Hurwitz Spaces for Hyperelliptic Curve Covers

–joint work with G. Frey

1. Introduction

Motivation: We want to study curve covers

$$f : C \to \mathbb{P}^1$$

over a field K satisfying the following conditions:

(i) C is a smooth hyperelliptic curve of genus $g_C = 3$;

(ii) f has degree $\deg(f) = 4$;

(iii) f has ramification type $(2,2)^4(2,1,1)^4$;

(iv) f has monodromy group $G_f \simeq S_4$.

Tasks: 1) Find explicit equations for such curve covers.

2) Describe the Hurwitz space of such covers, i.e., determine the space which classifies equivalence classes of such covers.

Remark: 1) Covers of the above type are of interest in cryptography in connection with Ben Smith's attack on the security of hyperelliptic genus 3 curves over $\mathbb{F}_q$ (cf. G. Frey's lecture).

2) If we drop the condition "hyperelliptic" in the above hypotheses, then the answer to Task 2 can be obtained from the usual techniques of the theory of Hurwitz spaces (cf. Fried/Völklein) However, these techniques do not easily extend to include the above situation.

2. An example

Consider the polynomial

$$F(T,X) = 12TX^4 + 12T(2T-1)X^3 + (28T^2+27T-88)X^2 + 18T(2T-3)X - 3T(8T-17).$$

Facts: (i) The equation $F(T,X) = 0$ defines a smooth curve $C/\mathbb{Q}$ of genus 3 which has good reduction C_p at all primes $p > 5$ except for

$$p \in S_1 := \{11, 13, 17, 19, 47, 191\}.$$

(ii) The projection $(T,X) \mapsto T$ defines a cover $f : C \to \mathbb{P}^1_{\mathbb{Q}}$ of degree 4, as well as degree 4 covers $f_p : C_p \to \mathbb{P}^1_{\mathbb{F}_p}$ (for $p > 5$).

(iii) f has ramification type $(2,2)$ at $T = 0, 1, -1, 2$ and simple ramification type $(2,1,1)$ at 4 other points (over $\overline{\mathbb{Q}}$). Moreover, the same is true for f_p if $p > 5$ except for

$$p \in S_2 = S_1 \cup \{7, 31, 379\}.$$

(iv) The Galois group of F over $\mathbb{Q}(T)$ is $\mathrm{Gal}(F) \simeq S_4$, i.e., the monodromy group of f is $G_f \simeq S_4$. Moreover, the same is true for f_p if $p > 19$, except when $p \in S_2$.

Remark: By considering other examples, one can show that curve covers satisfying conditions (i)–(iv) exist over K whenever $\mathrm{char}(K) > 7$.

3. Hyperelliptic Hurwitz spaces (General Theory)

Fix: an integer $n \geq 3$ and a field K, and consider K-covers

$$f : C \to \mathbb{P}^1_K$$

satisfying the following conditions:

(i) C/K is a smooth hyperelliptic curve of genus $g_C = n - 1$;

(ii) $\deg(f) = n$, and $f \circ \omega_C \neq f$, where ω_C is the hyperelliptic involution of C.

Definition: The set $\mathcal{H}_n(K)$ of isomorphism classes of such covers is called the *Hurwitz space of hyperelliptic covers of degree* n (and of genus $n - 1$).

Rigidification: Consider the set $\mathcal{H}_n^{\mathrm{rig}}(K)$ of isomorphism classes of triples (C, f, π) with $(C, f) \in \mathcal{H}_n(K)$ and a fixed hyperelliptic cover

$$\pi : C \to \mathbb{P}^1_K.$$

Note: Since π is unique up to an automorphism of $\mathrm{Aut}(\mathbb{P}^1_K)$,

$$\mathcal{H}_n(K) \ = \ \mathrm{Aut}(\mathbb{P}^1_K)\backslash\mathcal{H}_n^{\mathrm{rig}}(K).$$

Observation: Given $(C, f, \pi) \in \mathcal{H}_n^{\mathrm{rig}}(K)$, $\exists!$ morphism

$$j_C : C \to \mathbb{P}^1_K \times \mathbb{P}^1_K \ \text{ such that } \ f = pr_1 \circ j_C, \ \pi = pr_2 \circ j_C,$$

where $pr_i : \mathbb{P}^1_K \times \mathbb{P}^1_K \to \mathbb{P}^1_K$ is the i^{th} projection map. Also:

