

LOG-CONCAVITY OF ASYMPTOTIC MULTIGRADED HILBERT SERIES

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ABSTRACT. We study the linear map sending the numerator of the rational function representing the Hilbert series of a module to that of its r -th Veronese submodule. We show that the asymptotic behaviour as r tends to infinity depends on the multidegree of the module and the underlying positively multigraded polynomial ring. More importantly, we give a polyhedral description for the asymptotic polynomial and prove that the coefficients are log-concave.

1. INTRODUCTION

Although motivated by multigraded Hilbert series, our main result only involves linear operators on a multivariate power series. To be explicit, let $A := [\mathbf{a}_1 \cdots \mathbf{a}_n]$ be an integer $(d \times n)$ -matrix of rank d such that the only nonnegative vector in the kernel is the zero vector (i.e. $\mathbf{a}_1, \dots, \mathbf{a}_n$ is an acyclic vector configuration; see [Zie, §6.2]). Equivalently, the rational function $1/(\prod_{1 \leq j \leq n} (1 - \mathbf{t}^{\mathbf{a}_j}))$ has a unique expansion as a multivariate formal power series; cf. Lemma 8.16 in [MS]. For each positive integer r , consider the linear operator Φ_r induced by sifting out all terms with the exponent vector divisible by r in the power series expansion of a rational function. More precisely, Φ_r acts on $F(\mathbf{t}) \in \mathbb{Z}[\mathbf{t}^{\pm 1}] := \mathbb{Z}[t_1, t_1^{-1}, \dots, t_d, t_d^{-1}]$ as follows:

$$\text{if } \frac{F(\mathbf{t})}{\prod_{1 \leq j \leq n} (1 - \mathbf{t}^{\mathbf{a}_j})} = \sum_{\mathbf{w}} c_{\mathbf{w}} \mathbf{t}^{\mathbf{w}}, \text{ then } \sum_{\mathbf{w}} c_{r\mathbf{w}} \mathbf{t}^{\mathbf{w}} = \frac{\Phi_r[F(\mathbf{t})]}{\prod_{1 \leq j \leq n} (1 - \mathbf{t}^{\mathbf{a}_j})}.$$

The goal of this article is to understand $\Phi_r[F(\mathbf{t})]$ for $r \gg 0$.

To state our result, $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^d$ denotes the linear map determined by A . Let Z be the associated zonotope; it is the image under α of the unit hypercube $[0, 1]^n \subset \mathbb{R}^n$. For each $\mathbf{u} \in \mathbb{Z}^d$, set $P(\mathbf{u}) := \alpha^{-1}(\mathbf{u}) \cap [0, 1]^n$. We say that the map α is *degenerate* if there exists \mathbf{u} in the boundary of Z such that $\dim P(\mathbf{u}) = n - d$. By identifying the rational polytope $P(\mathbf{u})$ with a translate $P(\mathbf{u}) + \mathbf{x}$ lying in $\ker(A) = \alpha^{-1}(\mathbf{0})$, the *normalized volume* $\text{vol}_{n-d}(P(\mathbf{u}))$ equals $(n - d)!$ times the Euclidean volume of $P(\mathbf{u}) + \mathbf{x}$ with respect to the lattice $\alpha^{-1}(\mathbf{0}) \cap \mathbb{Z}^n$. Let m be the greatest common divisor of the maximal minors of A ; in other words, the sublattice $\mathbb{Z}A \subseteq \mathbb{Z}^d$ generated by the columns of A has index m .

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Theorem 1.1. *If α is nondegenerate and $F(\mathbf{t}) = \sum_{\mathbf{v} \in \mathbb{Z}^d} f_{\mathbf{v}} \mathbf{t}^{\mathbf{v}} \in \mathbb{Z}[\mathbf{t}^{\pm 1}]$, then we have*

$$\limsup_{r \rightarrow \infty} \frac{\Phi_r[F(\mathbf{t})]}{r^{n-d}} = \left(\frac{F(\mathbf{1})}{(n-d)!} \right) K_A(\mathbf{t}), \text{ where } K_A(\mathbf{t}) := \sum_{\mathbf{u} \in \text{int}(Z) \cap \mathbb{Z}^d} \text{vol}_{n-d}(P(\mathbf{u})) \mathbf{t}^{\mathbf{u}}.$$

Moreover, the coefficients of $K_A(\mathbf{t})$ are log-concave, quasi-concave, and sum to $m^{n-d} (n-d)!$.

When $A = [1 \cdots 1]$, Theorem 1.1 specializes to Theorem 5.1 in [DF2]. In this case, $K_A(\mathbf{t})$ is the Eulerian polynomial $\sum_{i \geq 0} \langle n-1 \atop i \rangle t^{i+1}$, where $\langle n-1 \atop i \rangle$ counts the permutations of $\{1, \dots, n-1\}$ with exactly i ascents. If A is a totally unimodular matrix (i.e. each subdeterminant of A is ± 1 or 0), then we have $K_A(\mathbf{t}) \in \mathbb{Z}[\mathbf{t}^{\pm 1}]$; see Remark 2.2. Thus, for any appropriate matrix A , one might regard $K_A(\mathbf{t})$ as a generalization of the Eulerian polynomial. However, $K_A(\mathbf{t})$ is not obviously related to the multivariate Eulerian polynomials in [BHVW, §4.3] or the mixed Eulerian numbers in [Pos, §16]. Nevertheless, the proof of Theorem 1.1 (see Step 1 in §2) implies that, for any $F(\mathbf{t}) = \sum_{\mathbf{v} \in \text{int}(mZ)} f_{\mathbf{v}} \mathbf{t}^{\mathbf{v}} \in \mathbb{Z}[\mathbf{t}^{\pm 1}]$ satisfying $F(\mathbf{1}) > 0$, there exists $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$, $\Phi_{mr}[F(\mathbf{t})]$ has nonnegative coefficients that are both log-concave and quasi-concave. Since quasi-concavity is the multivariate version of unimodality (see Step 2 in §2), Theorem 1.1 generalizes those parts of Theorem 1.4 in [BW] and Theorem 1.2 in [BS] that do not make explicit reference to the roots of $F(\mathbf{t})$. It is an open problem to effectively bound r_0 .

