

Monomial Ideals

Serkan Hoşten and Gregory G. Smith

Monomial ideals form an important link between commutative algebra and combinatorics. In this chapter, we demonstrate how to implement algorithms in *Macaulay 2* for studying and using monomial ideals. We illustrate these methods with examples from combinatorics, integer programming, and algebraic geometry.

An ideal I in $S = \mathbb{Q}[x_1, \dots, x_n]$ is called a monomial ideal if it satisfies any of the following equivalent conditions:

- (a) I is generated by monomials,
- (b) if $f = \sum_{\alpha \in \mathbb{N}^n} k_{\alpha} x^{\alpha}$ belongs to I then $x^{\alpha} \in I$ whenever $k_{\alpha} \neq 0$,
- (c) I is torus-fixed; in other words, if $(c_1, \dots, c_n) \in (\mathbb{Q}^*)^n$, then I is fixed under the action $x_i \mapsto c_i x_i$ for all i .

It follows that a monomial ideal is uniquely determined by the monomials it contains. Most operations are far simpler for a monomial ideal than for an ideal generated by arbitrary polynomials. In particular, many invariants can be effectively determined for monomial ideals. As a result, one can solve a broad collection of problems by reducing to or encoding data in a monomial ideal. The aim of this chapter is to develop the computational aspects of monomial ideals in *Macaulay 2* and demonstrate a range of applications.

This chapter is divided into five sections. Each section begins with a discussion of a computational procedure involving monomial ideals. Algorithms are presented as *Macaulay 2* functions. We illustrate these methods by solving problems from various areas of mathematics. In particular, we include the *Macaulay 2* code for generating interesting families of monomial ideals. The first section introduces the basic functions on monomial ideals in *Macaulay 2*. To demonstrate these functions, we use the Stanley-Reisner ideal associated to a simplicial complex to compute its f -vector. Next, we present two algorithms for finding a primary decomposition of a monomial ideal. In a related example, we use graph ideals to study the complexity of determining the codimension of a monomial ideal. The third section focuses on the standard pairs of a monomial ideal; two methods are given for finding the set of standard pairs. As an application, we use standard pairs to solve integer linear programming problems. The fourth section examines Borel-fixed ideals and generic initial ideals. Combining these constructions with distractions, we demonstrate that the Hilbert scheme $\text{Hilb}^{4t+1}(\mathbb{P}^4)$ is connected. Finally, we look at the chains of associated primes in various families of monomial ideals.

1 The Basics of Monomial Ideals

Creating monomial ideals in *Macaulay 2* is analogous to creating general ideals. The monomial ideal generated by a sequence or list of monomials can be constructed with the function `monomialIdeal`.

```
i1 : S = QQ[a, b, c, d];
i2 : I = monomialIdeal(a^2, a*b, b^3, a*c)
      2      3
o2 = monomialIdeal (a , a*b, b , a*c)
o2 : MonomialIdeal of S
i3 : J = monomialIdeal{a^2, a*b, b^2}
      2      2
o3 = monomialIdeal (a , a*b, b )
o3 : MonomialIdeal of S
```

The type `MonomialIdeal` is the class of all monomial ideals. If an entry in the sequence or list is not a single monomial, then `monomialIdeal` takes only the leading monomial; recall that every polynomial ring in *Macaulay 2* is equipped with a monomial ordering.

```
i4 : monomialIdeal(a^2+a*b, a*b+3, b^2+d)
      2      2
o4 = monomialIdeal (a , a*b, b )
o4 : MonomialIdeal of S
```

There are also several methods of associating a monomial ideal to an arbitrary ideal in a polynomial ring. The most important of these is the `initial` ideal — the monomial ideal generated by the leading monomials of all elements in the given ideal. When applied to an `Ideal`, the function `monomialIdeal` returns the initial ideal.

```
i5 : K = ideal(a^2, b^2, a*b+b*c)
      2  2
o5 = ideal (a , b , a*b + b*c)
o5 : Ideal of S
i6 : monomialIdeal K
      2      2      2
o6 = monomialIdeal (a , a*b, b , b*c )
o6 : MonomialIdeal of S
```

This is equivalent to taking the leading monomials of a Gröbner basis for `K`. In our example, the given generators for `K` are not a Gröbner basis.

```
i7 : monomialIdeal gens K
      2      2
o7 = monomialIdeal (a , a*b, b )
o7 : MonomialIdeal of S
```

One can also test if a general ideal is generated by monomials with the function `isMonomialIdeal`.

```
i8 : isMonomialIdeal K
o8 = false

i9 : isMonomialIdeal ideal(a^5, b^2*c, d^11)
o9 = true
```

The usual algebraic operations on monomial ideals are the same as on general ideals. For example, we have

```
i10 : I+J

o10 = monomialIdeal (a^2, a*b, b^2, a*c)
o10 : MonomialIdeal of S
```

Example: Stanley-Reisner Ideals and f -vectors

Radical monomial ideals — ideals generated by squarefree monomials — have a beautiful combinatorial interpretation in terms of simplicial complexes. More explicitly, a simplicial complex Δ on the vertex set $\{x_1, \dots, x_n\}$ corresponds to the ideal I_Δ in $S = \mathbb{Q}[x_1, \dots, x_n]$ generated by all monomials $x_{i_1} \cdots x_{i_p}$ such that $\{x_{i_1}, \dots, x_{i_p}\} \notin \Delta$. The ideal I_Δ is called the Stanley-Reisner ideal of Δ .

To illustrate the connections between Stanley-Reisner ideals and simplicial complexes, we consider the f -vector. Perhaps the most important invariant of a simplicial complex, the f -vector of a d -dimensional simplicial complex Δ is $(f_0, f_1, \dots, f_d) \in \mathbb{N}^{d+1}$, where f_i denotes the number of i -dimensional faces in Δ . From the monomial ideal point of view, the f -vector is encoded in the Hilbert series of the quotient ring S/I_Δ as follows:

Theorem 1.1. *If Δ is a simplicial complex with f -vector (f_0, \dots, f_d) , then the Hilbert series of S/I_Δ is*

$$H_{S/I_\Delta}(t) = \sum_{i=-1}^d \frac{f_i t^{i+1}}{(1-t)^{i+1}},$$

where $f_{-1} = 1$.

Proof. Following Stanley [24], we work with the fine grading and then specialize. The fine grading of S is the \mathbb{Z}^n -grading defined by $\deg x_i = \mathbf{e}_i \in \mathbb{Z}^n$, where \mathbf{e}_i is the i -th standard basis vector. The support of a monomial x^α is defined to be the set $\text{supp}(x^\alpha) = \{x_i : \alpha_i > 0\}$. Observe that $x^\alpha \neq 0$ in S/I_Δ if and only if $\text{supp}(x^\alpha) \in \Delta$. Moreover, the nonzero monomials x^α form a

\mathbb{Q} -basis of S/I_Δ . By counting such monomials according to their support, we obtain the following expression for the Hilbert series with the fine grading:

$$H_{S/I_\Delta}(\mathbf{t}) = \sum_{F \in \Delta} \sum_{\substack{\alpha \in \mathbb{N}^n \\ \text{supp}(\alpha) = F}} \mathbf{t}^\alpha = \sum_{F \in \Delta} \prod_{x_i \in F} \frac{t_i}{1 - t_i}.$$

Finally, by replacing each t_i with t , we complete the proof. \square

Since $H_{S/I_\Delta}(t)$ is typically expressed in the form $\frac{h_0 + h_1 t + \dots + h_d t^d}{(1-t)^{d+1}}$, we can obtain the f -vector by using the identity $\sum_i h_i t^i = \sum_{j=0}^d f_{j-1} t^j (1-t)^{d-j}$. In particular, we can compute f -vectors from Stanley-Reisner ideals as follows:

```
i11 : fvector = I -> (
      R := (ring I)/I;
      d := dim R;
      N := pincare R;
      t := first gens ring N;
      while 0 == substitute(N, t => 1) do N = N // (1-t);
      h := apply(reverse toList(0..d), i -> N_(t^i));
      f := j -> sum(0..j+1, i -> binomial(d-i, j+1-i)*h#(d-i));
      apply(toList(0..d-1), j -> f(j));
```

