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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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Abelian Varieties/ $\mathbb{C}$ and Theta-Divisors

E. Kani
Department of Mathematics and Statistics
Queens University
Jeffrey Hall
Kingston
K7L 3N6 Ontario
Canada

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MAIN BUILDING STRADA COSTIERA, 11 TEL. 22401 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224471 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 224559 TELEX 460392

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§1. Basic facts of complex tori (cf. [H1], ch. 1, §1)

Let X be a compact, connected complex Lie group of dimension g (i.e. a compact connected complex manifold of dim. g with a holomorphic group structure.

Fact 1. X is a complex torus in two ways:

a) Let $V = T_e(X) \simeq \mathbb{C}^g$ denote the tangent space of X at the identity. Then the exponential map

$$\exp : V \to X$$

induces an isomorphism of complex Lie groups:

(1) $$V/\Lambda \xrightarrow{\sim} X,$$

where $\Lambda = \mathrm{Ker}(\exp)$ is a lattice in V.

b) Let $W = H^0(X, \Omega^1)$ denote the space of holo. 1-forms. Then the period map

$$p : H_1(X, \mathbb{Z}) \to W^* = \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$$
$$\gamma \mapsto (\omega \mapsto \textstyle\int_\gamma \omega)$$

is injective, and its image is a lattice $\Lambda_1 \subset W^*$.

Thus the map

$$X \to W^*/\Lambda_1$$
$$x \mapsto (\omega \mapsto \textstyle\int_0^x \omega \mod \Lambda_1)$$

is well-defined, and one checks that this is an isomorphism. Thus one has the canonical identification

(2) $$X \xrightarrow{\sim} H^1(X, \Omega^1)^*/pH_1(X, \mathbb{Z})$$

Note that these two descriptions are inverse to each other via the canonical identification

$$T_e(X) = V = W^* = H^1(X, \Omega^1)^*$$

which is obtained by dualizing the map

(3) $$\begin{matrix} T_e(X)^* & \xrightarrow{\sim} & H^0(X, \Omega^1)_{inv} = H^0(X, \Omega^1) \\ \alpha & \mapsto & \omega_\alpha \end{matrix}$$

where ω_α denotes the translation-invariant holomorphic 1-form defined by $(\omega_\alpha)_x = T_{-x}^*(\alpha)$. Here $T_x : X \to X$ denotes the translation map $T_x(y) = x + y$.

Fact 2. $H^r(X, \mathbb{Z}) \simeq \mathrm{Alt}^r(\Lambda, \mathbb{Z}), \quad \forall r \geq 0$

Let $\pi : V \to X$ denote the projection map. Then (V, π) is clearly the universal covering space of X, and so we have

(4) $$\pi_1(X) = \Lambda \quad (\simeq \mathbb{Z}^{2g}).$$

Thus

(5a) $$H^1(X,\mathbb{Z}) = \mathrm{Hom}(\pi_1(X),\mathbb{Z}) = \mathrm{Hom}(\Lambda,\mathbb{Z}) = \mathrm{Alt}^1(\Lambda,\mathbb{Z}).$$

Furthermore, cupproduct induces a map

$$\Lambda^r H^1(X,\mathbb{Z}) \to H^r(X,\mathbb{Z})$$

which one checks to be an isomorphism by applying the Künneth formula to $\mathbb{C}^g/\Lambda \simeq (S^1)^{2g}$ (homeomorphism). Thus we obtain the identification

(5b) $$H^r(X,\mathbb{Z}) \xleftarrow{\sim} \Lambda^r \mathrm{Hom}(\Lambda,\mathbb{Z}) = \underset{\substack{\text{space of alternating } r\text{-forms} \\ \alpha:\,\underbrace{\Lambda\times\cdots\times\Lambda}_{r}\to\mathbb{Z}}}{\underset{\|}{\mathrm{Alt}^r(\Lambda,\mathbb{Z})}}$$

<u>Fact 3</u>. $\mathrm{Hom}(V/\Lambda, V'/\Lambda') = \{\lambda \in \mathrm{Hom}_{\mathbb{C}}(V,V') : \lambda(\Lambda)\subset\Lambda'\}$

Let $X' = V/\Lambda'$ be another complex torus, and consider

$\mathrm{Hom}(X,X') := \{h : X \to X'$ s.th. h is holo. with $h(0)=0\}$.

