Intersections of Humbert Surfaces and Binary Quadratic Forms

Ernst Kani
Queen's University

Québec-Maine Conference, Laval University
9 October 2016

Outline

1. Introduction

- Let:
  $M_g/\mathbb{C}$ be the moduli space of genus g curves $/\mathbb{C}$, i.e.
  $M_g(\mathbb{C})$ corresponds to isomorphism classes of such curves.
- Question: What is the dimension (and structure) of subvarieties (subschemes) of M_g defined by "special properties" of curves?
- Examples: 1) Curves with extra automorphisms;
  2) Curves with non-constant morphisms to non-rational curves;
  3) Curves C whose Jacobians J_C have non-trivial endomorphisms, i.e. $\mathrm{End}(J_C) \neq \mathbb{Z}$.
- Note: Example 2 is a special case of Example 3.

1. Introduction – 2

- Remark: Via the map $C \mapsto \langle C \rangle = (J_C, \lambda_\theta)$, where $\lambda_\theta : J_C \xrightarrow{\sim} \hat{J}_C$ is the θ-polarization, we can view $M_g(\mathbb{C}) \subset A_g(\mathbb{C})$, where A_g is the moduli space which classifies isomorphism classes of principally polarized (p.p.) abelian varieties (A, λ) of dimension g. Thus, Example 3 can be transported to A_g.
- Humbert (1900): For each positive integer $n \equiv 0, 1 \pmod 4$, $\exists$ a surface $H_n \subset A_2$ (called a Humbert surface) such that:
  (i) $\mathrm{End}(A) \neq \mathbb{Z} \Leftrightarrow (A, \lambda) \in H_n$, for some n;
  (ii) $M_2 = A_2 \setminus H_1$;
  (iii) $\exists f : C \to E \Leftrightarrow (J_C, \lambda_\theta) \in H_{N^2}$, for some $N \geq 2$.
- Remark: In [ECAS], property (iii) was refined to:
  (iii′) $(J_C, \lambda_\theta) \in H_{N^2} \Leftrightarrow \exists f : C \to E, \deg(f) = N, f$ minimal.

1. Introduction – 3

- Questions: 1) How can we describe/analyze the components of the intersection $H_n \cap H_m$ of two distinct Humbert surfaces? (Of particular interest: the case $n = N^2$.)

  2) How many such components are there?

- Basic idea: As will be explained below, each integral, positive definite quadratic form q defines a closed subscheme

$$H(q) \subset A_2,$$

called a generalized Humbert scheme.

1. Introduction – 4

- Properties: 1) $H(q)$ depends only on the GL_r-equivalence class of the quadratic form $q = q(x_1, \dots, x_r)$.
  2) We have that $H(q) \neq A_2$, but $H(q)$ may be empty.
  3) The usual Humbert surface is $H_n := H(nx^2)$.
  4) It follows easily from the definition of $H(q)$ (given below) that if $n \neq m$, then

$$H_n \cap H_m = \bigcup_{q \to n,m} H(q), \tag{1}$$

  where the union is over all integral, positive definite binary quadratic forms q which represent both n and m primitively.
  Note: Up to equivalence, there are only finitely many forms q with this property because $|\mathrm{disc}(q)| \leq 4mn$.

1. Introduction – 5

- Questions: 1) When is $H(q) \neq \emptyset$?
  2) What is the (birational) structure of $H(q)$?
  3) For a given q, how can we construct the p.p. abelian surfaces (A, λ) in $H(q)$? Is there a "modular construction"?

2. Main Results I

- Notation: Write $q = [a, b, c]$ for a binary quadratic form

$$q(x, y) = ax^2 + bxy + cy^2.$$

Let Q denote the set of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of integral binary quadratic forms q which satisfy:
(i) q is positive-definite;
(ii) $q(x, y) \equiv 0, 1 \pmod 4$, $\forall x, y \in \mathbb{Z}$.
Moreover, for $n, m \in \mathbb{N}$ and $m \neq n$ put

$$Q(n) = \{q \in Q : q \to n\} \quad \text{and} \quad Q(n, m) := Q(n) \cap Q(m),$$

and for $x \geq 1$ put

$$Q'(n, x) := \bigcup_{m \leq x, m \neq n} Q(n, m).$$

2. Main Results I - 2

- Theorem 1: Let q be an integral binary quadratic form such that $q \to N^2$, for some $N \geq 1$. Then

  (2) $\quad H(q) \neq \emptyset \;\Leftrightarrow\; q \in Q \;\Leftrightarrow\; H(q)$ is an irreducible curve.

