Generalized Humbert Schemes and Intersections of Humbert Surfaces

1. Introduction

Let: $M_g/\mathbb{C}$ be the moduli space of genus g curves $/\mathbb{C}$, i.e. $M_g(\mathbb{C})$ corresponds to isomorphism classes of such curves.

Question: What is the dimension (and structure) of subvarieties (subschemes) of M_g defined by "special properties" of curves?

Examples: 1) Curves with extra automorphims;
2) Curves with non-constant morphisms to non-rational curves;
3) Curves C whose Jacobians J_C have non-trivial endomorphisms, i.e. $\mathrm{End}(J_C) \neq \mathbb{Z}$.

Remark: Via the map $C \mapsto \langle C \rangle = (J_C, \lambda_\theta)$, where $\lambda_\theta : J_C \xrightarrow{\sim} \hat{J}_C$ is the θ-polarization, we can view $M_g(\mathbb{C}) \subset A_g(\mathbb{C})$, where A_g is the moduli space which classifies isomorphism classes of principally polarized (p.p.) abelian varieties (A, λ) of dimension g. Thus, Example 3 can be transported to A_g.

Humbert (1900): For each positive integer $n \equiv 0, 1 \pmod 4$, $\exists$ a surface $H_n \subset A_2$ (called a Humbert surface) such that:

(i) $\mathrm{End}(A) \neq \mathbb{Z} \Leftrightarrow (A, \lambda) \in H_n$, for some n;

(ii) $M_2 = A_2 \setminus H_1$;

(iii) $\exists f : C \to E \Leftrightarrow (J_C, \lambda_\theta) \in H_{N^2}$, for some $N \geq 2$.

Remark: In [ECAS], property (iii) was refined to:

(iii′) $(J_C, \lambda_\theta) \in H_{N^2} \Leftrightarrow \exists f : C \to E, \deg(f) = N, f$ minimal.

Questions: 1) How can we describe/analyze the components of the intersection $H_n \cap H_m$ of two distinct Humbert surfaces?

2) More generally, how can we describe the set of curves satisfying a "special property"?

Basic idea: As will be explained below, each integral, positive definite quadratic form q defines a closed subscheme

$$H_g(q) \subset A_g,$$

called a generalized Humbert scheme, of the moduli space A_g.

Properties: 1) $H_g(q)$ depends only on the GL_r-equivalence class of the quadratic form $q = q(x_1, \ldots, x_r)$.

2) We have that $H_g(q) \neq A_g$, but $H_g(q)$ may be empty.

3) The usual Humbert surface is $H_n := H_2(nx^2)$.

4) It follows easily from the definitions that if $n \neq m$, then

$$H_n \cap H_m = \bigcup_{q \to n,m} H_2(q), \tag{1}$$

where the union is over all integral, positive definite binary quadratic forms q which represent both n and m primitively.

Note: Up to equivalence, there are only finitely many forms q with this property because $|\mathrm{disc}(q)| \leq 4mn$.

Questions: 1) When is $H(q) \neq \emptyset$?

2) What is the (birational) structure of $H(q)$?

3) For a given q, how can we construct the p.p. abelian surfaces (A, λ) in $H(q)$? Is there a "modular construction"?

2. Main Results I ($g = 2$)

Theorem 1: Let q be a positive quadratic form in r variables. If $H(q) := H_2(q) \neq \emptyset$, then it has codimension r in A_2; i.e.,

$$\dim H(q) \;=\; 3 - r.$$

Moreover, if q' is another positive quadratic form, then

$$(2) \qquad H(q) \;=\; H(q') \quad \Leftrightarrow \quad q \;\sim_{\mathrm{GL}_r}\; q'.$$

Remark: If $r = 1$, then $q(x) = nx^2$ with $n \geq 1$, and then

$$H_n := H(nx^2) \neq \emptyset \;\Leftrightarrow\; n \equiv 0, 1 \,(\mathrm{mod}\, 4).$$

Thus, the Humbert surfaces correspond to the case $r = 1$.

Notation: If $n, m, d \geq 1$ are integers with $(n, d) = 1$, then let

$$T(n, m, d) \;=\; \{[a, b, c] \in \mathbb{Z}^3 : \text{conditions (i)-(iii) below hold}\}$$

denote the set of integral binary quadratic forms $q = [a, b, c]$ satisfying the following conditions:

(i) $\mathrm{disc}(q) := b^2 - 4ac = -16m^2 d$;

(ii) $q \longrightarrow (mn)^2$;

(iii) $q \equiv 0, 1 \,(\mathrm{mod}\, 4)$.