Each $h \in \mathrm{Hom}(X,X')$ induces by description 1a) a linear map $\lambda = dh \in \mathrm{Hom}_{\mathbb{C}}(V,V')$ on the tangent spaces

which, by description 1b), satisfies $\lambda(\Lambda)\subset\Lambda'$. Conversely, each $\lambda \in \mathrm{Hom}_{\mathbb{C}}(V,V')$ with $\lambda(\Lambda)\subset\Lambda'$ defines a holo. map $\bar{\lambda} : X = V/\Lambda \to V'/\Lambda'$. Since V and V' are also the universal covering spaces of X and X', it follows that $\lambda \to \bar{\lambda}$ is injective, and so we obtain the indicated equality.

In particular:

1) Every $h \in \mathrm{Hom}(X,X')$ is a group homomorphism.

2) Every holo. map $f : X \to X'$ is of the form $f(x) = h(x) + y$, where $h \in \mathrm{Hom}(X,X')$ is a homomorphism and $y = f(0)$.

Furthermore:

3) The induced map

(6) $$\begin{aligned} \rho : \mathrm{Hom}(X,X') &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\Lambda') \\ h &\longmapsto dh|_{\Lambda} \end{aligned}$$

is <u>injective</u> (since Λ contains a $\mathbb{C}$-basis of V), and so $\mathrm{Hom}(X,X')$ is free $\mathbb{Z}$-module of finite rank; in fact we have

(7) $$\mathrm{rank}_{\mathbb{Z}}\,\mathrm{Hom}(X,X') \le \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\Lambda') = 4gg'.$$

Note that the above map (6) may be (canonically!) identified with the homology map

$$H_1 \quad \mathrm{Hom}(X, X') \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_1(X,\mathbb{Z}), H_1(X',\mathbb{Z}))$$
$$f \quad \mapsto \quad H_1(f)$$

via the identifications $\Lambda = \pi_1(X) = H_1(X,\mathbb{Z})$ and $\Lambda' \quad \pi_1(X') = H_1(X'\mathbb{Z})$.

<u>Fact 4</u> $H^q(X, \Omega^p) \simeq \Lambda^p V^* \otimes \Lambda^q \overline{V^*}$, where $V = T_e(X)$, $V^* = \mathrm{Hom}_{\mathbb{C}\text{-linear}}(V, \mathbb{C})$, $\overline{V^*} = \mathrm{Hom}_{\mathbb{C}\text{-antilin}}(V, \mathbb{C})$.

The identification (3) generalizes to yield sheaf isomorphisms

(8) $\qquad \mathcal{O}_X \otimes \Lambda^p V^* \xrightarrow{\sim} \Omega^p$,

from which we obtain

(9) $\qquad H^q(X, \Omega^p) \simeq H^q(X, \mathcal{O}_X) \otimes \Lambda^p V^*$.

Much more difficult, however, is to show that

(10) $\qquad H^q(X, \mathcal{O}_X) \simeq \Lambda^q \overline{V^*}$,

where $\overline{V^*} = \mathrm{Hom}_{\mathbb{C}\text{-antilin}}(V^*, \mathbb{C})$, from which fact 4 follows in view of (9). (For the proof of (10), cf. [M], pp 4-8).

<u>In particular</u>: $H^1(X, \Omega^1) \simeq V^* \otimes \overline{V^*} = \mathrm{Herm}(V)$,

where $\mathrm{Herm}(V) = \{ H : V \times V \to \mathbb{C} : \begin{smallmatrix} H(\cdot, v) \ \mathbb{C}\text{-linear}, \\ H(v, \cdot) \ \mathbb{C}\text{-antilinear} \end{smallmatrix} \}$ denotes the space of <u>hermitian</u> <u>forms</u> on V

<u>Fact 5</u> The above isomorphisms render the following diagram commutative:

$$\begin{tikzcd}
H^n(X, \mathbb{Z}) \arrow[r, "\sim"] \arrow[d, "\alpha"'] & \mathrm{Alt}^r(\Lambda, \mathbb{Z}) = \Lambda^r \mathrm{Hom}(\Lambda, \mathbb{Z}) \arrow[d, "\Lambda^r i"] \\
H^n(X, \mathbb{C}) \arrow[r, "\sim"] \arrow[d, "\beta"'] & \Lambda^r(V^* \oplus \overline{V^*}) = \bigoplus_{p+q=n} \Lambda^p V^* \otimes \Lambda^q \overline{V^*} \arrow[d, "p"] \\
H^n(X, \mathcal{O}_X) \arrow[r, "\sim"] & \Lambda^r(\overline{V^*})
\end{tikzcd}$$

Here, α, β are the maps induced by the inclusion of sheaves $\mathbb{Z} \subset \mathbb{C} \subset \mathcal{O}_X$, and $i : \mathrm{Hom}(\Lambda, \mathbb{Z}) \hookrightarrow V^* \oplus \overline{V^*}$ is the can. inclusion. Finally, p denotes the projection onto the $p=0, q=n$ factor.