- Corollary: If $N \geq 1$ and $m \neq N^2$, then the rule $q \mapsto H(q)$ defines a bijection

$$Q(N^2, m) \xrightarrow{\sim} \mathrm{Irr}(H_{N^2} \cap H_m),$$

  where $\mathrm{Irr}(V)$ denotes the set of irreducible components of an algebraic set V. Similarly, the above rule also induces a bijection

$$Q'(N^2, x) \xrightarrow{\sim} \mathrm{Irr}(H_{N^2} \cap (\bigcup_{m \leq x, m \neq N^2} H_m))$$

- Remark: This result allows us to translate problems about the components of intersections of Humbert surfaces into problems about binary quadratic forms.

2. Main Results I - 3

- Theorem 2: (a) If $m \equiv 0, 1 \pmod 4$ and $m \neq N^2$, then

$$|\mathrm{Irr}(H_{N^2} \cap H_m)| \;\geq\; \min([N^2/4], [m/4]).$$

(b) For any $N \geq 1$ we have

$$|\mathrm{Irr}(H_{N^2} \cap (\bigcup_{m \leq x, m \neq N^2} H_m))| \;=\; c_N x + O(1),$$

where $c_N = [\frac{N^2+4}{8}]$, if N is even, and $c_N = [\frac{N^2+1}{8}]$, if N is odd.

- Remark: The above result follows from the above Corollary together with the reduction theory of binary quadratic forms. By the same method one also obtains that

$$H_{N^2} \cap H_m \cap M_2 \;\neq\; \emptyset, \quad \text{if } m > 1, N > 1.$$

This implies the validity of a conjecture of Accola-Previato[AP].

2. Main Results I - 4

- Numerical Examples: By the Corollary of Theorem 1 and the reduction theory of binary quadratic forms, we obtain

$$\begin{aligned} H_1 \cap H_4 &= H[1,0,4], \\ H_1 \cap H_5 &= H[1,0,4], \\ H_4 \cap H_5 &= H[1,0,4] \cup H[4,0,5] \cup H[4,4,5], \\ H_9 \cap H_5 &= H[4,0,5] \cup H[5,2,9] \cup H[5,4,8]. \end{aligned}$$

and the number of irreducible components of $H_{N^2} \cap H_m$ is:

$N^2 \backslash m$	1	4	5	8	9	12	13	16	17	20	21	24	25
1	$*$	1	1	2	1	2	2	2	3	3	2	3	3
4	1	$*$	3	4	3	4	5	5	5	6	5	6	6
9	1	3	3	5	$*$	6	5	6	8	7	7	9	9
16	2	5	5	6	6	9	9	$*$	9	12	10	11	12
25	3	6	6	8	9	9	10	12	15	13	11	13	$*$

- Remark: Enea Milio wrote me that he/she was also able to derive the above table by factoring suitable theta-series.

3. Main Results II

- Theorem 3: Let $q \in Q(N^2)$. If $\mathrm{disc}(q) = -16d$, where $(d, N) = 1$, then the normalization $\tilde{H}(q)$ of $H(q)$ is the Fricke modular curve $X_0(d)^+$, i.e.

$$\tilde{H}(q) \simeq X_0(d)^+ := X_0(d)/\langle w_d \rangle, \quad \text{where} \quad w_d = \left(\begin{smallmatrix} 0 & -1 \\ d & 0 \end{smallmatrix}\right),$$

  except possibly when q is a (so-called) ambiguous form. In the exceptional cases we have that

$$\tilde{H}(q) \simeq X_0(d)^+/\langle \alpha \rangle,$$

  for some Atkin-Lehner involution α.
- Remark: See [MS] (2016) for the precise characterization of the exceptional cases (and for the recipe for determining α.)