- j_C is a closed immersion (so $C \simeq j_C(C)$);
- $D_C := j_C(C)$ is a divisor on the surface $\mathbb{P}^1_K \times \mathbb{P}^1_K$ and

$$D_C \sim D_{2,n} := 2(P \times \mathbb{P}^1_K) + n(\mathbb{P}^1_K \times P), \quad \text{for } P \in \mathbb{P}^1(K).$$

Proposition 1: The rule $(C, f, \pi) \mapsto D_C$ induces a bijection

$$\kappa_n : \mathcal{H}_n^{\mathrm{rig}}(K) \ \stackrel{\sim}{\rightarrow} \ |D_{2,n}|_K^{sm},$$

where $|D_{2,n}|_K^{sm} \subset |D_{2,n}|_K$ denotes the subset of smooth divisors in the linear system $|D_{2,n}|_K$.

Remarks: 1) Since $|D_{2,n}|_K \simeq \mathbb{P}^{3n+2}$, this means that we can identify $\mathcal{H}_n^{\mathrm{rig}}(K)$ with a non-empty, open subset of $\mathbb{P}^{3n+2}$.

2) If we fix homogeneous coordinates on $\mathbb{P}^1$, then each divisor $D \in |D_{2,n}|$ is given by an equation $F(T_0, T_1; X_0, X_1) = 0$, where F is homogeneous of degree 2 in T_0, T_1 and of degree n in X_0, X_1, i.e.,

$$F(T_0, T_1; X_0, X_1) \ = \ \sum_{i=0}^{n} \sum_{j=0}^{2} r_{ij} X_0^i X_1^{n-i} T_0^{2-j} T_1^j,$$

where $r_{ij} \in K$. For simplicity, we write this polynomial in its affine (de-homogenized) form

$$F(T, X) \ = \ \sum_{i=0}^{n} \sum_{j=0}^{2} r_{ij} X^{n-i} T^j.$$

Proposition 2: Let $C \in |D_{2,n}|$ be given by $F(T_0, T_1, X_0, X_1)$. If $\mathrm{char}(K) \neq 2$, then $C \in |D_{2,n}|^{sm}$ if and only if its discriminant

$$D_F^h(X_0, X_1) = A_1^2 - 4A_0A_2, \ \text{ where } \ A_j = \sum_{i=0}^{n} r_{ij} X_0^i X_1^{n-i},$$

is separable, i.e., D_F^h factors over $\overline{K}$ into $2n$ distinct linear factors.

4. Special hyperelliptic covers of genus 3.

Assume henceforth: $\mathrm{char}(K) \neq 2$.

Notation: Fix coordinates on $\mathbb{P}^1_K$. Let $P_\infty = (0:1)$, and write $P_a = (1:a)$ for $a \in K$. Moreover, put

$$P_{a,b} = (P_a, P_b) \in (\mathbb{P}^1 \times \mathbb{P}^1)(K), \quad \text{for } a, b \in K_\infty = K \cup \{\infty\}.$$

Furthermore, let $\mathcal{H}^{\mathrm{rig}}_{4,3}$ denote the subset of curves $C \in |D_{2,4}|^{sm}$ satisfying the following conditions:

$$\begin{array}{lrcl} (1) & f_C^*(P_0) & = & 2P_{0,\infty} + 2P_{0,0}, \\ (2) & f_C^*(P_1) & = & 2P_{1,1} + 2P_{1,\alpha}, \quad \text{for some } \alpha \in K, \alpha \neq 1 \\ (3) & f_C^*(P_{-1}) & = & 2D, \text{ for some } D \in \mathrm{Div}(C), \\ & & & D \neq P_{-1,\infty} + P_{-1,0}, D \neq 2P, \forall P. \end{array}$$

Here $f_C = (pr_1)_{|C} : C \to \mathbb{P}^1_K$ is the induced degree 4 cover.

Thus: Each $C \in \mathcal{H}^{\mathrm{rig}}_{4,3}$ is smooth of genus 3, and the cover f_C is ramified of type $(2,2)$ at the points $P_0, P_1, P_{-1} \in \mathbb{P}^1_K(K)$.