For example, we can demonstrate that the f -vector of the octahedron is $(6, 12, 8)$.

```
i12 : S = QQ[x_1 .. x_6];
i13 : octahedron = monomialIdeal(x_1*x_2, x_3*x_4, x_5*x_6)
o13 = monomialIdeal (x x , x x , x x )
           1 2   3 4   5 6
o13 : MonomialIdeal of S
i14 : fvector octahedron
o14 = {6, 12, 8}
o14 : List
```

More generally, we can recursively construct simplicial 2-spheres with $f_0 \geq 4$, starting with the tetrahedron, by pulling a point in the relative interior of a facet. This procedure leads to the following family:

```
i15 : simplicial2sphere = v -> (
      S := QQ[x_1..x_v];
      if v == 4 then monomialIdeal product gens S
      else (
          L := {};
          scan(1..v-4, i -> L = L | apply(v-i-3,
              j -> x_i*x_(i+j+4)));
          scan(2..v-3, i -> L = L | {x_i*x_(i+1)*x_(i+2)});
          monomialIdeal L));
i16 : apply({4,5,6,7,8}, j -> fvector simplicial2sphere(j))
o16 = {{4, 6, 4}, {5, 9, 6}, {6, 12, 8}, {7, 15, 10}, {8, 18, 12}}
o16 : List
```

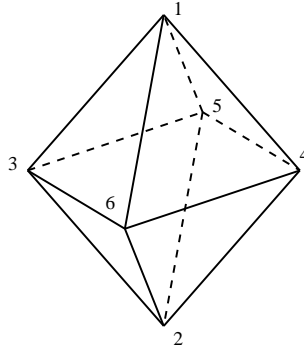


Fig. 1. The octahedron

In fact, it follows from Euler's formula that the f -vector of any simplicial 2-sphere has the form $(v, 3v - 6, 2v - 4)$ for $v \geq 4$. The problem of characterizing the f -vectors for triangulations of d -spheres is open for $d \geq 3$. One of the most important results in this direction is the upper bound theorem for simplicial spheres (Corollary 5.4.7 Bruns and Herzog [6]) which states that the cyclic polytope has the maximal number of i -faces for all i . We point out that Stanley's proof of this theorem depends heavily on these methods from commutative algebra.

On the other hand, the f -vectors for several major classes of simplicial complexes have been characterized. The Kruskal-Katona theorem (Theorem 8.32 in Ziegler [29]) gives necessary and sufficient conditions for a sequence of nonnegative integers to be an f -vector of a simplicial complex. Stanley [24] describes the f -vectors of pure shellable complexes and Cohen-Macaulay complexes. Given the Betti numbers of a simplicial complex, Björner and Kalai [5] specify the f -vectors. Finally, the g -theorem (Theorem 8.35 in Ziegler [29]) characterizes the f -vectors for boundary complexes of a simplicial convex polytope.

For a further study of Stanley-Reisner ideals see Bruns and Herzog [6] and Stanley [24]. For more information of f -vectors, see Ziegler [29] and Björner [4].

2 Primary Decomposition

A primary decomposition of an ideal I is an expression of I as a finite intersection of primary ideals; an ideal J is called primary if $r_1 r_2 \in J$ implies either $r_1 \in J$ or $r_2^\ell \in J$ for some $\ell > 0$. Providing an algorithm for computing the primary decomposition of an arbitrary ideal in a polynomial ring is quite difficult. However, for monomial ideals, there are two algorithms which are relatively simple to describe.

We first present a recursive method for generating an irreducible primary decomposition. It is based on the following two observations.

Lemma 2.1. *Let I be a monomial ideal in $S = \mathbb{Q}[x_1, \dots, x_n]$.*

- (1) *If I is generated by pure powers of a subset of the variables, then it is a primary ideal.*
- (2) *If r is minimal generator of I such that $r = r_1 r_2$ where r_1 and r_2 are relatively prime, then $I = (I + \langle r_1 \rangle) \cap (I + \langle r_2 \rangle)$.*

Proof. (1) This follows immediately from the definition of primary. (2) Since I is a monomial ideal, it is enough to show that I and $(I + \langle r_1 \rangle) \cap (I + \langle r_2 \rangle)$ contain the same monomials. A monomial r' belongs to $(I + \langle r_j \rangle)$ if and only if $r' \in I$ or r_j divides r' . Because r_1 and r_2 are relative prime, we have

$$r' \in (I + \langle r_1 \rangle) \cap (I + \langle r_2 \rangle) \Leftrightarrow r' \in I \text{ or } r_1 r_2 \text{ divides } r' \Leftrightarrow r' \in I. \quad \square$$

The following is an implementation of the resulting algorithm:

```
i17 : supp = r -> select(gens ring r, e -> r % e == 0);
i18 : monomialDecompose = method();
i19 : monomialDecompose List := L -> (
    P := select(L, I -> all(first entries gens I,
        r -> #supp(r) < 2) == false);
    if #P > 0 then (
        I := first P;
        m := first select(first entries gens I,
            r -> #supp(r) > 1);
        E := first exponents m;
        i := position(E, e -> e != 0);
        r1 := product apply(E_{0..i}, (gens ring I)_{0..i},
            (j, r) -> r^j);
        r2 := m // r1;
        monomialDecompose(delete(I, L) | {I+monomialIdeal(r1),
            I+monomialIdeal(r2)})
    else L);
i20 : monomialDecompose MonomialIdeal := I -> monomialDecompose {I};
```

Here is a small example illustrating this method.

```
i21 : S = QQ[a,b,c,d];
i22 : I = monomialIdeal(a^3*b, a^3*c, a*b^3, b^3*c, a*c^3, b*c^3)
o22 = monomialIdeal (a^3 b, a^3 c, a b^3, b^3 c, a c^3, b c^3)
o22 : MonomialIdeal of S
i23 : P = monomialDecompose I;
i24 : scan(P, J -> << endl << J << endl);
monomialIdeal (b, c)
monomialIdeal (a, c)
```

```

monomialIdeal (a3, b3, c3)
monomialIdeal (a, b)
monomialIdeal (a3, b3, c3)
monomialIdeal (a, b)
monomialIdeal (a3, b3, c3)
i25 : I == intersect(P)
o25 = true

```

As we see from this example, this procedure doesn't necessarily yield an irredundant decomposition.

The second algorithm for finding a primary decomposition of a monomial ideal I is based on the Alexander dual of I . The Alexander dual was first introduced for squarefree monomial ideals. In this case, it is the monomial ideal of the dual of the simplicial complex Δ corresponding to I . By definition the dual complex of Δ is $\Delta^\vee = \{F : F^c \notin \Delta\}$, where $F^c = \{x_1, \dots, x_n\} \setminus F$. The following general definition appears in Miller [16], [17]. If $I \subseteq \mathbb{Q}[x_1, \dots, x_n]$ is a monomial ideal and x^λ is the least common multiple of the minimal generators of I , then the Alexander dual of I is

$$I^\vee = \left\langle \prod_{\beta_i > 0} x_i^{\lambda_i + 1 - \beta_i} : \langle x_i^{\beta_i} : \beta_i \geq 1 \rangle \text{ is an irredundant irreducible component of } I \right\rangle.$$

In particular, the minimal generators of I^\vee correspond to the irredundant irreducible components of I . The next proposition provides a useful way of computing I^\vee given a set of generators for I .