Despite sharing most properties of the Eulerian polynomial, the Laurent polynomial $K_A(\mathbf{t})$ need not satisfy the multivariate analogues of real-zero univariate polynomials. For example, if

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix},$$

then we have $K_A(t_1, t_2, t_3) = t_1 t_3 + t_2$, which does not have the half-plane property; see Theorem 3.2 in [Bra]. Notably, the polynomial $K_A(t_1, t_2, t_3) = t_1 t_3 + t_2$ is neither real stable nor Hurwitz stable.

In contrast with [BW], [BS] or [DF2], we provide a new proof that the Eulerian numbers $\langle n \atop i \rangle$ are log-concave and unimodal. We use polyhedral geometry and the Prékopa-Leindler inequality (a.k.a. the functional form of the Brunn-Minkowski inequality) to establish the log-concavity for the coefficients of $K_A(\mathbf{t})$; see Proposition 2.1. Since this method does not appear in the survey articles [Sta2, Bre], our approach to log-concavity appears to be novel within algebraic combinatorics.

Motivation. Our primary motivation for Theorem 1.1 comes from commutative algebra and algebraic geometry. The matrix A defines a positive \mathbb{Z}^d -grading on the polynomial ring $S := \mathbb{C}[x_1, \dots, x_n]$ by setting $\deg(x_j) := \mathbf{a}_j$ for all $1 \leq j \leq n$; see Definition 8.7 in [MS]. For any finitely generated \mathbb{Z}^d -graded S -module $M = \bigoplus_{\mathbf{w} \in \mathbb{Z}^d} M_{\mathbf{w}}$, the Hilbert series of M can be expressed uniquely as a rational function of the form $F(\mathbf{t}) / (\prod_{1 \leq j \leq n} (1 - t^{\mathbf{a}_j}))$; see Theorem 8.20 in [MS]. The numerator $F(\mathbf{t})$ is the *Poincaré polynomial* or *K-polynomial* of M — it records the alternating sum of the multigraded Betti numbers for M ; see Definition 8.21 and Theorem 8.23 in [MS]. Applying the operator Φ_r to the numerator of Hilbert series for M yields the numerator of Hilbert series for the r -th Veronese submodule $\bigoplus_{\mathbf{w} \in \mathbb{Z}^d} M_{r\mathbf{w}}$. Theorem 1.1 shows that there is a unique asymptotic K -polynomial

for each M ; see §3 for the important case in which $F(\mathbf{1}) = 0$. Therefore, Theorem 1.1 gives some information about the solution to Problem 5.3 in [EL]. The results in both [EL] and this paper suggest that the asymptotic structure is surprisingly insensitive to the specific module.

Geometrically, the \mathbb{Z}^d -grading on S corresponds to the action of the torus $(\mathbb{C}^*)^d$ on \mathbb{A}^n , and a multigraded S -module M corresponds to a torus-equivariant sheaf on \mathbb{A}^n . In this setting, the numerator $F(\mathbf{t})$ is the class in the equivariant K -theory of \mathbb{A}^n represented by the corresponding sheaf; see Theorem 8.34 in [MS]. Hence, Theorem 1.1 implies that the Veronese operator Φ_r distinguishes the class $K_A(\mathbf{t})$ within equivariant K -theory. By taking a GIT quotient (e.g [MS, §10.4]), we see that the $(n-d)$ -dimensional toric variety $\mathbb{A}^n/(\mathbb{C}^*)^d$ is equipped with a distinguished K -theory class $K_A(\mathbf{t})$. A geometric explanation for this distinguished K -theory class would be interesting.

Our secondary motivation comes from the theory of lattice point enumeration. Given lattice polytopes P_1, \dots, P_d in \mathbb{R}^n and nonnegative integers w_1, \dots, w_d , the number of lattice points in the Minkowski sum of the w_i -dilates of the P_i is denoted $|(w_1P_1 + \dots + w_dP_d) \cap \mathbb{Z}^d|$. Ehrhart theory implies that the generating series $\sum_{(w_1, \dots, w_d)} |(w_1P_1 + \dots + w_dP_d) \cap \mathbb{Z}^d| t_1^{w_1} \dots t_d^{w_d}$ can be expressed in the form $F(\mathbf{t}) / (\prod_{1 \leq j \leq n} (1 - t^{a_j}))$. The numerator $F(\mathbf{t})$ is a multivariate h^* -vector (a.k.a. Ehrhart h -vector or δ -polynomial) for the collection P_1, \dots, P_d . Thus, Theorem 1.1 also yields a multivariate analogue of Corollary 1.3 in [BS].