Moreover: For $t \in K \setminus \{0, 1, -1\}$, let $\mathcal{H}^{\mathrm{rig}}_{4,4,t}$ denote the subset of those $C \in \mathcal{H}^{\mathrm{rig}}_{4,3}$ which are also ramified of type $(2,2)$ at P_t :

$$(4) \quad f_C^*(P_t) = 2D_t, \text{ with } D_t \neq 2P, \text{ for any } P \in C(\overline{K}).$$

Theorem 1: The Hurwitz space $\mathcal{H}^{\mathrm{rig}}_{4,3}$ is a smooth, rational variety of dimension. More precisely, $\mathcal{H}^{\mathrm{rig}}_{4,3}$ is covered by two open subsets which are isomorphic to open subsets of $\mathbb{A}^5$.

Remark: The curves $C \in \mathcal{H}^{\mathrm{rig}}_{4,3}$ can be described explicitly in terms of their associated equations $F(T,X) = 0$.

Notation: Let

$$\mathcal{H}^*_{4,4,t} = \{C \in \mathcal{H}^{\rm rig}_{4,3,t} : P_{-1,\infty} \notin C, P_{t,\infty} \notin C\}.$$

Theorem 2: The Hurwitz space $\mathcal{H}^*_{4,4,t}$ consists of two disjoint rational components:

$$\mathcal{H}^*_{4,3,t} \;=\; \mathcal{H}^*_{4,3,t,1} \,\dot\cup\, \mathcal{H}^*_{4,3,t,2}$$

Moreover, all the covers in $\mathcal{H}^*_{4,3,t,1}$ factor over a quadratic cover, whereas in general the covers in $\mathcal{H}^*_{4,3,t,2}$ do not admit such a factorization.

Remarks: 1) A similar result should also be true for $\mathcal{H}^{\rm rig}_{4,3,t}$ (in place of $\mathcal{H}^*_{4,3,t}$), but this has not been proved yet.

2) Due to the presence of certain exceptional (lower-dimensional) subvarieties, the proof of Theorem 2 is rather complicated.

5. Explicit equations.

Notation: For $r_{01}, r_{11}, r_{12}, t \in K$, put

$$\begin{aligned} a_0 &= 1 - 2r_{01} & a_3 &= r_{01}r_{11} + r_{01}r_{12} - r_{11} \\ a_1 &= r_{12} - r_{11} & a_5 &= (1 - r_{01})t + r_{01} \\ a_2 &= r_{12} + r_{11} & a_6 &= r_{12}t + r_{11} \\ \alpha &= -\tfrac{1}{2}(r_{11} + r_{12} + 2) \end{aligned}$$

For $a_0 a_5 \neq 0$, let

$$F_1(T, X) = AX^4 + BX^3 + CX^2 + \alpha BX + \alpha^2 A,$$

in which

$$\begin{aligned} A &= A(T) = r_{01}T + (1 - r_{01})T^2, \\ B &= B(T) = r_{11}T + r_{12}T^2, \\ C &= C(T) = r_{20} + r_{21}T + (\alpha^2 + 4\alpha + 1 - r_{20} - r_{21})T^2, \end{aligned}$$

with

$$r_{20} = \frac{ta_3^2}{4a_0a_5} \quad \text{and} \quad r_{21} = \frac{4a_0(4\alpha r_{01} + (\alpha+1)^2) - a_1^2}{8a_0}.$$

Moreover, if also $dq \neq 0$, where $d = 4\alpha a_0 a_3$ and

$$q = a_2(2r_{01}a_2 + a_1(2t - 3a_5) - 2a_5 r_{11}) + 2(t - 1)r_{11}^2,$$

then put

$$\begin{aligned} F_2(T, X) &= F_1(T, X) + \tfrac{d}{q} G(T, X), \quad \text{where} \\ G(T, X) &= (c_2(1 - T^2) + a_6 T(1 - T))X^2 \\ &\quad + c_3 T(1 - T)X + c_4 T(1 - T), \end{aligned}$$

with

$$c_2 = \frac{ta_3}{a_0}, \quad c_3 = \frac{a_1 a_6}{2a_0}, \quad c_4 = -\frac{\alpha a_1 a_2 a_5 a_6}{q}.$$