Here we identify $q = [a, b, c]$ with binary quadratic form

$$q(x, y) \;=\; ax^2 + bxy + cy^2.$$

Theorem 2: Let q be an integral binary quadratic form such that $q \longrightarrow N^2$, for some $N \geq 1$. Then

$$(3) \; H(q) \neq \emptyset \;\Leftrightarrow\; H(q) \text{ is an irreducible curve}$$
$$\Leftrightarrow\; q \in T(N/m, m, d), \text{ for some } m|N, d \geq 1 \text{ with } (N/m, d) = 1.$$

Corollary: If $m \equiv 0, 1 \pmod 4$ and $N \geq 1$, then

$$H_m \cap H_{N^2} \neq \emptyset.$$

Moreover, if $m > 1$ and $N > 1$, then

$$H_m \cap H_{N^2} \cap M_2 \neq \emptyset.$$

Proof. Wlog $m > 1$. Consider $q = [N^2, 2\varepsilon N, m] \in T(1, N, \frac{m-\varepsilon}{4})$, where $\varepsilon = \mathrm{rem}(m, 4)$. Then $H(q) \neq \emptyset$ by Theorem 2. Since $q \to N^2$ and $q \to m$, we see that $H(q) \subset H_m \cap H_{N^2}$. Moreover, since $q(x, y) = (Nx + \varepsilon y)^2 + (m - \varepsilon^2)y^2 > 1$ (when $N, m > 1$), we see that $q \not\to 1$. Thus $H(q) \not\subset H_1 = A_2 \setminus M_2$, and hence $H(q) \cap M_2 \neq \emptyset$.

Remark: The nature of the parameters (n, m, d) is partially explained by the following fact (which leads to the modular construction discussed below):

Theorem 3: Let C be a genus 2 curve and let $N \geq 2, d \geq 1$ be integers. The following conditions are equivalent:

(i) $\langle C \rangle \in H(q)$, for some $q \in T(N/m, m, d)$, where $m|N$ and $(N/m, d) = 1$;

(ii) $\exists$ two complementary elliptic subcovers $f_i : C \to E_i$ of degree N and a cyclic isogeny $h : E_1 \to E_2$ of degree d.

Note: An elliptic subcover is a minimal morphism $f : C \to E$ to some elliptic curve E. Two elliptic subcovers $f_i : C \to E_i$ are called *complementary* if the following sequence is exact:

$$0 \to J_{E_1} \xrightarrow{f_1^*} J_C \xrightarrow{(f_2)_*} J_{E_2} \to 0.$$

3. Some applications

Application 1: *Irreducible components of* $H_m \cap H_{N^2}$.

These can be computed by using (1)–(3) and the reduction theory of binary quadratic forms. For example,

$$\begin{aligned} H_5 \cap H_4 &= H[1,0,4] \cup H[4,0,5] \cup H[4,4,5], \\ H_5 \cap H_9 &= H[4,0,5] \cup H[5,2,9] \cup H[5,4,8], \end{aligned}$$

and the number of irreducible components of $H_m \cap H_{N^2}$ is:

$N^2 \backslash m$	1	4	5	8	9	12	13	16	17	20
1	$*$	1	1	2	1	2	2	2	3	3
4	1	$*$	3	4	3	4	5	5	5	6
9	1	3	3	5	$*$	3	2	2	4	2
16	2	6	5	2	2	4	3	$*$	2	4
25	3	5	4	4	3	1	4	3	6	3

Application 2: *Curves with extra automorphisms*

Theorem 4: Let C be a curve of genus 2. Then

(a) $4 \mid |\mathrm{Aut}(C)| \quad \Leftrightarrow \quad \langle C \rangle \in H_4$.

(b) $8 \mid |\mathrm{Aut}(C)| \quad \Leftrightarrow \quad \langle C \rangle \in H[4,0,4]$.

(c) $12 \mid |\mathrm{Aut}(C)| \quad \Leftrightarrow \quad \langle C \rangle \in H[4,4,4]$.

(d) $24 \mid |\mathrm{Aut}(C)| \quad \Leftrightarrow \quad \langle C \rangle \in H[4,0,4] \cap H[4,4,4]$.