§2. Line bundles on X

To construct line bundles on $X = V/\Lambda$, let us start with the trivial line bundle $\tilde{L} = V \times \mathbb{C}$ on V. If $\tilde{L}$ admits a Λ-action of the form

$$(1) \qquad \lambda \cdot (v, z) = (v + \lambda, e_\lambda(v) \cdot z),$$

where $\lambda \in \Lambda$, $v \in V$, $z \in \mathbb{C}$ and $e_\lambda(v) \in \mathbb{C}^\times$, then we can consider the quotient

$$L(\{e_\lambda\}) = V \times \mathbb{C} / \underset{\uparrow}{\Lambda} \text{ action via (1).}$$

One easily checks:

(2) $pr_1 : L(\{e_\lambda\}) \to V/\Lambda$ is a holo. line bundle on X

$\Leftrightarrow$ $\{e_\lambda\} \in Z^1(\Lambda, H^0(V, \mathcal{O}^*))$ is a 1-cocycle,

i.e. $e_\lambda \in H^0(V, \mathcal{O}^*)$, $\forall \lambda \in \Lambda$ and we have

$$(2a) \qquad e_{\lambda+\lambda'}(v) = e_\lambda(z + \lambda') \cdot e_{\lambda'}(v), \quad \forall \lambda\lambda' \in \Lambda, \ v \in V.$$

In fact, since every line bundle on V is trivial (because $H^1(X, \mathcal{O}^*) = 0$)*), one sees easily by pulling line bundles on X back to V that every holomorphic line bundle $L \in Pic(X)$ on X arises in this way. Moreover, one checks easily that we have an isomorph.:

$$(3) \qquad H^1(\Lambda, H^0(V, \mathcal{O}^*)) \xrightarrow{\sim} H^1(X, \mathcal{O}^*) = Pic(X).$$

Here, the group on the left is the usual 1st cohomology group $H^1 = Z^1/B^1$ in group cohomology.

We now want to arrive at a convenient representation of this cohomology group. To this end, let

$$Herm(V, \Lambda) = \{ H \in Herm(V) : (\mathrm{Im}\, H)|_{\Lambda \times \Lambda} \subset \mathbb{Z} \}$$

and for a hermitian form $H \in Herm(V, \Lambda)$ let

$$Ch^{1/2}(H) = \{ \chi : \Lambda \to \mathbb{C}_1^\times \text{ s.th. } (4)_H \text{ below holds} \}$$

Here, $\mathbb{C}_1^\times = \{ z \in \mathbb{C} : |z| = 1 \}$ and the condition condition here is

$$(4)_H \qquad \chi(\lambda_1 + \lambda_2) = \chi(\lambda_1)\chi(\lambda_2)\, e(\tfrac{1}{2} E(\lambda_1, \lambda_2)),$$

---

*) Since $\forall q > 0$ $H^q(V, \mathcal{O}) = 0$ ($\bar{\partial}$-Poincaré lemma) and $H^q(V, \mathbb{Z}) = 0$ ($V \simeq \mathbb{C}^g$ contractible), it follows from the exponential sequence that $H^q(V, \mathcal{O}^*) = 0$.

where, as usual, $\mathbf{e}(z) = \exp(2\pi i z)$ and $E = \mathrm{Im}(H)$. Note that since $\mathbf{e}(\frac{1}{2}E(\lambda_1,\lambda_2)) = \pm 1$, each χ^2 is a character (when $\chi \in Ch^{1/2}(H)$), so the χs are "square roots of characters", which justifies the notation $Ch^{1/2}(H)$.

Consider now a pair (H,χ), where $H \in \mathrm{Herm}(V,\Lambda)$ and $\chi \in Ch^{1/2}(H)$. Then, as is easily checked,

$$(5) \quad e_\lambda^{(H,\chi)}(v) = \chi(\lambda)\, \mathbf{e}\left(-\tfrac{i}{2} H(v,\lambda) - \tfrac{i}{4} H(\lambda,\lambda)\right)$$

is a cocycle $\{e_\lambda^{(H,\chi)}\} \in Z^1(\Lambda, H^0(V, \mathcal{O}_V^*))$ and hence gives rise to a holomorphic line bundle

$$L(H,\chi) := L(\{e_\lambda^{(H,\chi)}\}).$$

Let

$$P = P(V,\Lambda) = \{(H,\chi) : H, \chi \text{ as above}\}$$

denote the set of such pairs. We can make P into a group via the addition law

$$(H_1,\chi_1) + (H_2,\chi_2) = (H_1 + H_2, \chi_1 \chi_2).$$

We then have:

<u>Theorem 2.1</u> (Appell-Humbert). The map $(H,\chi) \mapsto L(H,\chi)$ induces a group homomorphism

$$L : P(V,\Lambda) \xrightarrow{\sim} \mathrm{Pic}(X) = H^1(X, \mathcal{O}^*).$$

More precisely, we have the following commutative diagram with exact rows and columns

(6)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(\Lambda, \mathbb{C}_1^\times) \arrow[r, "\alpha"] \arrow[d, "\lambda"', "\wr"] & P(V,\Lambda) \arrow[r, "\beta"] \arrow[d, "L"', "\wr"] & \mathrm{Herm}(V,\Lambda) \arrow[r] \arrow[d, "\wr"', "p"] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r, "c_1"] \arrow[d] & \mathrm{Ker}(H^2(\mathbb{Z},\mathbb{Z}) \to H^2(X,\mathcal{O})) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which:

$$\mathrm{Herm}(V,\Lambda) \overset{\text{recall}}{=} \{ H \in \mathrm{Herm}(V) : (\mathrm{Im} H)(\Lambda \times \Lambda) \subset \mathbb{Z} \}$$

$$\mathrm{Pic}^0(X) = \mathrm{Ker}(c_1 : H^1(X,\mathcal{O}^*) = \mathrm{Pic}(X) \to H^2(X,\mathbb{Z}))$$

$$\alpha(\chi) = (0,\chi) \in P(V,\Lambda)$$

$$\beta(H,\chi) = \beta(H) \in \mathrm{Herm}(V,\Lambda)$$

$$\lambda(\chi) = L(0,\chi)$$

$$\rho(H) = \mathrm{Im}(H)|_{\Lambda\times\Lambda} \in \mathrm{Alt}^2(\Lambda,\mathbb{Z}) = H^2(X,\mathbb{Z})$$

In particular we have the following formula for the first Chern class of $L(H,\chi)$:

$$(7) \qquad c_1(L(H,\chi)) = E|_{\Lambda\times\Lambda} \in \mathrm{Alt}^2(\Lambda,\mathbb{Z}) \underset{\substack{\uparrow \\ \text{fact 2}}}{=} H^2(X,\mathbb{Z}),$$

where, as before, $E = \mathrm{Im}(H)$.

Remark 22 Recall that if $L \in \mathrm{Pic}(X)$ is a line bundle on a cpt complex space X, then its first Chern class is defined as

$$c_1(L) = \delta(L),$$

where

$$\delta : H^1(X,\mathcal{O}^*) = \mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$$

is the boundary map of the long exact sequence induced by the exponential sequence

$$(8) \qquad 0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{e} \mathcal{O}^* \to 0 .$$

Pf Sketch (of Appell-Humbert): Clearly, the diagram (6) commutes and has exact rows.

Using fact 5 of §1 one sees easily that ρ is an isomorphism.

To see that λ is injective, use the fact that if $f \in H^0(V,\mathcal{O}^*)$ is bounded, then f is constant.

The surjectivity of λ follows by a suitable diagram chase and observing that the map $H^1(X,\mathbb{C}) \to H^1(X,\mathcal{O}_X)$ is surjective (cf. fact 5).

Since ρ and λ are isomorphisms, and the rows are exact, it follows that L is also an isomorphism.

The Theorem of Appell-Humbert has many consequences.

1. The dual torus $\hat{X} = \mathrm{Pic}^0(X)$

In the course of proving the A-H Theorem we had established the isomorphism

$$\lambda \quad \mathrm{Hom}(\Lambda, \mathbb{C}_1^\times) \xrightarrow{\sim} \mathrm{Pic}^0(X).$$

Note that $\hat{X} = \mathrm{Hom}(\Lambda\ \mathbb{C}_1^\times)$ is itself a complex torus (also of dimension g), so the group $\mathrm{Pic}^0(X)$ carries a natural top. structure.

On the other hand, from the long exact sequence associated to the exponential sequence we obtain

$$(9) \qquad \hat{X} = \mathrm{Pic}^0(X) = \mathrm{Ker}(\delta) = \mathrm{Coker}(H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}))$$
$$= H^1(X,\mathcal{O}) / H^1(X,\mathbb{Z})$$

which shows that $\hat{X}$ is again a complex torus.