Proposition 2.2. *If I is a monomial ideal and x^λ is the least common multiple of the minimal generators of I , then the generators for I^\vee are those generators of the ideal $(\langle x_1^{\lambda_1+1}, \dots, x_n^{\lambda_n+1} \rangle : I)$ that are not divisible by $x_i^{\lambda_i+1}$ for $1 \leq i \leq n$.*

Proof. See Theorem 2.1 in Miller [16]. □

Miller's definition of Alexander dual is even more general than the one above. The resulting algorithm for computing this general Alexander dual and primary decomposition are implemented in *Macaulay 2* as follows. For the Alexander dual we use, the list **a** that appears as an input argument for **dual** should be list of exponents of the least common multiple of the minimal generators of I .

```

i26 : code(dual, MonomialIdeal, List)

```

```

o26 = -- ../../m2/monideal.m2:260-278
      dual(MonomialIdeal, List) := (I,a) -> ( -- Alexander dual
        R := ring I;
        X := gens R;
        aI := lcmOfGens I;
        if aI != a then (
          if #aI != #a
          then error (
            "expected list of length ",
            toString (#aI));
          scan(a, aI,
            (b,c) -> (
              if b<c then
                error "exponent vector not large enough"
            ));
        );
        S := R/(I + monomialIdeal apply(#X, i -> X#i^(a#i+1)));
        monomialIdeal contract(
          lift(syz transpose vars S, R),
          product(#X, i -> X#i^(a#i))))

i27 : code(primaryDecomposition, MonomialIdeal)

o27 = -- ../../m2/monideal.m2:286-295
      primaryDecomposition MonomialIdeal := (I) -> (
        R := ring I;
        aI := lcmOfGens I;
        M := first entries gens dual I;
        L := unique apply(#M, i -> first exponents M_i);
        apply(L, i -> monomialIdeal apply(#i, j -> (
          if i#j == 0 then 0_R
          else R_j^(aI#j+1-i#j)
        )))
      )

```

This direct algorithm is more efficient than our recursive algorithm. In particular, it gives an irredundant decomposition. For example, when we use it to determine a primary decomposition for the ideal I above, we obtain

```

i28 : L = primaryDecomposition I;

i29 : scan(L, J -> << endl << J << endl);

      3 3 3
monomialIdeal (a , b , c )

monomialIdeal (b, c)

monomialIdeal (a, b)

monomialIdeal (a, c)

i30 : I == intersect L

o30 = true

```

For a family of larger examples, we consider the tree ideals:

$$\left\langle \left(\prod_{i \in F} x_i \right)^{n-|F|+1} : \emptyset \neq F \subseteq \{x_1, \dots, x_n\} \right\rangle.$$

These ideals are so named because their standard monomials (the monomials not in the ideal) correspond to trees on $n + 1$ labeled vertices. We determine the number of irredundant irreducible components as follows:

```
i31 : treeIdeal = n -> (
      S = QQ[vars(0..n-1)];
      L := delete({}, subsets gens S);
      monomialIdeal apply(L, F -> (product F)^(n - #F + 1)));

i32 : apply(2..6, i -> #primaryDecomposition treeIdeal i)

o32 = (2, 6, 24, 120, 720)

o32 : Sequence
```

Example: Graph Ideals and Complexity Theory

Monomial ideals also arise in graph theory. Given a graph G with vertices $\{x_1, \dots, x_n\}$, we associate the ideal I_G in $\mathbb{Q}[x_1, \dots, x_n]$ generated by the quadratic monomials $x_i x_j$ such that x_i is adjacent to x_j . The primary decomposition of I_G is related to the graph G as follows. Recall that a subset $F \subseteq \{x_1, \dots, x_n\}$ is called a *vertex cover* of G if each edge in G is incident to at least one vertex in F .

Lemma 2.3. *If G is a graph and \mathcal{C} is the set of minimal vertex covers of G then the irreducible irredundant primary decomposition of I_G is $\bigcap_{F \in \mathcal{C}} P_{F^c}$, where P_{F^c} is the prime ideal $\langle x_i : x_i \notin F^c \rangle = \langle x_i : x_i \in F \rangle$.*

Proof. Since each generator of I_G corresponds to an edge in G , it follows from the `monomialDecompose` algorithm that I_G has an irreducible primary decomposition of the form: $I_G = \bigcap P_{F^c}$, where F is a vertex cover. To obtain an irredundant decomposition, one clearly needs only the minimal vertex covers. □

As an application of graph ideals, we examine the complexity of determining the codimension of a monomial ideal. In fact, following Bayer and Stillman [3], we prove

Proposition 2.4. *The following decision problem is NP-complete:*

$$\begin{array}{l} \text{Given a monomial ideal } I \subseteq \mathbb{Q}[x_1, \dots, x_n] \text{ and} \\ m \in \mathbb{N}, \text{ is } \text{codim } I \leq m? \end{array} \quad (\text{CODIM})$$

By definition, a decision problem is NP-complete if all other problems in the class NP can be reduced to it. To prove that a particular problem is NP-complete, it suffices to show: (1) the problem belongs to the class NP; (2) some known NP-complete problem reduces to the given decision problem (see Lemma 2.3 in Garey and Johnson [8]). One of the “standard NP-complete” problems (see section 3.1 in Garey and Johnson [8]) is the following:

$$\begin{array}{l} \text{Given a graph } G \text{ and } m \in \mathbb{N}, \text{ is there a vertex} \\ \text{cover } F \text{ such that } |F| \leq m? \end{array} \quad (\text{VERTEX COVER})$$

Proof of Proposition. (1) Observe that a monomial ideal I has codimension at most m if and only if $I \subseteq P_{F^c}$ for some F with $|F| \leq m$. Now, if I has codimension at most m , then given an appropriate choice of F , one can verify in polynomial time that $I \subseteq P_{F^c}$ and $|F| \leq m$. Therefore, the CODIM problem belongs to the class NP.

(2) Lemma 2.3 implies that I_G has codimension m if and only if G has a vertex cover of size at most m . In particular, the VERTEX COVER problem reduces to the CODIM problem. \square

Thus, assuming $P \neq NP$, there is no polynomial time algorithm for finding the codimension of a monomial ideal. Nevertheless, we can effectively compute the codimension for many interesting examples.

To illustrate this point, we consider the following family of examples. Let $S = \mathbb{Q}[X]$ denote the polynomial ring generated by the entries of a generic $m \times n$ matrix $X = [x_{i,j}]$. Let I_k be the ideal generated by the $k \times k$ minors of X . Since the Hilbert function of S/I_k equals the Hilbert function of $S/\text{in}(I_k)$ (see Theorem 15.26 in Eisenbud [7]), we can determine the codimension I_k by working with the monomial ideal $\text{in}(I_k)$. Because Sturmfels [25] shows that the set of $k \times k$ -minors of X is the reduced Gröbner basis of I_k with respect to the lexicographic term order induced from the variable order

$$x_{1,n} > x_{1,n-1} > \cdots > x_{1,1} > x_{2,n} > \cdots > x_{2,1} > \cdots > x_{m,n} > \cdots > x_{m,1},$$

we can easily calculate $\text{in}(I_k)$. In particular, in *Macaulay 2* we have

```
i33 : minorsIdeal = (m,n,k) -> (
      S := QQ[x_1..x_(m*n), MonomialOrder => Lex];
      I := minors(k, matrix table(m, n, (i,j) -> x_(i*n+n-j)));
      forceGB gens I;
      I);

i34 : apply(2..8, i -> time codim monomialIdeal minorsIdeal(i,2*i,2))
-- used 0.02 seconds
-- used 0.05 seconds
-- used 0.1 seconds
-- used 0.36 seconds
-- used 1.41 seconds
-- used 5.94 seconds
-- used 25.51 seconds

o34 = (3, 10, 21, 36, 55, 78, 105)

o34 : Sequence
```

The properties of I_k are further developed in chapter 11 of Sturmfels [26] and chapter 7 of Bruns and Herzog [6]

For more on the relationships between a graph and its associated ideal, see Villarreal [28], Simis, Vasconcelos and Villarreal [23], and Ohsugi and Hibi [19].

```
i35 : erase symbol x;
```

3 Standard Pairs

In this section, we examine a combinatorial object associated to a monomial ideal. In particular, we present two algorithms for computing the standard pairs of a monomial ideal from its minimal generators. Before giving the definition of a standard pair, we consider an example.