Theorem 3: Let $t \in K^\bullet := K \setminus \{0, 1, -1\}$. If $C \in \mathcal{H}^*_{4,3,t,1}$, then $\exists!\ r_{01}, r_{11}, r_{12} \in K$ such that the associated equation $F_1(T, X) = 0$ gives C. Moreover, the discriminant

$$D_{F_1}(X) := A_1^2 - 4A_0A_2, \quad \text{where } A_j = \sum_{i=0}^{4} r_{ij} X^{4-i}$$

is separable of degree 8 and the following inequalities hold:

(5) $\qquad \alpha \neq 1, \quad a_1^2 \neq 16^2 a_0^2 \alpha \quad \text{and} \quad a_6^2 \neq 16 a_5^2 \alpha.$

Conversely, if $F_1(T, X)$ is as above (including (5) and the discriminant condition), then the equation $F_1(T, X) = 0$ defines a curve $C \in \mathcal{H}^*_{4,3,t,1}$.

Theorem 4: (a) Let $t \in K^\bullet$ and let $r_{01}, r_{11}, r_{12} \in K$ satisfy $a_0 a_5 dq \neq 0$ and the inequalities

(6) $\quad \alpha \neq 1,\ a_1^2 \neq 16^2 a_0^2(\alpha - \beta),\ a_6^2 \neq 16 a_5^2(\alpha - (t-1)\beta),$

where $\beta = \frac{da_6}{a_0 q}$. Then the associated equation $F_2(T, X) = 0$ defines a curve $C \in \mathcal{H}^*_{4,3,t,2}$, provided that its discriminant $D_{F_2}(X)$ is separable of degree 8.

(b) The set of curves C obtained by the equations of part (a) form an open subset $\mathcal{H}'_{4,3,t,2}$ of $\mathcal{H}^*_{4,3,t,2}$. The complement

$$\mathcal{H}''_{4,3,t,2} = \mathcal{H}^*_{4,3,t,2} \setminus \mathcal{H}'_{4,3,t,2}$$

consists of two disjoint rational varieties of dimension 2.

Remark: In our paper we give the explicit equations for the two families which describe the two components of $\mathcal{H}''_{4,3,t,2}$.

Remark: The proofs of the above theorems are very computational and use MAPLE to simplify complicated algebraic expressions. They also use the following technical fact which allows us to analyze the $(2,2)$-ramification condition.

Lemma: Let $Q(X) = AX^4 + BX^3 + CX^2 + DX + E \in K[X]$, where $A \neq 0$. The following are equivalent:

(i) $Q(X) = Aq(X)^2$, for some $q(X) = X^2 + bX + c$;

(ii) $8A^2D = B\Delta$ and $64EA^3 = \Delta^2$, where $\Delta = 4AC - B^2$.

Moreover, if this holds, then

$$b = B/(2A) \quad \text{and} \quad c = \Delta/(8A^2),$$

and so $q(X)$ has distinct roots in $\overline{K}$ if and only if

$$B^2 - 2A\Delta = 3B^2 - 8AC \neq 0.$$

6. Ramification types.

Definition: A curve cover $f : C \to C_0$ has ramification type $(e_1, \ldots, e_r)$ at $P_0 \in C_0(K)$ if $e_1 \geq \ldots, e_r \geq 1$ with $e_1 > 1$ and if there exist distinct points $P_1, \ldots, P_r \in C(\overline{K})$ such that

$$f^*(P_0) = \sum_{i=1}^{r} e_i P_i.$$

The list of ramification types of all points is called the ramification type of the cover.

Example: If $C \in \mathcal{H}^{\text{rig}}_{4,3,t}$, then the associated cover $f_C : C \to \mathbb{P}^1$ has ramification type $(2,2)$ at the points P_0, P_1, P_{-1} and $P_t \in \mathbb{P}^1(K)$.

Notation: If $F(T,X) \in K[T,X]$ is a polynomial, then let

$$D_{F,X}(T) \ = \ \text{disc}_X(D(F)) \in K[T]$$

denote the discriminant of F (viewed as a polynomial in X).