2. The theorem of the square

The pullback $T_x^* L$ of a line bundle $L \in \mathrm{Pic}(X)$ w.r.t. the translation map $T_x : X \to X$, $T_x(y) = x+y$ is given explicitly as follows:

$$(10) \qquad T_x^* L(H,\chi) \cong L(H, \mathbf{e}(E(v,)) \chi), \text{ for } v \in \pi^{-1}(x).$$

From this we see that for any $L \in \mathrm{Pic}(X)$ and $x \in X$ we have

$$(11) \qquad \phi_L(x) := T_x^*(L) \otimes L^{-1} \in \mathrm{Pic}^0(X),$$

so that ϕ_L defines a map

$$\phi_L : X \to \mathrm{Pic}^0(X) = \hat{X}.$$

The Theorem of the Square asserts that this is a homomorphism, i.e. that

$$(12) \qquad T_{x+y}^*(L) \otimes L \simeq T_x^*(L) \otimes T_y^*(L).$$

Again, this follows readily from (9) (and -H):

Write $L = L(H,\chi)$; then for $v \in \pi^{-1}(x)$, $w \in \pi^{-1}(y)$ we have:

$$T_{x+y}^*(L) \otimes L \approx L(H, \mathbf{e}(E(v+w, \cdot))\chi) \otimes L(H,\chi)$$
$$\simeq L(2H, \mathbf{e}(E(v, \cdot))\chi\ \mathbf{e}(E(w,\cdot))\chi)$$

$$= L(H, e(-E(\cdot,v))\chi)\cdot L(H, e(-E(\cdot,w))\chi)$$
$$\simeq T_x^*(L)\otimes T_y^*(L),$$ which proves (1).

We can also easily determine the kernel of ϕ_L:

$$K(L) := Ker(\Phi_L) = \{x\in X : T_x^* L \simeq L\}.$$

Indeed, since $e(E(v,\cdot))\equiv 1 \Leftrightarrow E(\ ,\ )\in\mathbb{Z}, \forall\lambda\in\Lambda$, it follows that

(13) $$K(L) = V(H)/\Lambda,$$

where $V(H)=\{v\in V : E(v,\lambda)\in\mathbb{Z}, \forall\lambda\in\Lambda\}$.
In particular we see.

(14) $K(L)$ is finite $\Leftrightarrow$ $V(H)$ is a lattice
$\Leftrightarrow$ H (or, equivalently, E) is non-degenerate.

3. The theorem of the cube

The line bundles $L(H,\chi)$ satisfy the following functoriality property: If $L = L(H,\chi)$ is a line bundle on $X' = V'/\Lambda'$ and $\bar\lambda : X \to X'$ is induced by $\lambda\in Hom_{\mathbb{C}}(V,V')$, then

(15) $$\bar\lambda^* L(H,\chi) = L(\lambda^* H, \lambda^*\chi).$$

We can use this to prove the theorem of the cube:

Th. 2.3. Given a complex space Y and holo maps $f, g, h : Y \to X$, where X is a complex torus. Then, for any $L\in Pic(X)$ we have

(16) $$(f+g+h)^*(L)\otimes(f^*(L)\otimes g^*(L)\otimes h^*(L)) \simeq (f+g)^*(L)\otimes(f+h)^*(L)\otimes(g+h)^*(L).$$

To prove this, consider the line bundle

(17) $$\mathcal{D}_n(L) = \bigotimes_{\emptyset\neq I\subset\{1,..,n\}} (m_I^* L)^{\otimes(-1)^{n+\#I}}$$

on X^n, when $m_I : X^n \to X$ is the

map $m_I(x_1, \dots, x_n) = \sum_{i \in I} x_i$. Then (16) is clearly equivalent to the assertion

$$(18) \qquad (f, g, h)^* \mathcal{D}_3(L) \simeq \mathcal{O} \quad ,$$

(where $(f, g, h) : Y \to X \times X \times X$). Now in fact we have

$$(15) \qquad \mathcal{D}_n(L) \simeq \mathcal{O}_{X^n} \quad , \quad \forall n \geqslant 3 ,$$

because for $L = L(H, \chi)$ we have $\mathcal{D}_n(L) \simeq L(\mathcal{D}_n(H), \mathcal{D}_n(\chi))$, and $\mathcal{D}_n(H)$ and $\mathcal{D}_n(\chi)$ are easily computed to be trivial. (Here, for any map $h : X' \to X$

$$\mathcal{D}_n(h) = \sum (m_I^* h),$$

and this is easily seen to be trivial.)

Remark 2.4. For the line bundle $\mathcal{D}_n(L)$ etc, cf. [M-B2], p. 12 ff.