Multivariate formal power series of the form $F(\mathbf{t}) / (\prod_{1 \leq j \leq n} (1 - t^{a_j}))$ also arise naturally in many other areas of mathematics. Perhaps the most ubiquitous source is the vector partition function $\psi_A: \mathbb{N}^d \rightarrow \mathbb{N}$ associated to A ; $\psi_A(\mathbf{u})$ counts the number of nonnegative integer vectors $\mathbf{x} \in \mathbb{N}^n$ such that $A\mathbf{x} = \mathbf{u}$. The generating series $\sum_{\mathbf{u}} \psi_A(\mathbf{u}) \mathbf{t}^{\mathbf{u}}$ equals $1 / (\prod_{1 \leq j \leq n} (1 - t^{a_j}))$. As [Stu] indicates, vector partition functions appear in representation theory, approximation theory, and statistics. Reinterpreting Φ_r in each of these areas will yield new insights into $K_A(\mathbf{t})$.

Basic examples. We end this section with two examples illustrating Theorem 1.1.

Example 1.2. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, so that $d = 2$, $n = 3$, and $m = \gcd(2, 4, 2) = 2$. For all $r > 2$, we have $\Phi_{2r}[1] = (r-1)t_1^2t_2^2 + (r-1)t_1^2t_2 + (r-1)t_1t_2^2 + rt_1t_2 + t_1 + t_2 + 1$, which implies that $\lim_{r \rightarrow \infty} \frac{\Phi_{2r}[1]}{2r} = \frac{1}{2}t_1^2t_2^2 + \frac{1}{2}t_1^2t_2 + \frac{1}{2}t_1t_2^2 + \frac{1}{2}t_1t_2$. However, we also have $\Phi_{2r+1}[1] = rt_1^2t_2^2 + rt_1t_2 + 1$, which shows that $\lim_{r \rightarrow \infty} \frac{\Phi_r[1]}{r}$ does not exist. Nevertheless, we obtain $\limsup_{r \rightarrow \infty} \frac{\Phi_r[1]}{r} = \frac{1}{2}t_1^2t_2^2 + \frac{1}{2}t_1^2t_2 + \frac{1}{2}t_1t_2^2 + \frac{1}{2}t_1t_2$.

The zonotope Z associated to A is $\text{conv}\{(0, 0), (2, 0), (0, 2), (3, 1), (1, 3), (3, 3)\}$. Its interior lattice points are $(1, 1), (2, 1), (1, 2), (2, 2)$ and the associated polytopes are the line segments:

$$\begin{aligned} P(1, 1) &= \text{conv}\{(0, 1, 0), (\frac{1}{2}, 0, \frac{1}{2})\}, & P(2, 1) &= \text{conv}\{(\frac{1}{2}, 1, 0), (1, 0, \frac{1}{2})\}, \\ P(1, 2) &= \text{conv}\{(0, 1, \frac{1}{2}), (\frac{1}{2}, 0, 1)\}, & P(2, 2) &= \text{conv}\{(1, 0, 1), (\frac{1}{2}, 0, 1)\}. \end{aligned}$$

Since $\alpha^{-1}(\mathbf{0}) \cap \mathbb{Z}^3 = \{(i, -2i, i) : i \in \mathbb{Z}\}$ and the coefficient of $\mathbf{t}^{\mathbf{u}}$ in $K_A(\mathbf{t})$ is the normalized volume of $P(\mathbf{u})$, we have $K_A(t_1, t_2) = \frac{1}{2}t_1^2t_2^2 + \frac{1}{2}t_1^2t_2 + \frac{1}{2}t_1t_2^2 + \frac{1}{2}t_1t_2$. The coefficients sum to 2. ◊

Example 1.3. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so that $d = 2$, $n = 3$, and $m = 1$. It follows that

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\Phi_r[1]}{r} &= \lim_{r \rightarrow \infty} \frac{(r-1)t_1+1}{r} = t_1, & \limsup_{r \rightarrow \infty} \frac{\Phi_r[t_1]}{r} &= \lim_{r \rightarrow \infty} \frac{(r-1)t_1}{r} = t_1, \\ \limsup_{r \rightarrow \infty} \frac{\Phi_r[t_2]}{r} &= \lim_{r \rightarrow \infty} \frac{(r-1)t_1t_2+t_2}{r} = t_1t_2, & \limsup_{r \rightarrow \infty} \frac{\Phi_r[t_1t_2]}{r} &= \lim_{r \rightarrow \infty} \frac{rt_1t_2}{r} = t_1t_2. \end{aligned}$$

Although each of the limits exists, they do not simply depend up to a scalar on the matrix A . The zonotope $Z = \text{conv}\{(0, 0), (2, 0), (0, 1), (2, 1)\}$ has no interior lattice points, but both of the polytopes $P(1, 0) = \text{conv}\{(1, 0, 0), (0, 1, 0)\}$ and $P(1, 1) = \text{conv}\{(1, 0, 1), (0, 1, 1)\}$ have dimension $n - d = 1$. In particular, the map α is degenerate. \diamond

Further examples, open problems, and other connections are discussed in §3. The proof of Theorem 1.1 is given in §2.

2. THE PROOF OF THE MAIN THEOREM

We divide the proof of Theorem 1.1 into three steps: Step 1 establishes that the limit exists and provides a polyhedral interpretation for $K_A(\mathbf{t})$, Step 2 uses the polyhedral interpretation to prove the log-concavity of the coefficients, and Step 3 gives a geometric explanation for the sum of the coefficients. For brevity, we write $\prod_j := \prod_{1 \leq j \leq n}$ and $\sum_j := \sum_{1 \leq j \leq n}$.

