Intersections of Humbert Surfaces and Binary Quadratic Forms

Ernst Kani
Queen's University

BIRS, Banff
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Outline

1. Introduction

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  $M_g(\mathbb{C})$ corresponds to isomorphism classes of such curves.

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- Let:
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  $M_g(\mathbb{C})$ corresponds to isomorphism classes of such curves.
- Question: What is the dimension (and structure) of subvarieties (subschemes) of M_g defined by "special properties" of curves?
- Examples: 1) Curves with extra automorphisms
  2) Curves with non-constant morphisms to non-rational curves;
  3) Curves C whose Jacobians J_C have non-trivial endomorphisms, i.e. $\mathrm{End}(J_C) \neq \mathbb{Z}$.
- Note: Example 2 is a special case of Example 3.

1. Introduction – 2

- Remark: By Torelli we have an injection

$$j : M_g(\mathbb{C}) \hookrightarrow A_g(\mathbb{C}),$$

where A_g is the moduli space which classifies isomorphism classes of principally polarized (p.p.) abelian varieties (A, λ) of dimension g.
Explicitly: $j(C) := (J_C, \lambda_\theta)$, where $\lambda_\theta : J_C \xrightarrow{\sim} \hat{J}_C$ is the θ-polarization.

Thus, Question/Example 3 can be transported to A_g. For $g = 2$ this question was answered by Humbert (1900).

1. Introduction – 3

- Humbert (1900): For each positive integer $n \equiv 0, 1 \pmod 4$, $\exists$ an irreducible surface $H_n \subset A_2$ (called a Humbert surface) such that:

  (i) $\mathrm{End}(A) \neq \mathbb{Z} \Leftrightarrow (A, \lambda) \in H_n$, for some n;

  (ii) $M_2 = A_2 \setminus H_1$;

  (iii) $\exists f : C \to E \Leftrightarrow (J_C, \lambda_\theta) \in H_{N^2}$, for some $N \geq 2$.

- Remark: In [EC] (1994), property (iii) was refined to:

  (iii′) $(J_C, \lambda_\theta) \in H_{N^2} \Leftrightarrow \exists f : C \to E, \deg(f) = N, f$ minimal.

- Note: $f : C \to E$ is minimal $\Leftrightarrow$ f does not factor over a non-trivial isogeny of E.

1. Introduction – 4

- Questions: 1) How can we describe/analyze the components of the intersection $H_n \cap H_m$ of two distinct Humbert surfaces? (Of particular interest: the case $n = N^2$.)

  Note: The intersection $H_{N^2} \cap H_{m^2} \cap M_2$ classifies curves C with two minimal morphisms $f_1 : C \to E_1$ and $f_2 : C \to E_2$ of degrees N and m.

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- Basic idea: As will be explained below, each integral, positive definite quadratic form q defines a closed subscheme

$$H(q) \subset A_2,$$

called a generalized Humbert scheme.

1. Introduction – 5

- Properties: 1) $H(q)$ depends only on the GL_r-equivalence class of the quadratic form $q = q(x_1, \ldots, x_r)$.

  2) We have that $H(q) \neq A_2$, but $H(q)$ may be empty.

  3) The usual Humbert surface is $H_n := H(nx^2)$.

  4) It follows easily from the definition of $H(q)$ (given below) that if $n \neq m$, then

  $$H_n \cap H_m = \bigcup_{q \to n,m} H(q), \tag{1}$$

  where the union is over all integral, positive definite binary quadratic forms q which represent both n and m primitively.

  Note: Up to equivalence, there are only finitely many forms q with this property because $|\mathrm{disc}(q)| \leq 4mn$.

1. Introduction – 6

- Questions: 1) When is $H(q) \neq \emptyset$?

  2) What is the (geometric) structure of $H(q)$? Is $H(q)$ irreducible?

  3) For a given q, how can we construct the p.p. abelian surfaces (A, λ) in $H(q)$? Is there a "modular construction"?

2. Main Results I

- Notation: Write $q = [a, b, c]$ for a binary quadratic form

$$q(x, y) = ax^2 + bxy + cy^2.$$

Let Q denote the set of integral binary quadratic forms q which satisfy:

(i) q is positive-definite;

(ii) $q(x, y) \equiv 0, 1 \pmod 4$, $\forall x, y \in \mathbb{Z}$.