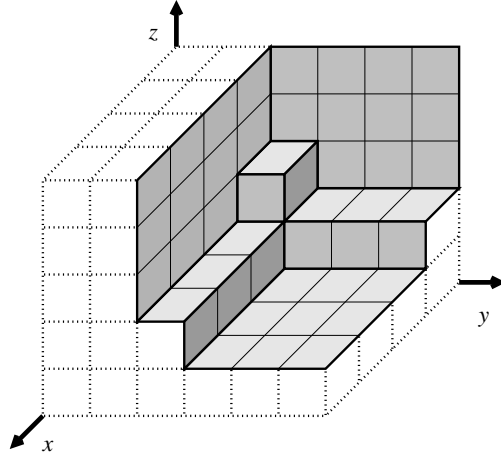


Fig. 2. Staircase diagram for $I = \langle xy^3z, xy^2z^2, y^3z^2, y^2z^3 \rangle$

Example 3.1. Let $I = \langle xy^3z, xy^2z^2, y^3z^2, y^2z^3 \rangle$ in $\mathbb{Q}[x, y, z]$. We identify the monomials in $\mathbb{Q}[x, y, z]$ with the lattice points in \mathbb{N}^3 ; see Figure 2. The standard monomials of I , those monomials which are not in I , can be enumerated as follows: (i) monomials corresponding to lattice points in the xy -plane, (ii) monomials corresponding to lattice points in the xz -plane, (iii) monomials corresponding to lattice points in the plane parallel to the xz -plane containing $(0, 1, 0)$, (iv) monomials corresponding to lattice points on the line parallel to the y -axis containing $(0, 0, 1)$, (v) monomials corresponding to lattice point on the line parallel to the x -axis containing $(0, 2, 1)$, and (vi) the monomial y^2z^2 .

Following Sturmfels, Trung and Vogel [27], we make the following definitions. Given a monomial x^α and a subset $F \subseteq \{x_1, \dots, x_n\}$, we index the set of monomials of the form $x^\alpha \cdot x^\beta$ where $\text{supp}(x^\beta) \subseteq F$ by the pair (x^α, F) . A standard pair of a monomial ideal I is a pair (x^α, F) satisfying the following three conditions:

- (1) $\text{supp}(x^\alpha) \cap F = \emptyset$,

- (2) all of the monomials represented by this pair are standard, and
(3) $(x^\alpha, F) \not\subseteq (x^\beta, G)$ for any other pair (x^β, G) satisfying the first two conditions.

Hence, the six standard pairs

$$(1, \{x, y\}), (1, \{x, z\}), (y, \{x, z\}), (z, \{y\}), (y^2z, \{x\}), (y^2z^2, \emptyset)$$

in Example 3.1 correspond to (i)–(vi).

Observe that the set of standard pairs of I gives an irreducible decomposition of $I = \bigcap \langle x_i^{\alpha_i+1} : x_i \notin F \rangle$, where the intersection is over all standard pairs (x^α, F) . Moreover, the prime ideal $P_F := \langle x_i : x_i \notin F \rangle$ is an associated prime of I if and only if there exists a standard pair of the form (\bullet, F) ; see Sturmfels, Trung and Vogel [27] for details.

Our first algorithm for computing the set of standard pairs is taken from Hoşten and Thomas [15]. The ideas behind it are as follows: given a witness $w_1 = x^\alpha$ for the associated prime $P_F := \langle x_i : x_i \notin F \rangle$, that is $(I : x^\alpha) = P_F$, set $w_2 = \prod_{x_i \in \text{supp}(w_1) \cap F^c} x_i^{\alpha_i}$. It follows that (w_2, F) is a standard pair of I . Now, consider the standard pairs of the slightly larger ideal $I + \langle w_1 \rangle$. Clearly (w_2, F) is not a standard pair of this ideal because w_1 “destroys” it. This larger ideal might have standard pairs which cover standard monomials in (w_2, F) that are not in the pair (w_1, F) . However, all other standard pairs are the same as the original ideal I . Thus, the problem of finding all standard pairs of I reduces to determining if a standard pair of $I + \langle w_1 \rangle$ is a standard pair for I . To decide if a pair (x^β, G) of $I + \langle w_1 \rangle$ is a standard pair of I , we first check that P_F is an associated prime of I . If this is true, we determine if (w_2, F) is covered by (x^β, G) .

The *Macaulay 2* version of this algorithm takes the following form:

```
i36 : stdPairs = I -> (
  S := ring I;
  X := gens S;
  std := {};
  J := I;
  while J != S do (
    w1 := 1_S;
    F := X;
    K := J;
    while K != 0 do (
      g1 := (ideal mingens ideal K)_0;
      x := first supp g1;
      w1 = w1 * g1 // x;
      F = delete(x, F);
      K = K : monomialIdeal(g1 // x);
      L := select(first entries gens K,
        r -> not member(x, supp r));
      if #L > 0 then K = monomialIdeal L
      else K = monomialIdeal 0_S;);
    w2 := w1;
    scan(X, r -> if not member(r, supp w1) or member(r, F)
      then w2 = substitute(w2, {r => 1}));
    P := monomialIdeal select(X, r -> not member(r, F));
```

```

if (I:(I:P) == P) and (all(std, p ->
    (w2 % (first p) != 0) or not
    isSubset(supp(w2 // first p) | F, last p)))
then std = std | {{w2, F}};
J = J + monomialIdeal(w1););
std);

```

We can compute the standard pairs of Example 3.1 using this *Macaulay 2* function:

```

i37 : S = QQ[x,y,z];
i38 : I = monomialIdeal(x*y^3*z, x*y^2*z^2, y^3*z^2, y^2*z^3);
o38 : MonomialIdeal of S
i39 : scan(time stdPairs I, P -> << endl << P << endl);
-- used 0.66 seconds
{y, {x, z}}
{1, {x, z}}
  2 2
{y z , {}}
{z, {y}}
  2
{y z, {x}}
{1, {x, y}}

```

Our second algorithm is taken from section 3.2 of Saito, Sturmfels and Takayama [22]. The proposition below provides the main ingredient for this algorithm. If I is a monomial ideal and $F \subseteq \{x_1, \dots, x_n\}$, we write I_F for the monomial ideal in $\mathbb{Q}[x_i : x_i \notin F]$ obtained by replacing each $x_i \in F$ with 1 in every minimal generator of I .

Proposition 3.2. *For (x^α, F) to be a standard pair of I , it is necessary and sufficient that (x^α, \emptyset) be a standard pair of I_F .*

Proof. Lemma 3.1 in Sturmfels, Trung and Vogel [27].

The definition of a standard pair implies that (x^α, \emptyset) is a standard pair of I_F if and only if x^α is one of the finitely many monomials contained in $(I_F : P_F^\infty)$ but not contained in I_F , where $P_F = \langle x_i : x_i \notin F \rangle$. Since ideal quotients and saturations are implemented in *Macaulay 2*, this reduces the problem to finding a set D which contains F for every associated prime P_F of I . One approach is to simply compute the associated primes of I from a primary decomposition.