Step 1. To begin, we describe the matrix associated to the linear operator Φ_r . Despite being defined via multivariate formal power series, Φ_r may be understood in terms of a linear operator on Laurent polynomials. Specifically, let $\Psi_r \in \text{End}(\mathbb{Z}[\mathbf{t}^{\pm 1}])$ be the linear operator that discards the terms with exponent vectors that are not componentwise divisible by r and divides each of the remaining exponent vectors by r . Consider a Laurent polynomial $F(\mathbf{t}) \in \mathbb{Z}[\mathbf{t}^{\pm 1}]$ such that $F(\mathbf{t}) / (\prod_j (1 - \mathbf{t}^{\mathbf{a}_j})) = \sum_{\mathbf{w}} c_{\mathbf{w}} \mathbf{t}^{\mathbf{w}}$. The linear operator Ψ_r lifts to an endomorphism on multivariate formal power series. Applying Ψ_r to this rational function yields

$$\begin{aligned} \frac{\Phi_r[F(\mathbf{t})]}{\prod_j (1 - \mathbf{t}^{\mathbf{a}_j})} &= \sum_{\mathbf{w}} c_{r\mathbf{w}} \mathbf{t}^{\mathbf{w}} = \Psi_r \left[\frac{F(\mathbf{t})}{\prod_j (1 - \mathbf{t}^{\mathbf{a}_j})} \right] \\ &= \Psi_r \left(\frac{F(\mathbf{t}) \prod_j (1 + \mathbf{t}^{\mathbf{a}_j} + \mathbf{t}^{2\mathbf{a}_j} + \dots + \mathbf{t}^{(r-1)\mathbf{a}_j})}{\prod_j (1 - \mathbf{t}^{r\mathbf{a}_j})} \right) \\ &= \frac{\Psi_r[F(\mathbf{t}) \prod_j (1 + \mathbf{t}^{\mathbf{a}_j} + \mathbf{t}^{2\mathbf{a}_j} + \dots + \mathbf{t}^{(r-1)\mathbf{a}_j})]}{\prod_j (1 - \mathbf{t}^{\mathbf{a}_j})}. \end{aligned}$$

Hence, we have $\Phi_r[F(\mathbf{t})] = \Psi_r[F(\mathbf{t}) \prod_j (1 + \mathbf{t}^{\mathbf{a}_j} + \mathbf{t}^{2\mathbf{a}_j} + \dots + \mathbf{t}^{(r-1)\mathbf{a}_j})]$; this is a multivariate version of Lemma 3.2 in [BS]. To express Φ_r as a matrix with respect to the monomial basis, set

$$C_r(\mathbf{u}, \mathbf{v}) := \left| \{ \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n \cap [0, r - 1]^n : \sum_j x_j \mathbf{a}_j = r\mathbf{u} - \mathbf{v} \} \right|.$$

Since $F(\mathbf{t}) := \sum_{\mathbf{v} \in \mathbb{Z}^A} f_{\mathbf{v}} \mathbf{t}^{\mathbf{v}}$, we have $\Phi_r[F(\mathbf{t})] = \sum_{\mathbf{u}} (\sum_{\mathbf{v} \in \mathbb{Z}^A} C_r(\mathbf{u}, \mathbf{v}) f_{\mathbf{v}}) \mathbf{t}^{\mathbf{u}}$. As $\mathbf{v} \in \mathbb{Z}^A$, we also have $C_r(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{u} \notin \mathbb{Z}^A$. However, the sublattice $\mathbb{Z}^A \subseteq \mathbb{Z}^d$ has index m , so $m\mathbf{u} \in \mathbb{Z}^A$. Hence, it is enough to consider

$$\Phi_{mr}[F(\mathbf{t})] = \sum_{\mathbf{u}} (\sum_{\mathbf{v} \in \mathbb{Z}^A} C_{mr}(\mathbf{u}, \mathbf{v}) f_{\mathbf{v}}) \mathbf{t}^{\mathbf{u}}.$$

We next relate the integer coefficients $C_{mr}(\mathbf{u}, \mathbf{v})$ to the normalized volume of the rational polytope $P(\mathbf{u}) := \{\mathbf{x} \in [0, 1]^n : \sum_j x_j \mathbf{a}_j = \mathbf{u}\} = \alpha^{-1}(\mathbf{u}) \cap [0, 1]^n \subseteq \mathbb{R}^n$. For any $\mathbf{u} \in \mathbb{Z}^d$ and any $\mathbf{v} \in \mathbb{Z}A$, there exists \mathbf{y} and $\mathbf{z} \in \mathbb{Z}^n$ such that $m\mathbf{u} = \alpha(\mathbf{y})$ and $\mathbf{v} = \alpha(\mathbf{z})$. It follows that

$$C_{mr}(\mathbf{u}, \mathbf{v}) = |((mr - 1)P(\mathbf{u}) + \frac{1}{m}\mathbf{y} - \mathbf{z}) \cap \mathbb{Z}^n| = |((mr - 1)P(\mathbf{u}) + \frac{1}{m}\mathbf{y}) \cap \mathbb{Z}^n|.$$