Remark: The curves in these families have explicit equations:

(a) $y^2 = x(x-1)(x-\alpha)(x-\beta)(x-\alpha\beta)$ (Jacobi, 1832)

(b) $y^2 = x(1-x^2)(1-\kappa^2 x^2)$ (Legendre, 1832)

(c) $y^2 = x^6 + ax^3 + 1$ (Bolza, 1888)

(d) $y^2 = x(x^4-1)$ (Bolza, 1888; Burnside)

Application 3: *Curves with isogenous elliptic involutions*

An automorphism $\sigma \in \mathrm{Aut}(C)$ is called an elliptic involution if $\sigma^2 = 1$ and if $C_\sigma := C/\langle\sigma\rangle$ is an elliptic curve.

Theorem 5: Let C be genus 2 curve with hyperelliptic involution σ_C, and let $d \geq 1$ be an integer. Equivalent conditions:

(i) There exists an elliptic involution $\sigma \in \mathrm{Aut}(C)$ and a cyclic isogeny $h : C_\sigma \to C_{\sigma\sigma_C}$ of degree d;

(ii) $\langle C\rangle \in H[4, 0, 4d] \cup H[4, 4, 4d + 1] \cup H(d)$, where

$$H(d) = \begin{cases} H[4, 4, d + 1], & \text{if } d \equiv 3 \pmod 4, \\ H[4, 0, d], & \text{if } d \equiv 1 \pmod 4, d > 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Remark: In Accola/Previato [AP] (2006), p. 142, it was stated that a characterization of (i) “does not appear to be known”.

Application 4: *Curves with elliptic morphisms of degree* d

Let $\mathcal{L}_d \subset M_2$ be the moduli space of genus 2 curves C with a morphism $C \xrightarrow{f} E$ of $\deg(f) = d$ to some elliptic curve E, i.e.,

$$\mathcal{L}_d := \bigcup_{1<N|d} H_{N^2} \cap M_2.$$

Question ([AP]): When is $\mathcal{L}_d$ connected?

Answer: **Always!** Indeed, any two irreducible components of $\mathcal{L}_d$ meet by the Corollary of Theorem 2.

Application 5: *Jacobians isomorphic to a product $E_1 \times E_2$.*

Theorem 6: Let C be a genus 2 curve. TFAE:

(i) $J_C \simeq E_1 \times E_2$, for some elliptic curves E_1, E_2;

(ii) $(J_C, \lambda_\theta) \in H(q)$, for some $q \in T(N, 1, d)$, where $(N, d) = 1$.

Remark: Hayashida and Nishi (1965) studied (but did not fully resolve) the following question:

For which pairs (E_1, E_2) is $E_1 \times E_2 \simeq J_C$, for some C?

Using Theorem 6 (and deep results from the theory of quadratic forms), one can prove (cf. [MS], [JT]):

Theorem 7: (a) If $\mathrm{Hom}(E_1, E_2) = \mathbb{Z}f$ and $d = \deg(f)$, then there exists a curve C with $J_C \simeq E_1 \times E_2 \Leftrightarrow d > 1$ and d is not an idoneal number with $d \equiv 2, 4, 6 \pmod 8$. Thus, C exists whenever $d \notin L$, where

$$L = \{1, 2, 4, 6, 10, 12, 18, 22, 28, 30, 42, 58, 60, 70, 78, 102, 130, 190, 210, 330, 462, d^*\},$$

and where $d^* > 10^9$ is one further possible value with this property. Moreover, if the Euler/Gauss Conjecture (or if the Generalized Riemann Hypothesis (GRH)) holds, then no such d^* exists.

(b) If $\mathrm{rank}(\mathrm{Hom}(E_1, E_2)) > 1$, then there exists a curve C with $J_C \simeq E_1 \times E_2$ except for finitely many pairs (E_1, E_2) of isomorphism classes of elliptic curves. (Exact #: 46 pairs.)

More precisely, there exist (up to isomorphism) precisely 15 (explicit!) surfaces A with Picard rank $\rho(A) \geq 4$ such that $A \not\simeq J_C$, for any curve C. (These are all product surfaces.)

4. The Refined Humbert Invariant (General Theory)

Key Observation: The Néron-Severi group $\mathrm{NS}(A)$ of a p.p. abelian variety (A, λ) comes equipped with a canonical integral quadratic form $q_{(A,\lambda)}$ (called the refined Humbert invariant).

Let A/K be an abelian variety with $\dim(A) = g$ (K a field),
$\mathrm{NS}(A) = \mathrm{Pic}(A)/\mathrm{Pic}^0(A)$ be its Néron-Severi group,
$\lambda : A \xrightarrow{\sim} \hat{A}$ be a principal polarization.