The method `standardPairs` uses this algorithm to determine the set of standard pairs for a monomial ideal.

```

i40 : code(standardPairs, MonomialIdeal, List)

```

```

o40 = -- ../../m2/monideal.m2:318-341
      standardPairs(MonomialIdeal, List) := (I,D) -> (
        R := ring I;
        X := gens R;
        S := {};
        k := coefficientRing R;
        scan(D, L -> (
          Y := X;
          m := vars R;
          Lset := set L;
          Y = select(Y, r -> not Lset#?r);
          m = substitute(m, apply(L, r -> r => 1));
          -- using monoid to create ring to avoid
          -- changing global ring.
          A := k (monoid [Y]);
          phi := map(A, R, substitute(m, A));
          J := ideal mingens ideal phi gens I;
          Jsat := saturate(J, ideal vars A);
          if Jsat != J then (
            B := flatten entries super basis (
              trim (Jsat / J));
            psi := map(R, A, matrix{Y});
            S = join(S, apply(B, b -> {psi(b), L}));
          ));
        S)

i41 : time standardPairs I;
      -- used 0.83 seconds

```

As an example, we will compute the standard pairs of the permutahedron ideal. Let $S = \mathbb{Q}[x_1, \dots, x_n]$ and let \mathfrak{S}_n be the symmetric group of order n . We write ρ for the vector $(1, 2, \dots, n)$ and $\sigma(\rho)$ for the vector obtained by applying $\sigma \in \mathfrak{S}_n$ to the coordinates of ρ . The n -th permutahedron ideal is $\langle x^{\sigma(\rho)} : \sigma \in \mathfrak{S}_n \rangle$. We compute the number of standard pairs for $2 \leq n \leq 5$.

```

i42 : permutohedronIdeal = n -> (
      S := QQ[X_1..X_n];
      monomialIdeal terms det matrix table(n ,gens S,
        (i,r) -> r^(i+1));

i43 : L = apply({2,3,4,5}, j -> standardPairs(permutohedronIdeal(j)));

i44 : apply(L, i -> #i)

o44 = {3, 10, 53, 446}

o44 : List

i45 : erase symbol x; erase symbol z;

```

Example: Integer Programming Problems

As an application of standard pairs, we show how to solve integer linear programming problems. Let A be a $d \times n$ matrix of nonnegative integers, let $\omega \in \mathbb{R}^n$ and fix $\beta \in \mathbb{Z}^d$. We focus on the following optimization problem

$$\text{IP}_{A,\omega}(\beta) : \text{ minimize } \omega \cdot \alpha \text{ subject to } A\alpha = \beta, \alpha \in \mathbb{N}^n.$$

We view this integer linear program as a family depending on the vector β . The algorithm we present for solving $\text{IP}_{A,\omega}(\beta)$ depends on the proposition below.

The toric ideal $I_A \subseteq S = \mathbb{Q}[x_1, \dots, x_n]$ associated to A is the binomial ideal generated by $x^\gamma - x^\delta$ where $\gamma, \delta \in \mathbb{N}^n$ and $A\gamma = A\delta$. We write $\text{in}_\omega(I_A)$ for the initial ideal of I_A with respect to the following order:

$$x^\gamma \prec_\omega x^\delta \iff \begin{cases} \omega \cdot \gamma < \omega \cdot \delta & \text{or} \\ \omega \cdot \gamma = \omega \cdot \delta & \text{and } x^\alpha \prec_{\text{rlex}} x^\gamma. \end{cases}$$

For more information on toric ideals and their initial ideals see Sturmfels [26].

- Proposition 3.3.** (1) *A monomial x^α is a standard monomial of $\text{in}_\omega(I_A)$ if and only if α is the optimal solution to the integer program $\text{IP}_{A,\omega}(A\alpha)$.*
 (2) *If (\bullet, F) is a standard pair of $\text{in}_\omega(I_A)$, then the columns of A corresponding to F are linearly independent.*

Proof. See Proposition 2.1 in Hoşten and Thomas [14] for the proof of the first statement. The second statement follows from Corollary 2.9 of the same article. \square

The first statement implies that the standard pairs of $\text{in}_\omega(I_A)$ cover all optimal solutions to all integer programs in $\text{IP}_{A,\omega}$. If α is the optimal solution to $\text{IP}_{A,\omega}(\beta)$ covered by the standard pair (x^γ, F) , then the second statement guarantees there exists a unique $\delta \in \mathbb{N}^n$ such that $A\delta = \beta - A\gamma$. Therefore, $\alpha = \delta + \gamma$. We point out that the complexity of this algorithm is dominated by determining the set of standard pairs of $\text{in}_\omega(I_A)$ which depends only on A and ω . As a result, this method is particularly well suited to solving $\text{IP}_{A,\omega}(\beta)$ as β varies.

To implement this algorithm in *Macaulay 2*, we need a function which returns the toric ideal I_A . Following Algorithm 12.3 in Sturmfels [26], we have

```
i47 : toBinomial = (b, S) -> (
      pos := 1_S;
      neg := 1_S;
      scan(#b, i -> if b_i > 0 then pos = pos*S_i^(b_i)
                  else if b_i < 0 then neg = neg*S_i^(-b_i));
      pos - neg);

i48 : toricIdeal = (A, omega) -> (
      n := rank source A;
      S = QQ[x_1..x_n, Weights => omega, MonomialSize => 16];
      B := transpose matrix syz A;
      J := ideal apply(entries B, b -> toBinomial(b, S));
      scan(gens S, r -> J = saturate(J, r));
      J);
```

Thus, we can solve $\text{IP}_{A,\omega}(\beta)$ using the following function.

```

i49 : IP = (A, omega, beta) -> (
  std := standardPairs monomialIdeal toricIdeal(A, omega);
  n := rank source A;
  alpha := {};
  Q := first select(1, std, P -> (
    F := apply(last P, r -> index r);
    gamma := transpose matrix exponents first P;
    K := transpose syz (submatrix(A,F) | (A*gamma-beta));
    X := select(entries K, k -> abs last(k) === 1);
    scan(X, k -> if all(k, j -> j>=0) or all(k, j -> j<=0)
      then alpha = apply(n, j -> if member(j, F)
        then last(k)*k_(position(F, i -> i === j))
        else 0));
    #alpha > 0));
  if #Q > 0 then (matrix {alpha})+(matrix exponents first Q)
  else 0);

```

We illustrate this with some examples.

```

i50 : A = matrix{{1,1,1,1,1},{1,2,4,5,6}}

```

```

o50 = | 1 1 1 1 1 |
      | 1 2 4 5 6 |

```

```

o50 : Matrix ZZ  $\leftarrow$  ZZ

```

```

i51 : w1 = {1,1,1,1,1};

```

```

i52 : w2 = {2,3,5,7,11};

```

```

i53 : b1 = transpose matrix{{3,9}}

```

```

o53 = | 3 |
      | 9 |

```

```

o53 : Matrix ZZ  $\leftarrow$  ZZ

```

```

i54 : b2 = transpose matrix{{5,16}}

```

```

o54 = | 5 |
      | 16 |

```

```

o54 : Matrix ZZ  $\leftarrow$  ZZ

```

```

i55 : IP(A, w1, b1)

```

```

o55 = | 1 1 0 0 1 |

```

```

o55 : Matrix ZZ  $\leftarrow$  ZZ

```

```

i56 : IP(A, w2, b1)

```

```

o56 = | 1 0 2 0 0 |

```

```

o56 : Matrix ZZ  $\leftarrow$  ZZ

```

```

i57 : IP(A, w1, b2)

```

```

o57 = | 2 1 0 0 2 |

```



```

o57 : Matrix ZZ <--- ZZ
      1      5
i58 : IP(A, w2, b2)
o58 = | 2 0 1 2 0 |
      1      5
o58 : Matrix ZZ <--- ZZ

```

4 Generic Initial Ideals

Gröbner basis calculations and initial ideals depend heavily on the given coordinate system. By making a generic change of coordinates before taking the initial ideal, we may eliminate this dependence. This procedure also endows the resulting monomial ideal with a rich combinatorial structure.