The enumerative theory for systems of linear diophantine equations establishes that the function

$$r \mapsto |((mr - 1)P(\mathbf{u}) + \frac{1}{m}\mathbf{y}) \cap \mathbb{Z}^n| = |((r - 1)(mP(\mathbf{u})) + (m - 1)P(\mathbf{u}) + \frac{1}{m}\mathbf{y}) \cap \mathbb{Z}^n|$$

is a quasi-polynomial; see Theorem 6.50 in [BG] or Theorem 4.6.11 in [Sta3]. Moreover, the leading coefficient of this quasi-polynomial is the relative volume of $mP(\mathbf{u})$ with respect to the lattice $\alpha^{-1}(rm\mathbf{u}) \cap \mathbb{Z}^n$; see Theorem 6.55 in [BG]. Since $m\mathbf{u} \in \mathbb{Z}A$ and $(mr - 1)P(\mathbf{u}) + \frac{1}{m}\mathbf{y} \subset \alpha^{-1}(rm\mathbf{u})$, the normalized volume $\text{vol}_{n-d}(P(\mathbf{u}))$ equals $(n-d)!/m^{n-d}$ times the coefficient of the degree $n-d$ term in this quasi-polynomial, so

$$\lim_{r \rightarrow \infty} \frac{C_{mr}(\mathbf{u}, \mathbf{v})}{(mr)^{n-d}} = \lim_{r \rightarrow \infty} \frac{|((mr - 1)P(\mathbf{u}) + \frac{1}{m}\mathbf{y}) \cap \mathbb{Z}^n|}{(mr)^{n-d}} = \frac{\text{vol}_{n-d}(P(\mathbf{u}))}{(n-d)!}.$$

By hypothesis, α is nondegenerate, so $P(\mathbf{u})$ has dimension $n - d$ if and only if \mathbf{u} lies in the interior of Z . Therefore, we have

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{\Phi_{mr}[F(\mathbf{t})]}{(mr)^{n-d}} &= \sum_{\mathbf{u}} \left(\sum_{\mathbf{v} \in \mathbb{Z}A} \left(\lim_{r \rightarrow \infty} \frac{C_{mr}(\mathbf{u}, \mathbf{v})}{(mr)^{n-d}} \right) f_{\mathbf{v}} \right) \mathbf{t}^{\mathbf{u}} \\ &= \frac{F(\mathbf{1})}{(n-d)!} \left(\sum_{\mathbf{u} \in \text{int}(Z) \cap \mathbb{Z}^d} \text{vol}_{n-d}(P(\mathbf{u})) \mathbf{t}^{\mathbf{u}} \right). \end{aligned}$$

It follows immediately from this polyhedral description that $K_A(\mathbf{t})$ inherits the symmetries of the zonotope Z . ▷

Step 2. The second step takes advantage of a well-known theorem from convex geometric analysis. To isolate the applicable result, recall that $g: \mathbb{R}^d \rightarrow \mathbb{R}$ is **log-concave** if $g(\mathbf{u}) > 0$ and $\log(g)$ is concave. Equivalently, for $s \in [0, 1]$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$, we have the inequality $g(s\mathbf{u} + (1-s)\mathbf{v}) \geq g(\mathbf{u})^s g(\mathbf{v})^{1-s}$.

Proposition 2.1. *Let $X \subset \mathbb{R}^n$ be a closed convex set and let $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^d$ be a surjective linear map. Consider the function $g: \mathbb{R}^d \rightarrow \mathbb{R}$ that assigns to $\mathbf{u} \in \mathbb{R}^d$ the volume of the fibre $\pi^{-1}(\mathbf{u}) \cap X$. It follows that g is log-concave.*

Proof. Choosing a basis for $\ker(\pi)$ defines an injective linear map $\varphi: \mathbb{R}^{n-d} \rightarrow \mathbb{R}^n$ such that $\pi \circ \varphi$ is the zero map. We may also choose an injective linear map $\eta: \mathbb{R}^d \rightarrow \mathbb{R}^n$ such that $\pi \circ \eta$ is the identity map, because π is surjective. For any $\mathbf{u} \in \mathbb{R}^d$, we set $X(\mathbf{u}) := \{\mathbf{p} \in \mathbb{R}^{n-d} : \varphi(\mathbf{p}) + \eta(\mathbf{u}) \in X\} \cong \pi^{-1}(\mathbf{u}) \cap X$, so that $g(\mathbf{u}) = \text{vol}(X(\mathbf{u}))$. Since X is convex, we have

$$sX(\mathbf{u}) + (1-s)X(\mathbf{v}) \subseteq X(s\mathbf{u} + (1-s)\mathbf{v})$$

for all $s \in [0, 1]$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. If $1_Y: \mathbb{R}^{n-d} \rightarrow \mathbb{R}$ denotes the indicator function for $Y \subseteq \mathbb{R}^{n-d}$, i.e. $1_Y(\mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{y} \in Y \\ 0 & \text{if } \mathbf{y} \notin Y \end{cases}$, then we obtain

$$1_{X(s\mathbf{u}+(1-s)\mathbf{v})}(s\mathbf{p} + (1-s)\mathbf{q}) \geq (1_{X(\mathbf{u})}(\mathbf{p}))^s (1_{X(\mathbf{v})}(\mathbf{q}))^{1-s}$$

for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{n-d}$. There exists a positive $k \in \mathbb{R}$ such that

$$g(\mathbf{u}) = k \int_{\mathbb{R}^{n-d}} 1_{X(\mathbf{u})}(\mathbf{p}) d\mathbf{p},$$

so the Prékopa-Leindler inequality (Theorem 7.1 in [Gar] or Theorem 6.4 in [Vil]) yields

$$\begin{aligned} g(s\mathbf{u} + (1-s)\mathbf{v}) &= k \int_{\mathbb{R}^{n-d}} 1_{X(s\mathbf{u}+(1-s)\mathbf{v})}(\mathbf{p}) d\mathbf{p} \\ &\geq k \left(\int_{\mathbb{R}^{n-d}} 1_{X(\mathbf{u})}(\mathbf{p}) d\mathbf{p} \right)^s \left(\int_{\mathbb{R}^{n-d}} 1_{X(\mathbf{v})}(\mathbf{p}) d\mathbf{p} \right)^{1-s} \\ &= (g(\mathbf{u}))^s (g(\mathbf{v}))^{1-s}. \end{aligned} \quad \square$$

To establish that the coefficient function of $K_A(\mathbf{t})$ is log-concave, we simply apply Proposition 2.1 when $X = [0, 1]^n$ and $\pi = \alpha$, because Step 1 shows that the coefficients of $K_A(\mathbf{t})$ equal the normalized volume of the fibres of the map $\alpha: [0, 1]^n \rightarrow Z$.