Proposition 3: If $F(T,X) = 0$ describes a curve C in $\mathcal{H}^{\text{rig}}_{4,3,t}$, then $\deg_T(D_{F,X}) = 12$ and

$$D^*_{F,X}(T) \ := \ D_{F,X}(T)/(T(T^2-1)(T-t))^2 \in K[T].$$

Moreover, f_C has ramification type $(2,2)^4(2,1,1)^4$ if and only if $D^*_{F,X}(T)$ is a separable polynomial, which is equivalent to

$$\text{disc}_T(D^*_{F,X}) \neq 0. \qquad (7)$$

Thus, the set of $C \in \mathcal{H}^{\text{rig}}_{4,3,t}$ with f_C of ramification type $(2,2)^4(2,1,1)^4$ is an open subset of $\mathcal{H}^{\text{rig}}_{4,3,t}$.

7. Monodromy groups.

Recall: By field theory, each separable cover $f : C \to C_0$ has a Galois hull

$$\tilde{f} : \tilde{C} \to C_0.$$

This a Galois cover which factors over f, i.e., $\tilde{f} = f \circ f'$, for some $f' : \tilde{C} \to C$, and which is minimal with these properties. The Galois group

$$G_f \;=\; \mathrm{Gal}(f)$$

is called the monodromy group of the cover f.

Proposition 4: Let $F(T, X) = 0$ define a curve $C \in \mathcal{H}^{\mathrm{rig}}_{4,3,t}$, and let $G_F = G_{f_C}$ be the monodromy group of the associated cover. Then the following are equivalent:

(i) $G_F \simeq D_4$ or $G_F \simeq S_4$;

(ii) $D^*_{F,X}(T)$ is not a square (in $\overline{K}(T)$).

On the other hand, if $D^*_{F,X}(T)$ is a square, then either $G_F \simeq A_4$ or $G_F \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In particular, G_F can never be a cyclic group.

Remark: A useful method for distinguishing between the D_4 and the S_4 case is to study the Lagrange resolvent (or cubic resolvent) of F.

8. The Lagrange Resolvent

Definition: The *Lagrange resolvent* of a general quartic

$$f(x) \;=\; ax^4 + bx^3 + cx^2 + dx + e$$

is the monic cubic polynomial $r_f(x)$ which is defined by

$$r_f(x) \;=\; x^3 - cx^2 + (bd - 4ae)x + a(4ce - d^2) - b^2e.$$

Remarks: 1) If f is monic, then this definition of r_f agrees with the usual definition. In general case we have (when $a \neq 0$) the relation

$$r_f(ax) \;=\; a^3 r_{\tilde{f}}(x),$$

where $\tilde{f}(x) = f(x)/a$ is the associated monic polynomial .

2) It is a remarkable and useful fact that

$$\mathrm{disc}(r_f) \;=\; \mathrm{disc}(f).$$

Proposition 5: Let $F(T, X) = 0$ define a curve $C \in \mathcal{H}^{\mathrm{rig}}_{4,3,t}$, and suppose that $D^*_{F,X}(T)$ is not a square. Then

$$G_F \simeq S_4 \;\Leftrightarrow\; r_F(X) \text{ is irreducible over } K(T).$$

Lemma: If $f(X) \in k[X]$ is an irreducible quartic of the form

$$f(X) \;=\; aX^4 + bX^3 + cX^2 + \alpha bX + \alpha^2 a,$$

then $\mathrm{Gal}_f \simeq D_4$ or $\mathrm{Gal}_f \simeq \mathbb{Z}/4\mathbb{Z}$ or $\mathrm{Gal}_f \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. $r_f(2a\alpha) = 0$, so r_f is reducible. Thus, the assertion follows from Proposition 4.11 of Hungerford's *Algebra*, p. 273.

Corollary: Let $C \in \mathcal{H}^*_{3,4,t,1}$ with associated polynomial $F_1(T, X)$. Then

$$G_{F_1} \simeq D_4 \quad \Leftrightarrow \quad D^*_{F,X} \text{ is not a square.}$$

On the other hand, if $D^*_{F,X}$ is a square, then $G_{F_1} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Theorem 5: Let $C \in \mathcal{H}'_{3,4,t,2}$ be defined by a polynomial equation $F_2(X, T) = 0$, with F_2 as in Theorem 4. Suppose that $D^*_{F_2,X}$ is separable, i.e, the discriminant condition (7) holds. Then

$$G_{F_2} \simeq S_4 \quad \Leftrightarrow \quad \alpha a_1 \neq 0.$$

Corollary: If $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > 7$, then there is a non-empty open subset $U_{3,4,t}$ of $\mathcal{H}'_{3,4,t,2}$ such that each $C \in U$ with its associated cover $f_C : C \to \mathbb{P}^1$ satisfies the conditions (i) – (iv) of the introduction.