Then we have a natural injection

$$\Phi_\lambda : \mathrm{NS}(A) \to \mathrm{End}_\lambda(A) := \{\alpha \in \mathrm{End}(A) : \hat{\alpha}\lambda = \lambda\alpha\}$$

given by $\Phi_\lambda(D) = \lambda^{-1} \circ \phi_D$. Moreover, Φ_λ is an isomorphism if K is algebraically closed (Mumford).

Proposition 0: (a) The rule

$$q_A(\alpha) = \tfrac{1}{2}\,\mathrm{tr}(\alpha^2),$$

where $\mathrm{tr} : \mathrm{End}(A) \to \mathbb{Z}$ is the usual trace map, defines an integral, positive definite quadratic form q_A on $\mathrm{End}_\lambda(A)$.

(b) The rule

$$q_{(A,\lambda)}(\alpha) = \tfrac{1}{4}(2g\ \mathrm{tr}(\alpha^2) - \mathrm{tr}(\alpha)^2)$$

defines an integral, positive definite quadratic form $q_{(A,\lambda)}$ on the quotient group $\overline{\mathrm{End}}_\lambda(A) := \mathrm{End}_\lambda(A)/\mathbb{Z}1_A$.

Definition: We call $q_{(A,\lambda)}$ the refined Humbert invariant of (A, λ).

Remarks: 1) If K is algebraically closed, then $\lambda = \phi_\theta$ for some (unique) $\theta \in \mathrm{NS}(A)$ and then Φ_λ induces an isomorphism

$$\bar{\Phi}_\lambda : \mathrm{NS}(A,\lambda) := \mathrm{NS}(A)/\mathbb{Z}\theta \ \stackrel{\sim}{\rightarrow}\ \overline{\mathrm{End}}_\lambda(A).$$

Thus, $q_{(A,\lambda)}$ can also be viewed as a quadratic form on $\mathrm{NS}(A,\lambda)$.

2) If A is an abelian surface (i.e., $g = 2$), then we have

$$q_{(A,\lambda)}(\Phi_\lambda(D)) = (D.\theta)^2 - 2(D.D), \quad \forall D \in \mathrm{NS}(A).$$

Moreover, if $\bar{D} \in \mathrm{NS}(A,\lambda)$ is primitive (i.e., if $\mathrm{NS}(A,\lambda)/\mathbb{Z}\bar{D}$ is torsionfree), then it was shown in [ECAS] (1994) that

$$N \ = \ q_{(A,\lambda)}(\bar{D})$$

is the classical Humbert invariant of A (which Humbert defined in the case $K = \mathbb{C}$ via the period matrix of A).

Note that if $\mathrm{rank}(\mathrm{NS}(A)) > 2$, then (A,λ) has infinitely many different (classical) Humbert invariants N associated to it.

5. Generalized Humbert Schemes

Observation: The refined Humbert invariant $q_{(A,\lambda)}$ can be used to define closed subschemes $H_g(q)$ of the moduli space A_g.

Definition: If (M_1, q_1) and (M_2, q_2) are two quadratic $\mathbb{Z}$-modules, then we say that (M_1, q_1) primitively represents (M_2, q_2) if there exists a linear injection $f : M_2 \rightarrow M_1$ such that

$$f \circ q_1 = q_2 \quad \text{and} \quad M_1/f(M_2) \text{ is torsionfree.}$$

If this is the case, then we write $q_1 \rightarrow q_2$.

Note: If $n \in \mathbb{Z}$, then $q_1 \rightarrow n$ (in the sense of the earlier notation) if and only if $q_1 \rightarrow q_2 := nx^2$.

Notation: If q is an integral, positive-definite quadratic form (on $\mathbb{Z}^r$), then we put

$$H_g(q) := \{(A, \lambda) \in A_g(\overline{K}) : q_{(A,\lambda)} \rightarrow q\}.$$

Theorem 0: $H_g(q)$ is a closed subscheme of A_g, provided that $\text{char}(K)^2 \nmid \text{disc}(q)$.

Examples: 1) As was already mentioned, the classical Humbert surface is $H_n = H_2(nx^2)$ (when $K = \mathbb{C}$).

2) If $(A, \lambda) \in A_g$, then $\text{NS}(A) \not\simeq \mathbb{Z} \Leftrightarrow (A, \lambda) \in H_g(nx^2)$, for some $n \geq 1$.

Open Questions: 1) When is $H_g(q) \neq \emptyset$?

2) Determine $\dim H_g(q)$ (when $H_g(q) \neq \emptyset$).