We finish this step by showing that the coefficient function of $K_A(\mathbf{t})$ is quasi-concave. A function $g: \mathbb{R}^d \rightarrow \mathbb{R}$ is *quasi-concave* if, for all $e \in \mathbb{R}$, its superlevel set $\{\mathbf{u} \in \mathbb{R}^d : g(\mathbf{u}) \geq e\}$ is convex. Equivalently, for $s \in [0, 1]$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, we have $g(s\mathbf{u} + (1-s)\mathbf{v}) \geq \min(g(\mathbf{u}), g(\mathbf{v}))$. In particular, every positive log-concave function is quasi-concave. ▷

Step 3. Lastly, we reinterpret the coefficients of $K_A(\mathbf{t})$ as the normalized volumes of certain regions in the hypercube $[0, m]^{n-d}$. This perspective is inspired by the well-known interpretation for the Eulerian numbers as the normalized volume of “slabs” in the hypercube; see [Sta1]. In fact, when $A = [1 \cdots 1]$, we recover this interpretation.

By reordering the columns and making a unimodular change of basis on \mathbb{Z}^d , it suffices to consider the case in which $A = [H \mid B']$, where H is a $(d \times d)$ -matrix in Hermite normal form, $\det(H) = m$, and $B' = [\mathbf{b}_1 \cdots \mathbf{b}_d]^t$ is a $(d \times (n-d))$ -matrix; see [Sch, §4.1]. Let B be the integer block $(n \times d)$ -matrix $\begin{bmatrix} mH^{-1}B' \\ -mI_{n-d} \end{bmatrix}$, so that $AB = 0$. If $m = 1$, then $H = I_d$ and B is the Gale dual of A ; see [Zie, §6.4]. Thus, we may assume that A has this block structure.

For $\mathbf{u} \in \mathbb{Z}^d$, consider regions

$$R(\mathbf{u}) := \{\mathbf{p} \in [0, m]^{n-d} : m(u_i - 1) \leq \mathbf{b}_i \cdot \mathbf{p} \leq mu_i \text{ for all } 1 \leq i \leq d\} \subset \mathbb{R}^{n-d}.$$

By definition, each region $R(\mathbf{u})$ is a rational polytope and, combined together, these regions partition the hypercube $[0, m]^{n-d}$. Hence, the union $\bigcup_{\mathbf{u}} R(\mathbf{u})$ has normalized volume equal to $m^{n-d}(n-d)!$.

To complete the proof, it is enough to show that the coefficients of $K_A(\mathbf{t})$ also correspond to the normalized volumes of the regions $R(\mathbf{u})$. If J is the integer $(n \times d)$ -matrix $\begin{bmatrix} mH_0^{-1} \\ \end{bmatrix}$, then we obtain $AJ = mI_d$. Moreover, the inequalities $m(u_i - 1) \leq \mathbf{b}_i \cdot \mathbf{p} \leq mu_i$ hold if and only if the inequalities $1 \geq u_i - \frac{1}{m}\mathbf{b}_i \cdot \mathbf{p} \geq 0$ hold. It follows that the affine map $\mathbf{p} \mapsto \frac{1}{m}(J\mathbf{u} - B\mathbf{p})$ sends the region $R(\mathbf{u})$ into the rational polytope $P(\mathbf{u}) = \{\mathbf{x} \in [0, 1]^n : \sum_j x_j \mathbf{a}_j = \mathbf{u}\} = \alpha^{-1}(\mathbf{u}) \cap [0, 1]^n$. The inverse map is associated to the $((n-d) \times n)$ -matrix $L := [0 \mid I_{n-d}]$ which satisfies $LJ = 0$ and $LB = -mI_{n-d}$. Since Step 1 establishes that the coefficients

of $K_A(\mathbf{t})$ equal the normalized volumes of the polytopes $P(\mathbf{u})$, we conclude that $K_A(\mathbf{1}) = m^{n-d}(n-d)!$, as required. \square

Remark 2.2. Suppose that A is a totally unimodular matrix which implies that $m = 1$. If $\mathbf{u} \in \mathbb{Z}^d$, then $P(\mathbf{u})$ is a lattice polytope (see Theorem 19.1 in [Sch]) and $C_r(\mathbf{u}, \mathbf{v}) = |((r-1)P(\mathbf{u})) \cap \mathbb{Z}^n|$. Thus, the normalized volume $\text{vol}_{n-d}(P(\mathbf{u}))$ is a nonnegative integer, which implies that $K_A(\mathbf{1}) = (n-d)!$ and $K_A(\mathbf{t}) \in \mathbb{Z}[\mathbf{t}^{\pm 1}]$. Finding a combinatorial description for these coefficients remains an open problem. One tantalizing possibility is that the coefficients naturally enumerate some partition of the symmetric group on $n-d$ letters.

3. OTHER CONNECTIONS

This section uses two additional examples to highlight some potential applications for Theorem 1.1. To help orient future research, we also state a few open problems.