Moreover, for $n \in \mathbb{N}$ let

$$Q(n) = \{q \in Q : q \to n\}$$

denote the set of forms $q \in Q$ which *primitively represent* n, i.e.,

$$q(x, y) = n, \quad \text{for some } x, y \in \mathbb{Z} \text{ with } \gcd(x, y) = 1.$$

2. Main Results I - 2

- Theorem 1: Let q be an integral binary quadratic form and let $N \geq 1$. Then:

$$H(q) \neq \emptyset \text{ and } H(q) \subset H_{N^2} \quad \Leftrightarrow \quad q \in Q(N^2).$$

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- Corollary: If $m \equiv 0, 1 \pmod 4$ and $N \geq 1$, then

$$H_m \cap H_{N^2} \neq \emptyset.$$

  Moreover, if $m > 1$ and $N > 1$, then

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- Proof. Wlog $m > 1$. Consider $q = [N^2, 2\varepsilon N, m]$, where $\varepsilon = \mathrm{rem}(m, 4)$. Then $H(q) \neq \emptyset$ by Theorem 1 because $q \in Q(N^2)$. Moreover, since $q \to N^2$ and $q \to m$, we have by (1) that $H(q) \subset H_m \cap H_{N^2}$.

2. Main Results I - 3

- Remark. This corollary implies that the moduli space

$$M_2(1,n) = \bigcup_{1<N|n} H_{N^2} \cap M_2$$

of curves admitting a morphism $f : C \to E$ of degree n to some elliptic curve E is connected.

This answers a question posed by Accola-Previato[AP] (2006).

Note: The space $M_2(1,n)$ was studied by Lange[La] (1976).

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3. Main Results II

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- Definition. A quadratic form q is said to be *of type* (N, m, d) if $q \in Q(N^2)$ and if $m|N$ and

$$disc(q) = -16m^2 d \quad \text{and} \quad \gcd(d, N/m) = 1.$$

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- Lemma: If $q \in Q(N^2)$, then there exists unique positive integers $m|N$ and d such that q has type (N, m, d).

3. Main Results II - 2

- Theorem 3: Let $q = [a, b, c] \in Q(N^2)$ have type (N, m, d), and put
$$c_m(q) = \gcd(a, b, c, m).$$
(a) $H(q)$ has at most $2^{\omega(c_m(q))}$ irreducible components, provided that $8 \nmid c_m(q)$. Here $\omega(n) = |\{p|n\}|$.
(b) If $d > N^4/(4m^2)$ and if $c_m(q)$ is odd, then $H(q)$ has precisely $2^{\omega(c_m(q))}$ irreducible components, except when $q \sim [N^2, 0, 4d]$.

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- Remarks: 1) Clearly, Theorem 3(a) $\Rightarrow$ Theorem 2.
2) If $8|c_m(q)$, then $H(q)$ has at most $2^{\omega(c_m(q))+1}$ irreducible components, and the analogue of part (b) holds (but there are more exceptions.) Moreover, the number of components can also be determined in the exceptional cases.

3. Main Results II - 4

- Numerical Examples: By the reduction theory of binary quadratic forms and the above results (and more), we obtain:

$$\begin{aligned} H_1 \cap H_4 &= H[1,0,4], \\ H_1 \cap H_5 &= H[1,0,4], \\ H_4 \cap H_5 &= H[1,0,4] \cup H[4,0,5] \cup H[4,4,5], \\ H_9 \cap H_5 &= H[4,0,5] \cup H[5,2,9] \cup H[5,4,8]. \end{aligned}$$

Also, the number of irreducible components of $H_{N^2} \cap H_m$ is:

$N^2 \backslash m$	1	4	5	8	9	12	13	16	17	20	21	24	25
1	$*$	1	1	2	1	2	2	2	3	3	2	3	3
4	1	$*$	3	4	3	4	5	5	5	6	5	6	6
9	1	3	3	5	$*$	6	5	6	8	7	8	10	9
16	2	5	5	6	6	9	9	$*$	9	12	10	11	12
25	3	6	7	8	9	9	10	12	15	16	11	13	$*$

Note: The numbers in red are those for which the intersection $H_{N^2} \cap H_m$ contains reducible $H(q)$'s.