To describe this structure, we introduce the following definitions and notation. Let $S = \mathbb{Q}[x_0, \dots, x_n]$. If $\mathbf{g} = [g_{i,j}] \in \text{GL}_{n+1}(\mathbb{Q})$ and $f \in S$, then $\mathbf{g} \cdot f$ denotes the standard action of the general linear group on S : $x_i \mapsto \sum_{j=0}^n g_{i,j} x_j$. For an ideal $I \subseteq S$, we define $\mathbf{g} \cdot I = \{\mathbf{g} \cdot f \mid f \in I\}$. Let B denote the Borel subgroup of $\text{GL}_{n+1}(\mathbb{Q})$ consisting of upper triangular matrices. A monomial ideal I is called *Borel-fixed* if it satisfies any of the following equivalent conditions:

- (a) $\mathbf{g} \cdot I = I$ for every $\mathbf{g} \in B$;
- (b) if r is a generator of I divisible by x_j then $\frac{rx_i}{x_j} \in I$ for all $i < j$;
- (c) $\text{in}(\mathbf{g} \cdot I) = I$ for every \mathbf{g} in some open neighborhood of the identity in B .

For a proof that these conditions are equivalent, see Proposition 1.25 in Green [9].

In *Macaulay 2*, the function `isBorel` tests whether a monomial ideal is Borel-fixed.

```

i59 : S = QQ[a,b,c,d];
i60 : isBorel monomialIdeal(a^2, a*b, b^2)
o60 = true
i61 : isBorel monomialIdeal(a^2, b^2)
o61 = false

```

The function `borel` generates the smallest Borel-fixed ideal containing the given monomial ideal.

```

i62 : borel monomialIdeal(b*c)
      2      2
o62 = monomialIdeal (a , a*b, b , a*c, b*c)
o62 : MonomialIdeal of S
i63 : borel monomialIdeal(a,c^3)
      3      2      2      3
o63 = monomialIdeal (a, b , b c, b*c , c )
o63 : MonomialIdeal of S

```

The next theorem provides the main source of Borel-fixed ideals.

Theorem 4.1 (Galligo). *Fix a term order on $S = \mathbb{Q}[x_0, \dots, x_n]$ such that $x_0 > \dots > x_n$. If I is a homogeneous ideal in S , then there is a Zariski open subset $U \subseteq \mathrm{GL}_{n+1}(\mathbb{Q})$ such that*

- (1) *there is a monomial ideal $J \subseteq S$ such that $J = \mathrm{in}(\mathbf{g} \cdot I)$ for all $\mathbf{g} \in U$;*
- (2) *the ideal J is Borel-fixed.*

The ideal J is called the generic initial ideal of I .

Proof. See Theorem 1.27 in Green [9]. □

The following method allows one to compute generic initial ideals.

```
i64 : gin = method();
i65 : gin Ideal := I -> (
      S := ring I;
      StoS := map(S, S, random(S^{0}, S^{numgens S:-1}));
      monomialIdeal StoS I);
i66 : gin MonomialIdeal := I -> gin ideal I;
```

This routine assumes that the random function generates a matrix in the Zariski open subset U . Since we are working over a field of characteristic zero this occurs with probability one. For example, we can determine the generic initial ideal of two generic homogeneous polynomials of degree p and q in $\mathbb{Q}[a, b, c, d]$.

```
i67 : genericForms = (p,q) -> ideal(random(p,S), random(q,S));
i68 : gin genericForms(2,2)
      2      3
o68 = monomialIdeal (a , a*b, b )
o68 : MonomialIdeal of S
i69 : gin genericForms(2,3)
      2      2      4
o69 = monomialIdeal (a , a*b , b )
o69 : MonomialIdeal of S
```

Although the generic initial ideal is Borel-fixed, some non-generic initial ideals may also be Borel-Fixed.

```
i70 : J = ideal(a^2, a*b+b^2, a*c)
      2      2
o70 = ideal (a , a*b + b , a*c)
o70 : Ideal of S
i71 : ginJ = gin J
      2      2      2
o71 = monomialIdeal (a , a*b, b , a*c )
o71 : MonomialIdeal of S
```

```

i72 : inJ = monomialIdeal J
      2      3      2
o72 = monomialIdeal (a , a*b, b , a*c, b c)
o72 : MonomialIdeal of S
i73 : isBorel inJ and isBorel ginJ
o73 = true

```

Finally, we show that the generic initial ideal does depend on the term order by computing lexicographic generic initial ideal for two generic forms of degree p and q in $\mathbb{Q}[a, b, c, d]$

```

i74 : S = QQ[a,b,c,d, MonomialOrder => Lex];
i75 : gin genericForms(2,2)
      2      4      2
o75 = monomialIdeal (a , a*b, b , a*c )
o75 : MonomialIdeal of S
i76 : gin genericForms(2,3)
      2      2      6      2      6      2      4
o76 = monomialIdeal (a , a*b , b , a*b*c , a*c , a*b*c*d , a*b*d )
o76 : MonomialIdeal of S

```

A more comprehensive treatment of generic initial ideals can be found in Green [9]. The properties of Borel-fixed ideals in characteristic $p > 0$ are discussed in Eisenbud [7].

Example: Connectedness of the Hilbert Scheme

Generic initial ideals are a powerful tool for studying the structure of the Hilbert scheme. Intuitively, the Hilbert scheme $\text{Hilb}^{p(t)}(\mathbb{P}^n)$ parameterizes subschemes $X \subseteq \mathbb{P}^n$ with Hilbert polynomial $p(t)$. For an introduction to Hilbert schemes see Harris and Morrison [11]. The construction of the Hilbert scheme $\text{Hilb}^{p(t)}(\mathbb{P}^n)$ can be found in Grothendieck's original article [10] or Altman and Kleiman [1]. While much is known about specific Hilbert schemes, the general structure remain largely a mystery. In particular, the component structure — the number of irreducible components, their dimensions, how they intersect and what subschemes they parameterize — is not well understood.

Reeves [21] uses generic initial ideals to establish the most important theorem to date on the component structure. The incidence graph of $\text{Hilb}^{p(t)}(\mathbb{P}^n)$ is defined as follows: to each irreducible component we assign a vertex and we connect two vertices if the corresponding components intersect. Reeves [21] proves that the distance (the number of edges in the shortest path) between any two vertices in the incidence graph of $\text{Hilb}^{p(t)}(\mathbb{P}^n)$ is at most $2 \deg p(t) + 2$. Her proof can be divided into three major steps.

Step I: connect an arbitrary ideal to a Borel-fixed ideal. Passing to an initial ideal corresponds to taking the limit in a flat family, in other words a path on the Hilbert scheme; see Theorem 15.17 in Eisenbud [7]. Thus, Theorem 4.1 shows that generic initial ideals connect arbitrary ideals to Borel-fixed ideals.

Step II: connect Borel-fixed ideals by projection. For a homogeneous ideal $I \subseteq S = \mathbb{Q}[x_0, \dots, x_n]$, let $\pi(I)$ denote the ideal obtained by setting $x_n = 1$ and $x_{n-1} = 1$ in I . With this notation, we have

Theorem 4.2. *If J is a Borel-fixed ideal, then the set of Borel-fixed ideals I , with Hilbert polynomial $p(t)$ and $\pi(I) = J$, consists of ideals defining subschemes of \mathbb{P}^n which all lie on a single component of $\text{Hilb}^{p(t)}(\mathbb{P}^n)$.*

Proof. See Theorem 6 in Reeves [21]. □

This gives an easy method for partitioning Borel-fixed ideals into classes, each of which must lie in a single component.

Step III: connect Borel-fixed ideals by distraction. Given a Borel-fixed ideal, we produce a new ideal via a two-step process called distraction. First, one polarizes the Borel-fixed ideal. The polarization of a monomial ideal $I \subset S$ is defined as:

$$\left\langle \prod_{i=0}^n \prod_{j=1}^{\alpha_i} z_{i,j} : \text{where } x_0^{\alpha_0} \cdots x_n^{\alpha_n} \text{ is a minimal generator of } I \right\rangle.$$

One then pulls the result back to an ideal in the original variables by taking a linear section of the polarization. Theorem 4.10 in Hartshorne [12] shows that the distraction is connected to the original Borel-fixed ideal. Now, taking the lexicographic generic initial ideal of the distraction yields a second Borel-fixed ideal. Reeves [21] proves that repeating this process, at most $\deg p(t) + 1$ times, one arrives at a distinguished component of $\text{Hilb}^{p(t)}(\mathbb{P}^n)$ called the lexicographic component. For more information on the lexicographic component see Reeves and Stillman [20].