§3. Thetafunctions

We now turn to examining the holomorphic sections of the line bundles $L = L(H, \chi)$. By general principles of quotient spaces and sections, we have a natural correspondence

$$(1) \qquad \begin{array}{ccc} H^0(X, L(H, \chi)) & = & H^0(V, V \times \mathbb{C})^\Lambda \\ f & \mapsto & \pi^* f \end{array}$$

of the space of holo sections of $L(H, \chi)$ with the space of Λ-invariant sections of $V \times \mathbb{C}$ (via the $\{e_\lambda^{(H,\chi)}\}$-action). Now we can also identify

$$H^0(V, V \times \mathbb{C}) \xrightarrow{\sim} H^0(V, \mathcal{O}) = \{\text{holo. maps } f : V \to \mathbb{C}\},$$

$$(s : V \to V \times \mathbb{C}) \mapsto f_s , \quad f_s(v) = pr_2(s(v)),$$

but this identification is incompatible with the group action. However, it is immediate that

$$s \in H^0(V, V \times \mathbb{C})^\Lambda \iff f = f_s \text{ satisfies:}$$

$$(2) \qquad f(v + \lambda) = e_\lambda^{(H,\chi)}(v) f(v) \quad , \quad \forall v \in V, \lambda \in \Lambda .$$

Thus we have a natural identification:

(3) $$H^0(X, L(H,\chi)) = Th(H,\chi),$$

where

$$Th(H,\chi) = \{\text{holo. } f : V \to \mathbb{C} \text{ satisfying } (2)\}.$$

Definition. The functions $f \in Th(H,\chi)$ are called (normalized) theta functions (with respect to (H,χ)).

Remark 3.0 If we consider more general cocycles $\{e_\lambda\} \in Z^1(\Lambda, H^0(V, \mathcal{O}^*_V))$ then an analogous assertion holds, i.e.

$$H^0(X, L(\{e_\lambda\})) = Th(\{e_\lambda\}),$$

where the space on the right denotes the space of holo functions $f : V \to \mathbb{C}$ satisfying

(2') $$f(v+\lambda) = e_\lambda(v)\, f(v), \quad \forall v \in V, \lambda \in \Lambda.$$

Such functions f are called unnormalized theta functions.

We first make some preliminary observations about $Th(H,\chi)$ (cf. [M], pp. 25 6):

1) If $R = Rad(H) = \{v \in V : H(v,w) = 0, \forall w \in V\}$
$= \{v \in V : E(v,w) = 0, \forall w \in V\}$
denotes the radical of H (or E) and $\bar{H} : \bar{V} \times \bar{V} \to \mathbb{C}$ the induced (non-degenerate!) hermitian form on $\bar{V} = V/R$, then $\bar{\Lambda} = V+R/R$ is a lattice in $\bar{V}$ and $X = \Lambda \to \mathbb{C}$ induces a map $\bar{\chi} : \bar{\Lambda} \to \mathbb{C}$ such that $(\bar{H}, \bar{\chi}) \in P(\bar{V}, \bar{\Lambda})$. Then, if $p . V \to \bar{V}$ denotes the projection map, one checks that

(4) $$\begin{array}{ccc} Th(\bar{H}, \bar{\chi}) & \xrightarrow{\sim} & Th(H,\chi) \\ f & \mapsto & p^* f \end{array}$$

is a bijection.

2) If H is not positive, then

(5) $$Th(H,\chi) = \{0\}.$$

3) By 1) and 2) we see:
If $L \simeq L(H,\chi)$ is ample, then
H is positive-definite (⇒ non-degenerate).

Theorem 3.1. A line bundle $L(H,\chi)$ is ample if and only if $H \in \mathrm{Herm}(V) = H^{1,1}(X)$ is positive-definite.

In particular, $X = V/\Lambda$ is projective ⇔ ∃ a pos. def. hermitian form H on V with $V(\Lambda\times\Lambda)\subset\mathbb{Z}$.

Pf. sketch (via Kodaira embedding theorem):
We had already seen that $L(H,\chi)$ ample ⇒ H positive. Conversely, suppose H is ample. Via our identifications (facts 4, 5) it follows that $H = c_1(L) \in H^2_{DR}(X) = H^2(X,\mathbb{C})$ defines a positive (1,1)-form. Thus $L(H,\chi)$ is a positive line bundle in the sense of Kodaira (cf. [G-H], p.148) and hence, by the Kodaira embedding theorem ([G-H], p.181), $L(H,\chi)$ is ample.

Remark 3.2. In place of using Kodaira's embedding theorem, one can also deduce Th. 3.1 from the following much more precise statement:

Theorem 3.3 (Lefschetz). Let $L \simeq L(H,\chi)$ be a line bundle such that H is positive-definite. Then $H^0(X, L^{\otimes k})$ has no base points for $k \geq 2$ and yields a projective embedding for $k \geq 3$.