4. The Refined Humbert Invariant

- Key Observation: The Néron-Severi group

$$\mathrm{NS}(A) \;=\; \mathrm{Div}(A)/\equiv$$

of a p.p. abelian variety (A, λ) comes equipped with a canonical integral quadratic form $q_{(A,\lambda)}$ (called the refined Humbert invariant).

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  of a p.p. abelian variety (A, λ) comes equipped with a canonical integral quadratic form $q_{(A,\lambda)}$ (called the refined Humbert invariant).

- Notation: Let A/K be an abelian surface over an algebraically closed field K. If $\lambda : A \to \hat{A}$ is a p.p., then $\lambda = \phi_\theta$ for some (unique) $\theta \in \mathrm{NS}(A)$. Put

$$\tilde{q}_{(A,\lambda)}(D) = (D.\theta)^2 - 2(D.D), \quad \forall D \in \mathrm{NS}(A).$$

  Then by the Hodge Index Theorem $\tilde{q}_{(A,\lambda)}$ defines a positive definite quadratic form $q_{(A,\lambda)}$ on the quotient group

$$\mathrm{NS}(A, \lambda) \; := \; \mathrm{NS}(A)/\mathbb{Z}\theta.$$

4. The Refined Humbert Invariant - 2

- Definition: We call $q_{(A,\lambda)}$ the refined Humbert invariant of (A, λ).
- Remark: If $\bar{D} \in \mathrm{NS}(A, \lambda)$ is primitive (i.e., if $\mathrm{NS}(A, \lambda)/\mathbb{Z}\bar{D}$ is torsionfree), then it was shown in [EC] (1994) that

  $$N = q_{(A,\lambda)}(\bar{D})$$

  is the classical Humbert invariant of A (which Humbert defined in the case $K = \mathbb{C}$ via the period matrix of A).
  Note that if $\mathrm{rank}(\mathrm{NS}(A)) > 2$, then (A, λ) has infinitely many different (classical) Humbert invariants N associated to it.

5. Generalized Humbert Schemes

- Observation: The refined Humbert invariant $q_{(A,\lambda)}$ can be used to define closed subschemes $H(q)$ of the moduli space A_2.

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- Definition: If (M_1, q_1) and (M_2, q_2) are two quadratic $\mathbb{Z}$-modules, then we say that (M_1, q_1) primitively represents (M_2, q_2) if there exists a linear injection $f : M_2 \to M_1$ such that

$$q_1 \circ f = q_2 \quad \text{and} \quad M_1/f(M_2) \text{ is torsionfree.}$$

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  If this is the case, then we write $q_1 \to q_2$.
- Notation: If q is an integral, positive-definite quadratic form (on $\mathbb{Z}^r$), then we put

  $$H(q) := \{(A, \lambda) \in A_2(\overline{K}) : q_{(A,\lambda)} \to q\}.$$

5. Generalized Humbert Schemes - 2

- Proposition 1: $H(q)$ is a closed subscheme of A_2, provided that $\mathrm{char}(K)^2 \nmid \mathrm{disc}(q)$.

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5. Generalized Humbert Schemes - 2

- Proposition 1: $H(q)$ is a closed subscheme of A_2, provided that $\mathrm{char}(K)^2 \not| \mathrm{disc}(q)$.
- Example: As was already mentioned, the classical Humbert surface is $H_n = H(nx^2)$ (when $K = \mathbb{C}$).
- Remark: It is possible to generalize the refined Humbert invariant $q_{(A,\lambda)}$ to p.p. abelian varieties (A, λ) of arbitrary dimension $g \geq 2$. Then the above definition of $H(q)$ extends to define closed subschemes of A_g.

5. Generalized Humbert Schemes - 3

- Generalization: Let (A, λ) be a p.p. abelian variety of dimension g. Then the rule $D \mapsto \lambda^{-1} \circ \phi_D$ defines a bijection

$$\Phi_A : \mathrm{NS}(A) \xrightarrow{\sim} \mathrm{End}_\lambda(A) := \{\alpha \in \mathrm{End}(A) : \lambda^{-1} \circ \hat{\alpha} \circ \lambda = \alpha\}.$$

Put, for $\alpha \in \mathrm{End}_\lambda(A)$,

$$q_{(A,\lambda)}(\alpha) = \frac{1}{4}(2g\,\mathrm{tr}(\alpha^2) - \mathrm{tr}(\alpha)^2).$$

Then $q_{(A,\lambda)}$ defines a positive definite quadratic form on

$$\mathrm{NS}(A, \lambda) = \mathrm{End}_\lambda(A)/\mathbb{Z}1_A$$

(which generalizes the case $g = 2$) and one can show that

$$H(q) := \{(A, \lambda) \in A_g : q_{(A,\lambda)} \to q\}$$

is a closed subscheme of A_g.