3. Main Results II - 2

- **Theorem 4:** If $q \in Q(N^2)$, then $\exists !\, m | N$ and $d \geq 1$ such that

$$\mathrm{disc}(q) = -16m^2 d \quad \text{and} \quad (N/m, d) = 1.$$

  Moreover, we have a finite surjective morphism

$$\beta_{q,N} : X_0(N,d) \rightarrow H(q),$$

  where $X_0(N,d)$ denotes the affine modular curve

$$X_0(N,d) := (\Gamma(N) \cap \Gamma_0(Nd)) \backslash \mathfrak{H}.$$

- **Remark:** It is easy to see that Theorem 4 $\Rightarrow$ Theorem 1.

4. The Refined Humbert Invariant

- Key Observation: The Néron-Severi group $\mathrm{NS}(A) = \mathrm{Div}(A)/\equiv$ of a p.p. abelian variety (A, λ) comes equipped with a canonical integral quadratic form $q_{(A,\lambda)}$ (called the refined Humbert invariant).
- Notation: Let A/K be an abelian surface over an algebraically closed field K. If $\lambda : A \to \hat{A}$ is a p.p., then $\lambda = \phi_\theta$ for some (unique) $\theta \in \mathrm{NS}(A)$. Put

$$\tilde{q}_{(A,\lambda)}(D) = (D.\theta)^2 - 2(D.D), \quad \forall D \in \mathrm{NS}(A).$$

Then by the Hodge Index Theorem $\tilde{q}_{(A,\lambda)}$ defines a positive definite quadratic form $q_{(A,\lambda)}$ on the quotient group

$$\mathrm{NS}(A, \lambda) := \mathrm{NS}(A)/\mathbb{Z}\theta.$$

4. The Refined Humbert Invariant - 2

- Definition: We call $q_{(A,\lambda)}$ the refined Humbert invariant of (A, λ).
- Remark: If $\bar{D} \in \mathrm{NS}(A, \lambda)$ is primitive (i.e., if $\mathrm{NS}(A, \lambda)/\mathbb{Z}\bar{D}$ is torsionfree), then it was shown in [ECAS] (1994) that

$$N = q_{(A,\lambda)}(\bar{D})$$

is the classical Humbert invariant of A (which Humbert defined in the case $K = \mathbb{C}$ via the period matrix of A). Note that if $\mathrm{rank}(\mathrm{NS}(A)) > 2$, then (A, λ) has infinitely many different (classical) Humbert invariants N associated to it.

5. Generalized Humbert Schemes

- Observation: The refined Humbert invariant $q_{(A,\lambda)}$ can be used to define closed subschemes $H(q)$ of the moduli space A_2.
- Definition: If (M_1, q_1) and (M_2, q_2) are two quadratic $\mathbb{Z}$-modules, then we say that (M_1, q_1) primitively represents (M_2, q_2) if there exists a linear injection $f : M_2 \to M_1$ such that

$$q_1 \circ f = q_2 \quad \text{and} \quad M_1/f(M_2) \text{ is torsionfree.}$$

  If this is the case, then we write $q_1 \to q_2$.
- Notation: If q is an integral, positive-definite quadratic form (on $\mathbb{Z}^r$), then we put

$$H(q) := \{(A, \lambda) \in A_2(\overline{K}) : q_{(A,\lambda)} \to q\}.$$

5. Generalized Humbert Schemes

- Proposition 1: $H(q)$ is a closed subscheme of A_2, provided that $\mathrm{char}(K)^2 \nmid \mathrm{disc}(q)$.
- Example: As was already mentioned, the classical Humbert surface is $H_n = H(nx^2)$ (when $K = \mathbb{C}$).
- Remark: It is possible to generalize the refined Humbert invariant $q_{(A,\lambda)}$ to p.p. abelian varieties (A, λ) of arbitrary dimension $g \geq 2$. Then the above definition of $H(q)$ extends to define closed subschemes of A_g.

6. The Modular Construction: Step 1

- **Step 1: The Basic Construction ([FK])**
- Theorem 5: Let $\mathrm{char}(K) \nmid N \geq 1$, and let $X(N)/K$ denote the affine modular curve of level N. Then there is a finite surjective morphism

$$\beta_N : X(N) \times X(N) \; \rightarrow \; H_{N^2}.$$

Moreover, the normalization $\tilde{H}_{N^2}$ of H_{N^2} is isomorphic to the quotient surface $(X(N) \times X(N))/\operatorname{Aut}(\beta_N)$.