Asymptotic equalities. When $F(\mathbf{1}) \neq 0$, Theorem 1.1 shows that some of the coefficients of $\Phi_r[F(\mathbf{t})]$, regarded as functions of r , are asymptotically equal to an explicit multiple of r^{n-d} . In many situations, Theorem 1.1 yields an asymptotic equality even though $F(\mathbf{1}) = 0$. Roughly speaking, the Taylor expansion of $F(\mathbf{t})$ about $\mathbf{t} = \mathbf{1}$ allows one to cancel factors from the denominator and, thereby, apply Theorem 1.1 to a submatrix of A . We demonstrate this approach in the following prototypical example.

Example 3.1. Let $S = \mathbb{C}[x_1, \dots, x_6]$ be the polynomial ring with the positive \mathbb{Z}^2 -grading induced by $A = \begin{bmatrix} 2 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 1 & 2 \end{bmatrix}$. Observe that $m = 1$. Fix the finitely generated \mathbb{Z}^2 -graded S -module $M = S/I$ where $I = \langle x_5^2 - x_4x_6, x_3x_5 - x_2x_6, x_3x_4 - x_2x_5, x_3^2 - x_1x_6, x_2x_3 - x_1x_5, x_2^2 - x_1x_6 \rangle$. Under the standard grading (i.e. when $A = [1 \ \dots \ 1]$), the module M would be the homogeneous coordinate ring of the Veronese surface in \mathbb{P}^5 . As a rational function, the Hilbert series of M is

$$\frac{F(\mathbf{t})}{\prod_j (1 - \mathbf{t}^{\mathbf{a}_j})} = \frac{(1 - t_1t_2)(1 - t_1^2t_2)(1 - t_1t_2^2)(1 + t_1t_2 + t_1^2t_2 + t_1t_2^2)}{(1 - t_1^2)(1 - t_1t_2)(1 - t_1^2t_2)(1 - t_2^2)(1 - t_1t_2^2)(1 - t_1^2t_2^2)}.$$

Since $\ell := \text{codim}(M) = 3 > 0$, it follows that $F(\mathbf{1}) = 0$.

In this context, an appropriate expansion of $F(\mathbf{t})$ comes from a choice of initial module. Each monomial initial module of M is homogeneous with respect to the \mathbb{Z}^n -grading arising from the identity matrix I_n ; see [MS, §2.2]. Since M and any initial module have the same Hilbert series, the Taylor expansion for the numerator of the \mathbb{Z}^n -graded Hilbert series produces an expression for $F(\mathbf{t})$ as a polynomial in the variables $(1 - \mathbf{t}^{\mathbf{a}_1}), \dots, (1 - \mathbf{t}^{\mathbf{a}_n})$; see Exercise 8.15 in [MS]. Moreover, the lowest terms in each such expression have degree ℓ , are square-free, and have non-negative coefficients; see Exercise 8.8 in [MS]. For instance, the monomial initial ideals $\langle x_2, x_3, x_5 \rangle^2$ and $\langle x_1, x_2, x_5^2 \rangle \cap \langle x_2^2, x_5, x_6 \rangle$ of I (chosen from among the 29 possibilities) yield the following expansions:

$$\begin{aligned} F(\mathbf{t}) &= 4(1 - \mathbf{t}^{\mathbf{a}_2})(1 - \mathbf{t}^{\mathbf{a}_3})(1 - \mathbf{t}^{\mathbf{a}_5}) - (1 - \mathbf{t}^{\mathbf{a}_2})^2(1 - \mathbf{t}^{\mathbf{a}_3})(1 - \mathbf{t}^{\mathbf{a}_5}) \\ &\quad - (1 - \mathbf{t}^{\mathbf{a}_2})(1 - \mathbf{t}^{\mathbf{a}_3})^2(1 - \mathbf{t}^{\mathbf{a}_5}) - (1 - \mathbf{t}^{\mathbf{a}_2})(1 - \mathbf{t}^{\mathbf{a}_3})(1 - \mathbf{t}^{\mathbf{a}_5})^2 \\ &= 2(1 - \mathbf{t}^{\mathbf{a}_1})(1 - \mathbf{t}^{\mathbf{a}_2})(1 - \mathbf{t}^{\mathbf{a}_5}) + 2(1 - \mathbf{t}^{\mathbf{a}_2})(1 - \mathbf{t}^{\mathbf{a}_5})(1 - \mathbf{t}^{\mathbf{a}_6}) \\ &\quad - (1 - \mathbf{t}^{\mathbf{a}_1})(1 - \mathbf{t}^{\mathbf{a}_2})(1 - \mathbf{t}^{\mathbf{a}_5})^2 - (1 - \mathbf{t}^{\mathbf{a}_1})(1 - \mathbf{t}^{\mathbf{a}_2})(1 - \mathbf{t}^{\mathbf{a}_5})(1 - \mathbf{t}^{\mathbf{a}_6}) \\ &\quad - (1 - \mathbf{t}^{\mathbf{a}_2})^2(1 - \mathbf{t}^{\mathbf{a}_5})(1 - \mathbf{t}^{\mathbf{a}_6}). \end{aligned}$$