6. The Modular Construction: Step 1

- **Step 1: The Basic Construction ([FK])**
- Theorem 4: Let $\mathrm{char}(K) \nmid N \geq 1$, and let $X(N)/K$ denote the affine modular curve of full level N. Then there is a finite surjective morphism

$$\beta_N : X(N) \times X(N) \rightarrow H_{N^2}.$$

Moreover, the normalization $\tilde{H}_{N^2}$ of H_{N^2} is isomorphic to the quotient surface $(X(N) \times X(N))/\operatorname{Aut}(\beta_N)$.

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  Moreover, the normalization $\tilde{H}_{N^2}$ of H_{N^2} is isomorphic to the quotient surface $(X(N) \times X(N))/\,\mathrm{Aut}(\beta_N)$.
- Remarks: 1) The morphism β_N is a variant of the "basic construction" of [FK].
  2) We have that $\deg(\beta_N) = |\,\mathrm{Aut}(\beta_N)|$ and that

$$\mathrm{Aut}(\beta_N) \ \simeq \ \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\} \rtimes \mathbb{Z}/2\mathbb{Z}.$$

  In particular, $|\,\mathrm{Aut}(\beta_N)| = |\,\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})|$, if $N \geq 3$.

6. The Modular Construction: Step 2

- **Step 2: The Modular Correspondences X_A^N**
- Notation: For $d \geq 1$, let $\mathcal{M}_d$ denote the set of primitive matrices of determinant d, so

$$\mathcal{M}_d = \Gamma(1)\alpha_d\Gamma(1), \quad \text{where } \Gamma(1) = \mathrm{SL}_2(\mathbb{Z}), \alpha_d = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}.$$

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- Fact: If $K = \mathbb{C}$, then for each $A \in \mathcal{M}_d$ there is an irreducible curve

$$X_A^N \subset X(N) \times X(N)$$

which depends only on the double coset $\pm\Gamma(N)A\Gamma(N)$.

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  which depends only on the double coset $\pm\Gamma(N)A\Gamma(N)$.
- Remark: Analytically, $X(N) = \Gamma(N)\backslash\mathfrak{H}$, and X_A^N is the image of the graph $\Gamma_A \subset \mathfrak{H} \times \mathfrak{H}$ of A (viewed as a fractional linear transformation on the upper half-plane $\mathfrak{H}$).

6. The Modular Construction: Step 3

- **Step 3: The structure of $H(q)$**
- Notation: For $A \in \mathcal{M}_d$ and $N \geq 1$ let

$$q_A^N = [N^2, 2mt, m^2(t^2 + 4d)/N^2].$$

Here, $t = \mathrm{trace}(BA)$, where $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $m|N$ is determined by the formula

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- Lemma: (a) q_A^N is a form of type (N, m, d).

  (b) If q is a form of type (N, m, d), then there is a (primitive) matrix $A \in \mathcal{M}_d$ such that $q \sim q_A^N$.

6. The Modular Construction: Step 3 (cont'd)

- Notation: For $A \in \mathcal{M}_d$ and $N \geq 1$ let

$$\overline{X}_A^N := \beta_N(X_A^N) \subset H_{N^2} \subset A_2$$

denote the image of the modular correspondence X_A^N in the Humbert surface H_{N^2}.

6. The Modular Construction: Step 3 (cont'd)

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denote the image of the modular correspondence X_A^N in the Humbert surface H_{N^2}.

- Theorem 5: If q is a binary form of type (N, m, d), then

$$H(q) = \bigcup_A \overline{X}_A^N, \tag{2}$$

where the union is over all $A \in \mathcal{M}_d$ such that $q_A^N \sim q$. This is a finite union because

$$gBA_1g^{-1} \equiv \pm BA_2 \,(\mathrm{mod}\, N),\, g \in \Gamma(1) \;\Rightarrow\; \overline{X}_{A_1}^N = \overline{X}_{A_2}^N. \tag{3}$$

7. The Structure of H(q)

- **Analysis of the structure of $H(q)$**
- Note: In view of Theorem 5, the study of the irreducible components of $H(q)$ leads to the following 3 problems:
  1. Determine the $SL_2(\mathbb{Z}/N\mathbb{Z})$-conjugacy classes of the matrices $A \bmod N$.
  2. Study the $\pm$-action on the conjugacy classes.
  3. Examine the converse of implication (3).