- Remarks: 1) The morphism β_N is a variant of the "basic construction" of [FK].
2) We have that $\deg(\beta_N) = |\operatorname{Aut}(\beta_N)|$ and that

$$\operatorname{Aut}(\beta_N) \; \simeq \; \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\} \rtimes \mathbb{Z}/2\mathbb{Z}.$$

In particular, $|\operatorname{Aut}(\beta_N)| = |\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})|$, if $N \geq 3$.

6. The Modular Construction: Step 1 (cont'd)

- Remarks: 3) The morphism β_N is constructed by using the modular interpretation of the curve $X(N)$, i.e., the fact that $X(N)$ represents the functor $\mathcal{X}(N)$ which classifies isomorphism classes of elliptic curves with (symplectic) level-N-structure. In particular,

$$\mathcal{X}(N)(K) = \{\langle E/K, \alpha \rangle\},$$

where E/K is an elliptic curve and

$$\alpha : E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$$

is a level-N-structure (of fixed determinant), and $\langle \cdot, \cdot \rangle$ denotes the isomorphism class of the pair $(E/K, \alpha)$.

6. The Modular Construction: Step 2

- **Step 2: The Modular Curve $X_{A,N}$**
- Notation: For $d \geq 1$, let $\mathcal{M}_d$ denote the set of primitive matrices of determinant d, so

$$\mathcal{M}_d = \Gamma(1)\alpha_d\Gamma(1), \quad \text{where } \Gamma(1) = \mathrm{SL}_2(\mathbb{Z}), \alpha_d = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right).$$

Moreover, for $A \in \mathcal{M}_d$ and $N \geq 1$, let $\mathcal{X}_{A,N}$ denote the moduli functor (or moduli problem) given by

$$\mathcal{X}_{A,N}(K) \;=\; \{\langle E_1/K, \alpha_1; E_2/K, \alpha_2; h\rangle\},$$

where $\langle E_i/K, \alpha_i\rangle \in \mathcal{X}(N)(K)$ for $i = 1, 2$, and $h : E_1 \to E_2$ is a cyclic isogeny of degree $d = \det(A)$ such that

$$\alpha_2 \circ h_{|E_1[N]} \;=\; [A]_N \circ \alpha_1,$$

where $[A]_N \in \mathrm{End}((\mathbb{Z}/N\mathbb{Z})^2)$ is defined by the matrix $A \,(\mathrm{mod}\, N)$ (via the canonical basis of $(\mathbb{Z}/N\mathbb{Z})^2$).

6. The Modular Construction: Step 2 (cont'd)

- Proposition 2: If $\mathrm{char}(K) \nmid Nd$ and $N \geq 3$, then the functor $\mathcal{X}_{A,N}$ is represented by an irreducible smooth affine curve

$$X_{A,N} \simeq X_0(N,d)_{/K}.$$

6. The Modular Construction: Step 3

- **Step 3: The Modular Correspondence $T_{A,N}$**
- Notation: Define the forget map

$$\tau_{A,N} : \mathcal{X}_{A,N} \to \mathcal{X}(N) \times \mathcal{X}(N)$$

by the rule

$$\tau_{A,N}(\langle E_1/K, \alpha_1; E_2/K, \alpha_2; h\rangle\}) = \langle E_1/K, \alpha_1; E_2/K, \alpha_2\rangle\}.$$

By the modular interpretation, this induces a morphism

$$\tau_{A,N} : X_{A,N} \to \ X(N) \times X(N).$$

- Proposition 3: $\tau_{A,N} : X_{A,N} \to T_{A,N} := \tau_{A,N}(X_{N,A})$ is the normalization of $T_{A,N}$.
- Remark: The curve $T_{A,N} \subset X(N) \times X(N)$ is the modular correspondence associated to the double coset $\Gamma(N)A\Gamma(N)$.

6. The Modular Construction: Step 4

- **Step 4: The Morphism $\beta_{A,N}$**
- Notation: For $A \in \mathcal{M}_d$ and $N \geq 1$, let $\beta_{A,N}$ be the composition

$$\beta_{A,N} := \beta_N \circ \tau_{A,N} : X_{A,N} \to X(N) \times X(N) \to H_{N^2}.$$

- Theorem 6: Let $A = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathcal{M}_d$, and put $A' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) A = \left(\begin{smallmatrix} x & y \\ -z & -w \end{smallmatrix}\right)$ and $m := N/\gcd(\mathrm{tr}(A'), y, z, N)$. Then

$$\beta_{A,N}(X_{A,N}) \ = \ H(q_{A',N}),$$

where

$$q_{A',N} \ = \ [N^2, 2m\,\mathrm{tr}(A'), m^2(\mathrm{tr}(A')^2 + 4d)/N^2].$$

- Remark: The proof uses the computations from [ESC].