Observe that the coefficient of the term $(1 - \mathbf{t}^{a_{s_1}})(1 - \mathbf{t}^{a_{s_2}})(1 - \mathbf{t}^{a_{s_3}})$ equals the multiplicity $\mu_{\mathbf{s}}$ of the minimal prime $\langle x_{s_1}, x_{s_2}, x_{s_3} \rangle$ for the initial ideal; cf. Definition 8.43 in [MS]. Since we have $\Phi_r[(1 - \mathbf{t}^{a_i})F(\mathbf{t})] = (1 - \mathbf{t}^{a_i})\Phi_r[F(\mathbf{t})]$, Theorem 1.1 gives

$$\lim_{r \rightarrow \infty} \frac{\Phi_r[F(\mathbf{t})]}{r^{n-\ell-d}} = \frac{1}{(n - \ell - d)!} \sum_{\mathbf{s}} \left(\frac{\mu_{\mathbf{s}}}{m_{A_{\widehat{\mathbf{s}}}}^{n-\ell-d}} K_{A_{\widehat{\mathbf{s}}}}(\mathbf{t}) \prod_{1 \leq i \leq \ell} (1 - \mathbf{t}^{a_{s_i}}) \right),$$

where $A_{\widehat{\mathbf{s}}}$ is the $(d \times (n - \ell))$ -submatrix of A in which the columns indexed by \mathbf{s} are omitted and $m_{A_{\widehat{\mathbf{s}}}}$ is the greatest common divisor of the maximal minors of $A_{\widehat{\mathbf{s}}}$. In particular, we have

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{\Phi_r[F(\mathbf{t})]}{r} &= \frac{1}{2} t_1 t_2 (t_1 + 1)(t_2 + 1)(t_1 t_2 + 1)(1 - t_1 t_2)(1 - t_1^2 t_2)(1 - t_1 t_2^2) \\ &= \frac{4}{4} \left(K_{\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}}(\mathbf{t}) \right) (1 - \mathbf{t}^{a_2})(1 - \mathbf{t}^{a_3})(1 - \mathbf{t}^{a_5}) \\ &= \frac{2}{2} \left(K_{\begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}}(\mathbf{t}) \right) (1 - \mathbf{t}^{a_1})(1 - \mathbf{t}^{a_2})(1 - \mathbf{t}^{a_5}) \\ &\quad + \frac{2}{2} \left(K_{\begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}}(\mathbf{t}) \right) (1 - \mathbf{t}^{a_2})(1 - \mathbf{t}^{a_5})(1 - \mathbf{t}^{a_6}). \end{aligned}$$

As the expansions for $F(\mathbf{t})$ vary, these equations lead to nontrivial relations among asymptotic K -polynomials associated to submatrices of A . ◊

This approach also suggests a way to understand lower-order asymptotics. Specifically, if $\limsup_{r \rightarrow \infty} \frac{\Phi_r[F(\mathbf{t})]}{r^{n-\ell-d}} = \frac{G(\mathbf{t})}{(n-\ell-d)!}$, then $(n - \ell - d)! \Phi_r[F(\mathbf{t})] - G(\mathbf{t})$ vanishes at $\mathbf{t} = \mathbf{1}$ and one should analyze $\limsup_{r \rightarrow \infty} \frac{(n-\ell-d)! \Phi_r[F(\mathbf{t})] - G(\mathbf{t})}{r^{n-\ell-d-1}}$. Providing an algebraic or geometric interpretation for this lower-order asymptotics is an open problem.

Stochastic matrices. By scaling the matrix associated to the linear operator Φ_r , we obtain a stochastic matrix. Specifically, the matrix $C(r) := r^{d-n} [C_r(\mathbf{u}, \mathbf{v})]$ where \mathbf{u} and \mathbf{v} range over $\text{int}(Z) \cap \mathbb{Z}^d$ is a nonnegative square matrix each of whose columns sum to 1. When $A = [1 \cdots 1]$, [Hol] (also see [DF1]) establishes that $C(r)$ has the following “amazing” properties:

- for all r , the stationary vector (i.e., the eigenvector with eigenvalue 1) corresponds to the coefficients of $K_A(\mathbf{t})/(n - d)!$;
- the matrix $C(r)$ has eigenvalues r^{-i} for $0 \leq i < n - d$ with explicit eigenvectors independent of r ;
- we have $C(r_1)C(r_2) = C(r_1 r_2)$.

The next example illustrates how these properties extend to our more general setting.

Example 3.2. Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$, so that

$$Z = \text{conv}\{(0, 0), (2, 0), (-1, 1), (2, 2), (-1, 3), (1, 3)\}$$

and $K_A(t_1, t_2) = t_1 t_2^2 + 2t_1 t_2 + 2t_2^2 + t_2$. Fixing $(1, 2), (1, 1), (2, 0), (1, 0)$ as the ordering for the interior lattice points yields the stochastic matrix

$$C(r) := r^{-3} \begin{bmatrix} \binom{r+2}{3} & \binom{r+1}{3} & \binom{r+1}{3} & \binom{r}{3} \\ 2\binom{r+1}{3} & 2\binom{r+1}{3} + \binom{r+1}{2} & 2\binom{r}{3} + \binom{r}{2} & 2\binom{r+1}{3} \\ 2\binom{r+1}{3} & 2\binom{r}{3} + \binom{r}{2} & 2\binom{r+1}{3} + \binom{r+1}{2} & 2\binom{r+1}{3} \\ \binom{r}{3} & \binom{r+1}{3} & \binom{r+1}{3} & \binom{r+2}{3} \end{bmatrix}.$$

The eigenvalues of $C(r)$ are $1, r^{-1}, r^{-1}, r^{-2}$ and the corresponding eigenvectors are simply $[1\ 2\ 2\ 1]^t$, $[1\ 0\ 0\ -1]^t$, $[0\ 1\ -1\ 0]^t$, $[1\ -1\ -1\ 1]^t$. \diamond

In the standard graded case, [Hol] and [DF1] relate the matrix $C(r)$ to the process of “carries” when adding integers and to shuffling cards, respectively. Do our more general stochastic matrices also correspond to known Markov chains? Regardless, it would be interesting to bound the rates of convergence for the associated Markov chains and thereby extend the results in [DF2, §3].

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