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  2. Study the $\pm$-action on the conjugacy classes.
  3. Examine the converse of implication (3).
- Solutions: 1) This is an easy extension of the work of Nobs[No] (1977).

  2) This is an easy exercise and leads to the exceptional cases of Theorem 3.

  3) This is more difficult because the failure of the converse leads to curves lying in the singular locus of H_{N^2}. However:

7. The Structure of H(q) - 2

- Theorem 6: Let q be a form of type (N, m, d) which satisfies the condition

$$|\{(x,y) \in \mathbb{Z}^2 : q(x,y) = N^2, \gcd(x,y) = 1\}| = 2. \tag{4}$$

Then the converse of (3) holds for the matrices $A_i \in \mathcal{M}_d$ with $q_{A_i}^N \sim q$.

7. The Structure of H(q) - 2

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- Remark: If $d > N^4/(4m^2)$, then the theory of quadratic forms (reduction theory) shows that (4) holds. ($\Rightarrow$ Theorem 3(b).)

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- Remark: If $d > N^4/(4m^2)$, then the theory of quadratic forms (reduction theory) shows that (4) holds. ($\Rightarrow$ Theorem 3(b).)
- Theorem 7: Let N be an odd prime and q a form of type (N, N, d) for which (4) does not hold. Then the converse of (3) holds if and only if $N \equiv 1 \mod 4$.

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  Then the converse of (3) holds for the matrices $A_i \in \mathcal{M}_d$ with $q_{A_i}^N \sim q$.
- Remark: If $d > N^4/(4m^2)$, then the theory of quadratic forms (reduction theory) shows that (4) holds. ($\Rightarrow$ Theorem 3(b).)
- Theorem 7: Let N be an odd prime and q a form of type (N, N, d) for which (4) does not hold. Then the converse of (3) holds if and only if $N \equiv 1 \mod 4$.
- Example: The form $q = [9, 6, 9]$ has type $(3, 3, 2)$ and condition (4) fails for q. Thus, the converse of (3) does not hold for q, and hence $H(q)$ is an irreducible curve lying in the singular locus of H_9.

8. Method of proof

- Definition. A N-presentation of a p.p. abelian surface (A, λ) is 4-tuple (E_1, E_2, ψ, π) where E_i/K are elliptic curves, $\psi : E_1[N] \to E_2[N]$ is an anti-isometry, and

$$\pi : E_1 \times E_2 \to A$$

is an isogeny such that $\mathrm{Ker}(\pi) = \mathrm{Graph}(-\psi)$ and

$$\pi^*\theta \equiv N(\theta_1 + \theta_2),$$

where θ is the theta-divisor of (A, λ) and $\theta_i = pr_i^*(0_{E_i})$.

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- Remark: It follows from the basic construction (cf. [FK]) that

$$(A, \lambda) \text{ has an } N\text{-presentation} \quad \Leftrightarrow \quad (A, \lambda) \in H_{N^2}.$$

8. Method of proof - 2

- Step 0: Given an N-presentation (E_1, E_2, ψ, π) of (A, λ), compute the refined Humbert invariant $q_{(A,\lambda)}$ of (A, λ). This was done in [ES]. (See [MJ] for a special case.)

8. Method of proof - 2

- Step 0: Given an N-presentation (E_1, E_2, ψ, π) of (A, λ), compute the refined Humbert invariant $q_{(A,\lambda)}$ of (A, λ). This was done in [ES]. (See [MJ] for a special case.)
- Step 1: Use the *modular interpretation* of $X(N)$ to construct the morphism

$$\beta_N : X(N) \times X(N) \rightarrow A_2.$$

Then $\mathrm{Im}(\beta_N) = H_{N^2}$ by the basic construction. Verify that β_N has finite fibres and that β_N is proper. Thus, β_N is finite.

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- Steps 2 and 3: Determine a useful *modular interpretation* of (the normalization of) the modular correspondence X_A^N. Use this and Step 0 to show that $\beta_N(X_A^N) \subset H(q)$ if and only if $q_A^N \sim q$.

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