6. The Modular Construction: Step 4 (cont'd)

- **Corollary:** If $q \in Q(N^2)$, then $\exists d \geq 1$ and a matrix $A \in \mathcal{M}_d$ such that $H(q) = \beta_{A,N}(X_{A,N}) = \beta_N(T_{A,N})$. In particular, $H(q)$ is an irreducible curve, provided that $\mathrm{char}(K) \nmid Nd$.
- **Remark:** This follows from Theorem 6 together with:
- **Lemma:** If $q \in Q(N^2)$, then $\exists!\, m|N$ and $d \geq 1$ such that

$$\mathrm{disc}(q) = -16m^2d \quad \text{and} \quad (N/m, d) = 1.$$

Moreover, there is a matrix $A \in \mathcal{M}_d$ such that

$$q \;\sim\; q_{A',N}, \quad \text{where } A' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A.$$

- **Remarks:** 1) Corollary $\Rightarrow$ Theorem 4 $\Rightarrow$ Theorem 1.
  2) If $\mathrm{char}(K) = 0$, then the above results show that

$$\{H(q) : q \in Q(N^2)\} \;=\; \{\beta_N(T_{A,N}) : A \in \bigcup_{d \geq 1} \mathcal{M}_d\}.$$

7. Appendix: The basic construction β_N

- Modular description: Define the morphism (of functors)

$$\beta_N : \mathcal{X}(N) \times \mathcal{X}(N) \;\to\; \mathcal{A}_2$$

as follows. Let $\langle E_i/K, \alpha_i \rangle \in \mathcal{X}(N)(K)$, where $i = 1, 2$, and put

$$\psi := \alpha_2^{-1} \circ \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)_N \circ \alpha_1 : E_1[N] \to E_2[N].$$

Let

$$\pi_\psi : E_1 \times E_2 \;\to\; A_\psi := (E_1 \times E_2)/\mathrm{Graph}(-\psi)$$

be the quotient map. Since $\psi : E_1[N] \to E_2[N]$ is an anti-isometry, there is a unique p.p. $\lambda_\psi : A_\psi \to \hat{A}_\psi$ such that

$$\hat{\pi}_\psi \circ \lambda_\psi \circ \pi_\psi \;=\; N(\lambda_{E_1} \otimes \lambda_{E_2}),$$

where $\lambda_{E_1} \otimes \lambda_{E_2}$ is the product polarization on $E_1 \times E_2$.

7. The basic construction - 2

- Thus: $(A_\psi, \lambda_\psi) \in \mathcal{A}_2(K)$, and so the rule

$$\beta_{N,K}(\langle E_1/K, \alpha_1; E_2/K, \alpha_2\rangle) \;=\; \langle A_\psi, \lambda_\psi\rangle$$

defines a map

$$\beta_{N,K} : (\mathcal{X}(N) \times \mathcal{X}(N))(K) \;\to\; \mathcal{A}_2(K).$$

Since this map is compatible with base-change (and extends to all K-schemes S), we obtain the desired morphism

$$\beta_N = \{\beta_{N,S}\}_S : \mathcal{X}(N) \times \mathcal{X}(N) \;\to\; \mathcal{A}_2$$

8. References

[AP] R. Accola, E. Previato, Covers of Tori: Genus 2. *Letters for Math. Phys.* **76** (2006), 135–161.

[FK] G. Frey, E.K., Curves of genus 2 and associated Hurwitz spaces. *Contemp. Math.* **487** (2009), 33–81.

[ECAS] E. K., Elliptic curves on abelian surfaces. *Manusc. math.* **84** (1994), 199–223.

[MS] E. K., The moduli spaces of Jacobians isomorphic to a product of two elliptic curves. *Collect. Math.* **67** (2016), 21–54.

[ESC] E. K., Elliptic subcovers of a curve of genus 2. Preprint, 2016, 41pp.