This proof (will not be given) depends in part on having a suitable basis at one's disposal. Here the first step is given by

Theorem 3.4 (Riemann-Roch): If $L = L(H,\chi)$ is positive (i.e. H is positive-definite), then

$$(6) \quad \dim H^0(X,L) = \sqrt{\det(E|_{\Lambda\times\Lambda})} = \sqrt{\# K(L)},$$

where $E = \mathrm{Im}(H)$. Thus, for any $n \geq 1$ we have:

(7) $\dim H^0(X, L^{\otimes n}) = n^g \dim H^0(X,L)$.

Remark 3.5 It is possible to deduce this theorem from the Hirzebruch–Riemann–Roch theorem,

$$\chi(\mathcal{O}(L)) = \deg(ch(L)\cdot td(X)) ,$$

once one knows in addition:

1) $\omega_X \simeq \mathcal{O}_X$ ($\Rightarrow td(X) = 1$)

2) $H^q(X, \mathcal{O}(L)) = 0$, $\forall q > 0$; this follows from Kodaira's vanishing theorem since $\omega_X \simeq \mathcal{O}_X$ and L is positive

3) $c_1(L)^g = g!\sqrt{\det(E_{|\Lambda\times\Lambda})}$.

Thus, Th 3.3 is truly a "Riemann-Roch theorem". However, the proof sketched below is much more (elementary and) explicit in that a canonical basis of $H^0(X,L)$ will be constructed. Since this basis lies at the heart of the theory of theta-functions, we sketch the construction. First:

Rough outline of proof of R-R:

1) For a suitable $\chi_0 \in Ch^{\frac{1}{2}}(H)$, construct a "basic" theta-function $\vartheta_0 \in Th(H, \chi_0)$.

2) There exists $v \in V$ such that $\chi = \chi_0\, e(E(v,\cdot))$. Then the (modified) translate $\vartheta = t_v^* \vartheta_0$ lies in $Th(H,\chi)$.

3) There is a finite subgroup $K_2 \subset K(L)$ such that $\{t_v^*\vartheta\}_{v\in K_2}$ is a basis of $Th(H,\chi)$.

Remark 3.6 As we shall see, the "basic Θ function" ϑ_0 above is a suitable modification of the classical Riemann's ϑ-function which is defined as follows.

Let $T \in \mathfrak{h}_g := \{T \in M_g(\mathbb{C}) : T^t = T$ (i.e. symmetric), and $\operatorname{Im} T > 0$ (i.e. positive definite)$\}$

$\swarrow$ g×g matrices $\swarrow$ transpose be an element of the Siegel upper ½-space $\mathfrak{h}_g$. Then the Riemann ϑ-function is defined by

$$(8)\quad \vartheta(\vec z, T) = \sum_{\vec n \in \mathbb{Z}^g} \mathbf{e}\left(\tfrac{1}{2}(\vec n^t T \vec n) + \vec n^t \vec z\right), \quad \vec z \in \mathbb{C}^g.$$

(Note that since $\operatorname{Im} T > 0$, the terms of the sum are bounded by $e^{-c\vec n^t n}$ with $c > 0$, so this series converges absolutely.)

Thus, in case $g = 1$, we have $\mathfrak{h}_g = \mathfrak{h} =$ usual upper ½-plane, and

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n^2 \tau + 2nz)}$$

is precisely Jacobi's ϑ-function. It is this latter function, which Jacobi denoted by ϑ "by accident", that gives the theory its name: "theta".

As a function of $\vec z \in \mathbb{C}^g$, the transformation laws of $\vartheta(\vec z, T)$ are as follows:

$$(9a)\quad \vartheta(\vec z + \vec n, T) = \vartheta(\vec z, T)$$

$$(9b)\quad \vartheta(\vec z + T\vec m, T) = \mathbf{e}(-\tfrac{1}{2}\vec m^t T \vec m + \vec m^t \vec z)\,\vartheta(\vec z, T),$$

for $\vec n, \vec m \in \mathbb{Z}^g$, i.e.

$$(9)\quad \vartheta(\vec z + \vec n + T\vec m, T) = \mathbf{e}(-\tfrac{1}{2}\vec m^t T \vec m + \vec m^t \vec z)\,\vartheta(\vec z, T).$$

To see that this is a theta function, let:

$\Lambda = \mathbb{Z}^g + T\mathbb{Z}^g \subset \mathbb{C}^g$ (lattice)

$T = X + iY$, $X, Y \in M_g(\mathbb{R})$

$H(\vec z_1, \vec z_2) = \vec z_1^{\,t} Y^{-1} \overline{(\vec z_2)}$ ← complex conjugate, pos. def. Herm. form;

(actually: $H \in \operatorname{Herm}(\mathbb{C}^g, \Lambda)$)

$\chi_0(\vec n + T\vec m) = \mathbf{e}(\tfrac{1}{2}\vec n^t m)$; $\chi_0 \in Ch^{1/2}(H)$.