Remark: However, $U_{3,4,t}$ is not the full (rigid) Hurwitz space of such covers because one of the two components of the complement $\mathcal{H}''_{3,4,t,2}$ also produces examples of curve covers satisfying (i) – (iv).

9. The associated (2,3)-cover.

Proposition 5: Let $f : C \to \mathbb{P}^1_K$ be a curve cover satisfying conditions (i) – (iv), and let $F(T, X) = 0$ be its defining equation. Let $r_F(T, X)$ be the Lagrange resolvent of F over $K(T)$.

(a) The curve $C_{r_F} : r_F(T, X) = 0$ is rational. If we fix a parametrization $(T(U), X(U))$ of C_{r_F}, then the rational function $T(U) \in K(U)$ defines a cubic cover

$$f_3 : \mathbb{P}^1 \to \mathbb{P}^1.$$

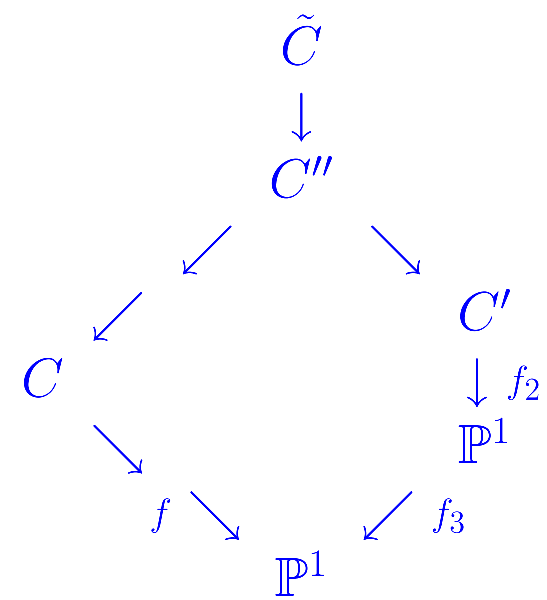
(b) Let C' be the (hyperelliptic) curve defined by the equation

$$Y^2 = X(U)^2 - 4A(T(U))E(T(U)),$$

where $A(T)$ and $E(T)$ are the highest and constant coefficients of $F(T, X)$. If $f_2 : C' \to \mathbb{P}^1$ denotes the associated hyperelliptic cover, then C' has genus 3, and the Galois hull $\tilde{f} : \tilde{C} \to \mathbb{P}^1$ factors over $f_3 \circ f_2$. Moreover, $\tilde{f}$ is also the Galois hull of $f_3 \circ f_2$.

Remarks: 1) MAPLE has a nice program which computes a parametrization of any rational plane curve $g(x, y) = 0$.

2) We thus have:



10. Connection with the attack of Ben Smith.

Given: A hyperelliptic cover $f_2 : C' \longrightarrow \mathbb{P}^1_{\mathbb{F}_q}$,

Construct: a curve $C/\mathbb{F}_q$ of genus 3 and a $(3,2)$-correspondence C'' between C and C'

$$\begin{array}{ccccc} & & C'' & & \\ & {\scriptstyle 3}\swarrow & & \searrow{\scriptstyle 2} & \\ C & & & & C' \end{array}$$

such that the induced homomorphism on the Jacobians is an isogeny:

$$T_{C''} : J_{C'} \to J_C.$$

Note: If C is NOT hyperelliptic, then the attack is successful (the cryptosystem based on C' is not secure).

Method (Donagi/Livné/Smith): Use the trigonal construction: construct a cubic (sub)cover $f_3 : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ such that $f_3 \circ f_2$ has a "special" ramification structure. Smith gives a geometric construction for obtaining C from f_3 and f_2.

Main idea (via Galois theory): The hypotheses imply that $f_6 := f_3 \circ f_2$ has monodromy group S_4. If $\tilde{f}_6 : \tilde{C} \longrightarrow \mathbb{P}^1$ is the Galois hull, then $C := \tilde{C}/S_3$ is the associated genus 3 curve.

Thus: The construction of §9 is inverse to that of Ben Smith.