3) When is $H_g(q)$ irreducible?

6. The Modular Construction $(g = 2)$

The Basic Construction ([FK]): Fix $N \geq 1$, and consider a tuple (E_1, E_2, ψ) consisting of two elliptic curves E_i and an isomorphism

$$\psi : E_1[N] \xrightarrow{\sim} E_2[N]$$

of the groups $E_i[N]$ of N-torsion points. Let

$$\pi_\psi : E_1 \times E_2 \to A_\psi := (E_1 \times E_2)/(\mathrm{Graph}(\psi))$$

be the quotient map, which is an isogeny of degree N^2.

If ψ is an anti-isometry (wrt. Weil pairings on $E_i[N]$) then $\exists! \theta_\psi \in \mathrm{NS}(A_\psi)$ such that

$$\pi_\psi^* \theta_\psi = N(\theta_1 + \theta_2), \quad \text{where } \theta_i = pr_i^*(0_{E_i}),$$

and then $(A_\psi, \lambda_{\theta_\psi}) \in A_2$ is a p.p. abelian surface. Thus, if

$$\mathcal{Z}_N \;=\; \{\langle E_1, E_2, \psi\rangle_N\}$$

denotes the set of isomorphism classes of such triples, then the rule $(E_1, E_2, \psi) \mapsto (A_\psi, \lambda_\psi)$ defines a map

$$\beta_N : \mathcal{Z}_N \;\to\; A_2,$$

called the "basic construction" in [FK].

Facts: 1) $\mathcal{Z}_N$ can be identified with the quotient variety

$$Z_N \;=\; (X(N) \times X(N))/(\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\pm 1),$$

where $X(N) = \Gamma(N)\backslash\mathfrak{H}$ is the modular curve, and β_N induces a morphism of varieties

$$\beta_N : Z_N \;\to\; A_2.$$

2) The image of β_N is the Humbert surface H_{N^2}, and the induced morphism

$$\beta_N : Z_N \;\longrightarrow\; \beta_N(Z_N) = H_{N^2}$$

is finite of (generic) degree 2.

Question: If $H(q) \subset H_{N^2}$, what is $\beta_N^{-1}(H(q))$?

Expect: $\beta_N^{-1}(H(q))$ is a union of certain modular curves on Z_N, i.e., those that are images of modular correspondences $T_{A,N}$ on $X(N) \times X(N)$.

7. The Case of Product Surfaces $(m = 1)$

Let $X_0(d) = \Gamma_0(d)\backslash\mathfrak{H}$, the Hecke modular curve and let $\mathcal{X}_0(d) = \{\langle E_1, E_2, h\rangle : h \in \mathrm{Hom}(E_1, E_2) \text{ is cyclic of degree } d\}$.

Note: It is known that $\exists$ natural bijection $\mathcal{X}_0(d) \leftrightarrow X_0(d)(\mathbb{C})$.

Fix N, and let k, d satisfy $k^2 d \equiv -1 \pmod N$. Then have map

$$\tau_{d,k,N} : \mathcal{X}_0(d) \;\to\; \mathcal{Z}_N$$

given by $\tau_{d,k,N}(\langle E_1, E_2, h\rangle) = \langle E_1, E_2, kh_{|E_1[N]}\rangle$. This induces a morphism

$$\tau_{d,k,N} : X_0(d) \;\to\; Z_N$$

which is birational onto its image. Consider the composition

$$\mu_{d,k,N} := \beta_N \circ \tau_{d,k,N} : X_0(d) \to Z_N \;\to\; A_2.$$

Theorem 8: We have that $\mu_{d,k,N}(X_0(d)) = H(q_{d,k,N})$, where

$$q_{d,k,N} = [N^2, 2kt, (k^2t^2 + 4d)/N^2] \text{ with } t = d(k^2d + 3).$$

Corollary: If $q \in T(N, 1, d)$, then $H(q) = \mu_{d,k,N}(X_0(d))$, for some k with $k^2 d \equiv -1 \pmod N$.

Lemma: A binary quadratic form q has type $(N, 1, d)$ if and only $q \sim q_{d,k,N}$ for some k with $dk^2 \equiv -1 \pmod N$.

Remark: The normalization $\tilde{H}(q)$ of $H(q)$ can be determined precisely, at least if $q \in T(N, 1, d)$. Here one has (cf. [MS]):

$$\tilde{H}(q) \;\simeq\; X_0(d)^+ := X_0(d)/\langle w_d\rangle, \quad \text{where } w_d = \left(\begin{smallmatrix} 0 & -1 \\ d & 0 \end{smallmatrix}\right),$$

except when q is a (so-called) ambiguous form. Then $\tilde{H}(q) \simeq X_0(d)^+/\langle\alpha\rangle$, for some Atkin-Lehner involution α.

