0=r_0<r_1<\cdots<r_\ell=1$ such that the ideal J_r remains unchanged as the parameter r ranges inside the open interval (r_j,r_{j+1}); we denote this ideal by $J_{(r_j,r_{j+1})}$. By Proposition 4.9, we have $\mathrm{in}_{(\mathbf{u}_2,\mathbf{v}_2)}(J_{r_j}) = J_{(r_j,r_{j+1})} = \mathrm{in}_{(\mathbf{u}_1,\mathbf{v}_1)}(J_{r_{j+1}})$. By applying Lemma 5.1, we see that $\operatorname{Kdim} J_{r_j} = \operatorname{Kdim} J_{(r_j,r_{j+1})} = \operatorname{Kdim} J_{r_{j+1}}$. Combining these equalities for $0\leqslant j<\ell$ completes the proof. □

Remark 5.3. Combining Proposition 5.2 with subsection 8.6.18 in McConnell and Robson [9], we see that the Gelfand–Kirillov dimension of a finitely generated R-module M does not depend on the choice of finite dimensional filtrations for R or M. In particular, the Gelfand–Kirillov dimension appearing in Theorem 1.2 is independent of the vector $(\mathbf{u},\mathbf{v})$.

We are now ready to prove:

Proof of Theorem 1.2. Fix a positive vector in $\mathrm{PR}(R)$ and let $\tilde{R}$ and $\tilde{M}$ be associated Rees ring and module. Now, suppose $\mathrm{Ch}_{(\mathbf{u},\mathbf{v})}(M)$ has an irreducible component of dimension p. Since

$$\mathrm{Ann}_S(\operatorname{gr}(M)) = \operatorname{gr}(\mathrm{Ann}_R(M)) = \mathrm{in}_{(\mathbf{u},\mathbf{v})}(\mathrm{Ann}_R(M)),$$

this means that the S-ideal $\mathrm{in}_{(\mathbf{u},\mathbf{v})}(\mathrm{Ann}_R(M))$ has a minimal prime $\mathfrak{p}$ of dimension p. By Proposition 4.3 and Proposition 4.5, we have

$$\mathrm{in}_{(\mathbf{u},\mathbf{v})}(\mathrm{Ann}_R(M)) = \mathrm{in}_{(1,\mathbf{u},\mathbf{v})}(\mathrm{Ann}_{\tilde{R}}(\tilde{M}))|_{\bar{x}_0=1}.$$

Because $\bar{x}_0$ is a non-zero-divisor on $S[\bar{x}_0]$, $\mathfrak{p}$ lifts to a minimal prime of dimension $p+1$ over $\mathrm{in}_{(1,\mathbf{u},\mathbf{v})}(\mathrm{Ann}_{\tilde{R}}(\tilde{M}))$. By Proposition 4.6, there exists a positive vector $(u_0,\mathbf{u}',\mathbf{v}')\in \mathrm{PR}(\tilde{\mathrm{R}})$ such that $\mathrm{in}_{(1,\mathbf{u},\mathbf{v})}(\mathrm{Ann}_{\tilde{R}}(\tilde{M})) = \mathrm{in}_{(u_0,\mathbf{u}',\mathbf{v}')}(\mathrm{Ann}_{\tilde{R}}(\tilde{M}))$. Theorem 3.7

implies that $\tilde{M}$ has a submodule L satisfying $\operatorname{GKdim} L = p + 1$ and Proposition 3.5 implies $\pi_0(L)$ is a submodule of M with Gelfand–Kirillov dimension p. $\square$

We recall Bernstein's inequality which is also called the "Weak Fundamental Theorem of Algebraic Analysis"—see Section 1.4 in Björk [3] for two distinct proofs.

Theorem 5.4 (Bernstein). *Let k be a field of characteristic zero. If A_n has the filtration induced by $(\mathbf{1}, \mathbf{1})$ (called the standard filtration) and M is a finitely generated A_n-module, then one has* $\operatorname{GKdim} M \geqslant n$.

Proof of Theorem 1.1. This follows immediately from Theorem 1.2 and Theorem 5.4. $\square$

References

[1] A. Assi, F.J. Castro-Jiménez, N.M. Granger, The Gröbner fan of an A_n-module, J. Pure Appl. Algebra 150 (2000) 27–39.
[2] W. Bruns, J. Herzog, Cohen–Macaulay Rings (revised edition), Cambridge Studies in Advanced Mathematics, Vol. 39, Cambridge University Press, New York, 1998.
[3] J.-E. Björk, Rings of Differential Operators, North-Holland, New York, 1979.
[4] J.-E. Björk, The Auslander condition on Noetherian rings, Séminaire d'Algèbra Paul Dubreil et Marie-Paul Malliavin (39ème Année), Lecture Notes in Mathematics, Vol. 1404, Springer, Berlin, 1989, pp. 137–173.
[5] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, F. Ehlers, Algebraic $\mathscr{D}$-modules, Perspectives in Mathematics, Vol. 2, Academic Press, Boston, MA, 1987.
[6] O. Gabber, The integrability of characteristic varieties, Amer. J. Math. 103 (1981) 445–468.
[7] O. Gabber, Equidimensionalité de la variété caractéristique, Exposé de O. Gabber redigé par T. Levasseur June 1982.
[8] H. Li, F. van Oystaeyen, Zariskian Filtrations, K-Monographs in Mathematics, Vol. 2, Kluwer Academic Publishers, Boston, 1996.
[9] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, Wiley Series in Pure and Applied Mathematics, Wiley, New York, 1988.
[10] T. Mora, L. Robbiano, The Gröbner fan of an ideal, J. Symbolic Comput. 6 (1988) 183–208.
[11] M. Sato, M. Kashiwara, T. Kawai, Microfunctions and pseudo-differential equations, Lecture Notes in Mathematics, Vol. 287, Springer, 1973, pp. 265–529.
[12] M. Saito, B. Sturmfels, N. Takayama, Gröbner Deformations of Hypergeometric Differential Equations, Algorithms and Computation in Mathematics, Vol. 6, Springer, New York, 1999.
[13] R.P. Stanley, Enumerative Combinatorics (Volume 1), Cambridge Studies in Advanced Mathematics, Vol. 49, Cambridge University Press, New York, 1997.
[14] B. Sturmfels, Gröbner Bases and Convex Polytopes, University Lecture Series, Vol. 8, American Mathematical Society, Providence, RI, 1996.