We can implement these operations in *Macaulay 2* as follows:

```
i77 : projection = I -> (
      S := ring I;
      n := numgens S;
      X := gens S;
      monomialIdeal mingens substitute(ideal I,
        {X#(n-2) => 1, X#(n-1) => 1});

i78 : polarization = I -> (
      n := numgens ring I;
      u := apply(numgens I, i -> first exponents I_i);
      l.lcm = max \ transpose u;
      Z := flatten apply(n, i -> apply(l.lcm#i, j -> z_{i,j}));
      R := QQ(monoid[Z]);
      Z = gens R;
      p := apply(n, i -> sum((l.lcm)_{0..i-1}));
      monomialIdeal apply(u, e -> product apply(n, i ->
        product(toList(0..e#i-1), j -> Z#(p#i+j)))));
```

```
i79 : distraction = I -> (
      S := ring I;
      n := numgens S;
      X := gens S;
      J := polarization I;
      W := flatten apply(n, i -> flatten apply(I.lcm#i,
          j -> X#i));
      section := map(S, ring J, apply(W, r -> r -
          random(500)*X#(n-2) - random(500)*X#(n-1)));
      section ideal J);
```

For example, we have

```
i80 : S = QQ[x_0 .. x_4, MonomialOrder => GLex];
i81 : I = monomialIdeal(x_0^2, x_0*x_1^2*x_3, x_1^3*x_4)

          2      2      3
o81 = monomialIdeal (x , x x x , x x )
          0      0 1 3   1 4

o81 : MonomialIdeal of S
i82 : projection I

          2      2      3
o82 = monomialIdeal (x , x x , x )
          0      0 1   1

o82 : MonomialIdeal of S
i83 : polarization I
o83 = monomialIdeal (z      z      , z      z      z      z      , z      ...
          {0, 0} {0, 1}   {0, 0} {1, 0} {1, 1} {3, 0}   {1 ...
o83 : MonomialIdeal of QQ [z      , z      , z      , z      , z      ...
          {0, 0}   {0, 1}   {1, 0}   {1, 1}   {1, 2} ...

i84 : distraction I

          2              2              2      ...
o84 = ideal (x  - 398x x  - 584x x  + 36001x  + 92816x x  + 47239x , - ...
          0          0 3          0 4          3          3 4          4      ...

o84 : Ideal of S
```

To illustrate Reeves' method, we show that the incidence graph of the Hilbert scheme $\text{Hilb}^{4t+1}(\mathbb{P}^4)$ has diameter at most 2. Note that the rational quartic curve in \mathbb{P}^4 has Hilbert polynomial $4t + 1$.

```
i85 : m = matrix table({0,1,2}, {0,1,2}, (i,j) -> (gens S)#(i+j))

o85 = | x_0 x_1 x_2 |
      | x_1 x_2 x_3 |
      | x_2 x_3 x_4 |

          3      3
o85 : Matrix S <--- S

i86 : rationalQuartic = minors(2, m);

o86 : Ideal of S

i87 : H = hilbertPolynomial(S/rationalQuartic);
```

```
i88 : hilbertPolynomial(S/rationalQuartic, Projective => false)
```

```
o88 = 4i + 1
```

```
o88 : QQ [i]
```

There are 12 Borel-fixed ideals with Hilbert polynomial $4t + 1$; see Example 1 in Reeves [21].

```
i89 : L = {monomialIdeal(x_0^2, x_0*x_1, x_0*x_2, x_1^2, x_1*x_2, x_2^2 ...
```

```
i90 : scan(#L, i -> << endl << i+1 << " : " << L#i << endl);
```

```
1 : monomialIdeal (x2, x2, x2, x2, x2, x2)
0 0 1 1 0 2 1 2 2
```

```
2 : monomialIdeal (x2, x2, x3, x2, x2, x2, x3, x3)
0 0 1 1 0 2 1 2 2 0 3
```

```
3 : monomialIdeal (x2, x2, x3, x2, x3, x3)
0 1 1 2 2 1 2 3
```

```
4 : monomialIdeal (x2, x4, x3, x2, x3)
0 1 1 2 2 2 3
```

```
5 : monomialIdeal (x5, x4, x3, x3)
0 1 2 2 3
```

```
6 : monomialIdeal (x2, x5, x4, x2, x2, x2)
0 1 1 2 2 1 3 2 3
```

```
7 : monomialIdeal (x2, x2, x5, x4, x4, x4, x4)
0 0 1 1 0 2 1 2 2 0 3 1 3 2 3
```

```
8 : monomialIdeal (x2, x5, x4, x2, x2, x2)
0 1 1 2 2 2 3 1 3
```

```
9 : monomialIdeal (x2, x2, x4, x2, x2, x2)
0 0 1 1 0 2 1 2 2 0 3 1 3
```

```
10 : monomialIdeal (x2, x2, x4, x2, x2, x2)
0 1 1 2 2 1 2 3 1 3
```

```
11 : monomialIdeal (x2, x4, x3, x2, x3)
0 1 1 2 2 1 3
```

```
12 : monomialIdeal (x6, x5, x4, x2, x2)
0 1 2 2 3 2 3
```

```
i91 : all(L, I -> isBorel I and hilbertPolynomial(S/I) == H)
```

```
o91 = true
```

The projection operation partitions the list L into 3 classes:

```

i92 : class1 = projection L#0

o92 = monomialIdeal (x2, x2x, x2, x2x, x2x, x2)
          0 0 1 1 0 2 1 2 2

o92 : MonomialIdeal of S

i93 : class2 = projection L#1

o93 = monomialIdeal (x2, x, x3x, x)
          0 1 1 2 2

o93 : MonomialIdeal of S

i94 : class3 = projection L#4

o94 = monomialIdeal (x4, x, x)
          0 1 2

o94 : MonomialIdeal of S

i95 : all(1..3, i -> projection L#i == class2)

o95 = true

i96 : all(4..11, i -> projection L#i == class3)

o96 = true

```

Finally, we use the distraction to connect the classes.

```

i97 : all(L, I -> I == monomialIdeal distraction I)

o97 = true

i98 : all(0..3, i -> projection gin distraction L#i == class3)

o98 = true

```

Therefore, the components corresponding to `class1` and `class2` intersect the one corresponding to `class3`. Note that `class3` corresponds to the lexicographic component.

5 The Chain Property

Hoşten and Thomas [14] recently established that the initial ideals of a toric ideal have an interesting combinatorial structure called the chain property. This structure is on the poset of associated primes where the partial order is given by inclusion. Since a monomial ideal $I \subset S = \mathbb{Q}[x_1, \dots, x_n]$ is prime if and only if it is generated by a subset of the variables $\{x_1, \dots, x_n\}$, the poset of associated primes of I is contained in the power set of the variables. We say that a monomial ideal I has the chain property if the following condition holds:

For any embedded prime $P_F = \langle x_i : x_i \notin F \rangle$ of I , there exists an associated prime $P_G \subset P_F$ such that $|G| = |F| - 1$.

In other words, there is a saturated chain from every embedded prime to some minimal prime. Experimental evidence suggests that, in fact, most initial ideals of prime ideals satisfy this saturated chain condition. Because of ubiquity and simplicity of this condition, we are interested in understanding which classes of initial ideals (or more generally monomial ideals) have the chain property.