8. The General Case $(m \geq 1)$

Note: Here we need to use the modular correspondences $T_{A,N}$ attached to arbitrary primitive matrices $A \in \mathcal{M}_d$, where

$$\mathcal{M}_d = \Gamma(1)\alpha_d\Gamma(1), \quad \text{with } \Gamma(1) = \mathrm{SL}_2(\mathbb{Z}), \alpha_d = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right).$$

Modular description of $T_{A,N}$: Let

$$\mathcal{T}_{A,N} = \{\langle E_1, \alpha_1; E_2, \alpha_2; h\rangle_N\}$$

where each $\alpha_i : E_i[N] \xrightarrow{\sim} V_N := (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$ is a level-N-structure (= symplectic isomorphism), $h : E_1 \to E_2$ is a cyclic isogeny of degree $d = \det(A)$ such that

$$\alpha_2 \circ h_{|E_1[N]} = [A]_N \circ \alpha_1,$$

where $[A]_N \in \mathrm{End}(V_N)$ is defined by the matrix $A \pmod N$ (via the canonical basis of V_N).

Notation: Define the map $\tau_{A,N} : \mathcal{T}_{A,N} \to Z_N$ by the rule

$$\tau_{A,N}(x) = \langle E_1, E_2, \psi_x\rangle_N$$

where

$$\psi_x := \alpha_2^{-1} \circ [\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)]_N \circ \alpha_1, \quad \text{if } x = \langle E_1, \alpha_1; E_2, \alpha_2; h\rangle_N.$$

By the modular interpretation, this induces a morphism

$$\tau_{A,N} : T_{A,N} := \Gamma_{A,N}\backslash\mathfrak{H} \to Z_N,$$

where $\Gamma_{A,N} := \Gamma(N) \cap A^{-1}\Gamma(N)A \geq \Gamma(Nd)$. Moreover, let

$$\mu_{A,N} := \beta_N \circ \tau_{A,N} : T_{A,N} \to Z_N \to A_2$$

be the composition.

Notes: 1) If $k^2d \equiv -1 \pmod N$, and if $\sigma_k \in \Gamma(1)$ satisfies $\sigma_k \equiv \left(\begin{smallmatrix} k^{-1} & 0 \\ 0 & k \end{smallmatrix}\right) \pmod N$, then $\tau_{-\sigma_k\alpha_d,N}$ factors over the previous morphism $\tau_{d,k,N}$ and both have the same image in $\mathcal{Z}_N$.

2) Different A's in $\mathcal{M}_d$ can define the same set $\tau_{A,N}(\mathcal{T}_{A,N})$ in $\mathcal{Z}_N$. Thus, it's useful to introduce the following terminology.

Definition: A matrix $A \in \mathcal{M}_d = \Gamma(1)\alpha_d\Gamma(1)$ is called (right) normalized if it is of the form

$$A = g\alpha_d = g\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right) \quad \text{with} \quad g \in \Gamma(1) = \mathrm{SL}_2(\mathbb{Z}).$$

Theorem 9: If $A = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathcal{M}_d$ is normalized, then

$$\mu_{A,N}(T_{A,N}) = H(q_{A,N}),$$

where $q_{A,N} \in T(N/m, m, d)$ with $m = N/\gcd(\mathrm{tr}(A), y, z, N)$, and is given explicitly by

$$q_{A,N} = [N^2, 2m(x-w), m^2(\mathrm{tr}(A)^2 - 4yz)/N^2].$$

Corollary: If $q \in T(N/m, m, d)$, then $H(q) = \mu_{A,N}(T_{A,N})$, for some normalized matrix $A \in \mathcal{M}_d$. In particular, $H(q)$ is an irreducible curve.

This follows from Theorem 9 together with the following generalization of the previous lemma:

Lemma′: If $q \in T(N/m, m, d)$, then there exists a normalized matrix $A \in \mathcal{M}_d$ such that $q \sim q_{A,N}$.

Remarks: 1) Theorem 9 and its Corollary imply the existence part of Theorem 2.

2) Thus, every (non-empty) $H(q)$ on H_{N_2} is the image (via β_N) of a modular correspondence on Z_N (or on $X(N) \times X(N)$).

9. References

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