More recently, Miller, Sturmfels and Yanagawa [18] provided a large class of monomial ideals with the chain property. A monomial ideal I is called generic when the following condition holds: if two distinct minimal generators r_1 and r_2 of I have the same positive degree in some variable x_i , there is a third generator r_3 which strictly divides the least common multiple of r_1 and r_2 . In particular, if no two distinct minimal generators have the same positive degree in any variable, then the monomial ideal is generic. Theorem 2.2 in Miller, Sturmfels and Yanagawa [18] shows that generic monomial ideals have the chain property.

Examples and Counterexamples

In this final section, we illustrate how to use *Macaulay 2* for further experimentation and investigation of the chain property. The following function determines whether a monomial ideal has the chain property:

```
i99 : hasChainProperty = I -> (
      L := ass I;
      radI := radical I;
      all(L, P -> radI : (radI : P) == P or (
        gensP := first entries gens P;
        all(gensP, r -> (
          Q := monomialIdeal delete(r, gensP);
          I : (I : Q) == Q)))));
```

Using `hasChainProperty`, we examine the initial ideals of four interesting classes of ideals related to toric ideals.

An Initial Ideal of a Toric Ideal. As mentioned above, Hoşten and Thomas proved that any initial ideal of a toric ideal satisfies the saturated chain condition. The following example demonstrates this phenomenon. Consider the matrix A :

```
i100 : A = matrix{{1,1,1,1,1,1,1}, {2,0,0,0,1,0,0}, {0,2,0,0,0,1,0}, { ...
o100 = | 1 1 1 1 1 1 1 |
      | 2 0 0 0 1 0 0 |
      | 0 2 0 0 0 1 0 |
      | 2 2 0 2 1 1 1 |

      4          7
o100 : Matrix ZZ <--- ZZ

i101 : IA = toricIdeal(A, {1,1,1,1,1,1,1})
```



```

o101 = ideal (x x2 - x2, x x2 - x2, x x2 - x2)
             3 4       7 2 3       6 1 3       5

o101 : Ideal of S

i102 : inIA = monomialIdeal IA

o102 = monomialIdeal (x x2, x x2, x x2, x x2, x x2, x x2)
                    1 3   2 3   3 4   2 5   4 5   4 6

o102 : MonomialIdeal of S

i103 : hasChainProperty inIA

o103 = true

```

An Initial Ideal of a Prime Ideal. Since toric ideals are prime, one naturally asks if the initial ideal of any prime ideal has the chain property. By modifying the previous example, we can show that this is not the case. In particular, making the linear change of coordinates by $x_4 \mapsto x_3 - x_4$, we obtain a new prime ideal J .

```

i104 : StoS = map(S, S, {x_1, x_2, x_3, x_3 - x_4, x_5, x_6, x_7});

o104 : RingMap S <--- S

i105 : J = StoS IA

o105 = ideal (x2 - x x2 - x2, x x2 - x2, x x2 - x2)
             3   3 4   7 2 3   6 1 3   5

o105 : Ideal of S

```

Taking the initial ideal with respect to the reverse lexicographic term order (the default order), we have

```

i106 : inJ = monomialIdeal J

o106 = monomialIdeal (x x2, x x2, x2, x x2, x x2, x x x2, x x2, x x x2 ...)
                    1 3   2 3   3   2 5   3 5   1 4 5   3 6   1 4 6 ...

o106 : MonomialIdeal of S

i107 : hasChainProperty inJ

o107 = false

```

An A -graded Monomial Ideal. Let A be a $d \times n$ matrix of nonnegative integers and let \mathbf{a}_i denote the i -th column of A . Consider the polynomial ring $S = \mathbb{Q}[x_1, \dots, x_n]$ with the \mathbb{Z}^d -grading defined by $\deg x_i = \mathbf{a}_i$. An ideal $I \subset \mathbb{Q}[x_1, \dots, x_n]$ is called A -graded provided it is homogeneous with respect to the A -grading and

$$\dim_{\mathbb{Q}} \left(\frac{S}{I} \right)_{\mathbf{b}} = \begin{cases} 1 & \text{if } \mathbf{b} \in \mathbb{N}A \\ 0 & \text{otherwise} \end{cases}$$

for all $\mathbf{b} \in \mathbb{N}^d$. Remark 10.1 in Sturmfels [26] shows that the initial ideal of the toric ideal I_A is A -graded. Altmann [2] shows that when A has rank 2 every A -graded monomial ideal has the chain property. However, Altmann [2] also provides a counterexample when A has rank 3. We can verify his example in *Macaulay 2* as follows:

```
i108 : A = matrix{{2,0,0,1,0,0,2,1,1,3,2,2,2,3,3,3},
                  {0,2,0,0,1,0,1,2,1,2,3,2,3,2,3,3},
                  {0,0,2,0,0,1,1,1,2,2,2,3,3,3,2,3}};
```

```
o108 : Matrix ZZ <--- ZZ
      3      16
```

In *Macaulay 2*, the first entry in degree vector of each variable must be positive. Hence, we append to A the sum of its rows to get a matrix whose columns will serve as the degrees of the variables.

```
i109 : D = A^{0}+A^{1}+A^{2} || A
```

```
o109 = | 2 2 2 1 1 1 4 4 4 7 7 7 8 8 8 9 |
      | 2 0 0 1 0 0 2 1 1 3 2 2 2 3 3 3 |
      | 0 2 0 0 1 0 1 2 1 2 3 2 3 2 3 3 |
      | 0 0 2 0 0 1 1 1 2 2 2 3 3 3 2 3 |
```

```
o109 : Matrix ZZ <--- ZZ
      4      16
```

```
i110 : D = entries transpose D;
```

```
i111 : S = QQ[vars(0..15), Degrees => D, MonomialSize => 16];
```

```
i112 : I = monomialIdeal(d*j, d*k, d*l, d*m, d*n, d*o, d*p, e*j, e*k,
                        e*l, e*m, e*n, e*o, e*p, f*j, f*k, f*l, f*m, f*n, f*o, f*p,
                        g*j, g*k, g*l, g*m, g*n, g*o, g*p, h*j, h*k, h*l, h*m, h*n,
                        h*o, h*p, i*j, i*k, i*l, i*m, i*n, i*o, i*p, g^2, g*h, g*i,
                        h^2, h*i, i^2, j^2, j*k, j*l, j*m, j*n, j*o, j*p, k^2, k*l,
                        k*m, k*n, k*o, k*p, l^2, l*m, l*n, l*o, l*p, m^2, m*n, m*o,
                        m*p, n^2, n*o, n*p, o^2, o*p, p^2, d^2, e^2, f^2, d*h, e*i,
                        f*g, f*d*i, d*e*g, e*f*h, c*d*g, a*e*h, b*f*i, c*e*g,
                        a*f*h, b*d*i, c*d*e, a*e*f, b*f*d, c*b*d, a*c*e, b*a*f,
                        c*b*g, a*c*h, b*a*i);
```

```
o112 : MonomialIdeal of S
```

To help convince you that I is an A -graded ideal, we compute the $\dim_{\mathbb{Q}} \left(\frac{S}{I}\right)_{\mathbf{a}_i}$ for $1 \leq i \leq 16$.

```
i113 : apply(D, d -> rank source basis(d, (S^1)/ ideal I))
```

```
o113 = {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

```
o113 : List
```

Finally, we check the chain property.

```
i114 : hasChainProperty I
```

```
o114 = false
```

The Vertex Ideal. Lastly, we consider a different family of monomial ideals arising from toric ideals. The vertex ideal V_A is defined as intersection all the

monomial initial ideals of the toric ideal I_A . Although there are (in general) infinitely many distinct term orders on a polynomial ring, an ideal has only finitely many initial ideals; see Theorem 1.2 in Sturmfels [26]. In particular, the above intersection is finite. Vertex ideals were introduced and studied by Hoşten and Maclagan [13]. However, the question “Does the vertex ideal V_A have the chain property?” remains open.

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