Then ϑ is "almost" in $Th(H, \chi_0)$: if we put

$$(10)\quad \vartheta_0(\vec z) = \mathbf{e}(-\tfrac{i}{4} z Y^{-1} \vec z)\,\vartheta(z, T),$$

then we have

$$(11)\quad \vartheta_0 \in Th(H, \chi_0);$$

cf. [L], p. 140.

Pf sketch of Th. 3.4 (Riemann-Roch)

Lemma 3.7 (Frobenius) Let $H \in \mathrm{Herm}(V,\Lambda)$ be positive definite. Then there exists a basis $\lambda_1, \ldots, \lambda_{2g}$ of Λ such that the (Gram) matrix of $E = \mathrm{Im} H$ w.r.t. basis is

$$(12) \qquad J_{\underline{\delta}} := \begin{pmatrix} 0 & \Delta_{\underline{\delta}} \\ -\Delta_{\underline{\delta}} & 0 \end{pmatrix},$$

where $\Delta_{\underline{\delta}} = \mathrm{diag}(\delta_1, \ldots, \delta_g)$ with $\delta_i \in \mathbb{N}$ and $\delta_1 | \delta_2 | \ldots | \delta_g$. Furthermore, the vector $\underline{\delta} = (\delta_1, \ldots, \delta_g)$ is uniquely determined by H.

Pf cf. e.g. [L], p. 92.

For a vector $\underline{\delta} = (\delta_1, \ldots, \delta_g)$ as in the lemma, put

$$K_1(\underline{\delta}) = \bigoplus_{i=1}^{g} \mathbb{Z}/\delta_i\mathbb{Z},$$

$$(13) \qquad K_1(\delta)^* = \mathrm{Hom}(K_1(\delta), \mathbb{C}^\times),$$

$$K(\delta) = K_1(\delta)^* \oplus K_1(\delta).$$

Note that $K(\delta)$ carries a unique symplectic form $\langle\, , \rangle$ defined by

$$\langle (h_1, x_1), (h_2, x_2) \rangle = h_1(x_2) - h_2(x_1).$$

Lemma 3.8 If $\{\lambda_1, \ldots, \lambda_{2g}\}$ is a basis of Λ as in Lemma 3.7, then

$$V(H) = \bigoplus_{i=1}^{g} \tfrac{1}{\delta_i}\lambda_i \oplus \bigoplus_{i=1}^{g} \tfrac{1}{\delta_i}\lambda_{g+i}$$

and hence there is a symplectic isomorphism*)

$$\lambda : K(\underline{\delta}) \to K(L) = V(H)/\Lambda.$$

Furthermore, if we put $e_i = \lambda_i / \delta_i$, $1 \leq i \leq g$, then $\{e_1, \ldots, e_g\}$ is a $\mathbb{C}$-basis of V and if we write

$$\lambda_i = \sum_{j=1}^{g} \omega_{ij} e_j$$

then the $2g \times g$-matrix $\Omega = (\omega_{ij})$ has the form

$$\Omega = \begin{pmatrix} \Delta_{\underline{\delta}} \\ T \end{pmatrix},$$

where $T \in \mathfrak{h}_g$.

---

*) Here, $K(L)$ has a symplectic structure via $\langle x,y \rangle = e(E(x,y))$

Remark 3.9 Conversely, if we fix a $\mathbb{C}$-basis $e_1, \dots, e_g$ of V and let $\underline{\delta} = (\delta_1, \dots, \delta_g)$ and T be given as in Lemma 3.8. Put

$$(14) \qquad \Omega = \begin{pmatrix} \Delta_{\underline{\delta}} \\ T \end{pmatrix} \quad \text{and} \quad \Lambda = \mathbb{Z}^{2g} \Omega \begin{pmatrix} e_1 \\ e_g \end{pmatrix} \subset V$$

Then, if we put

$$(15) \qquad H(\vec{z}, \vec{w}) = \vec{z}^{\,t} Y^{-1} \overline{(\vec{w})},$$

where $T = X + iY$, it follows that $H \in \mathrm{Herm}(V, \Lambda)$ and the above processes yield $\underline{\delta}$ and T (for a suitable choice of basis $\lambda_1, \dots, \lambda_{2g}$).

Pf sketch (cont'd).

step 1. Construction of ϑ_0.

Choose a symplectic basis $\lambda_1, \dots, \lambda_{2g}$ of Λ as in Lemma 3.7 and put

$$(16) \qquad V_1 = \bigoplus_{i=1}^{g} \mathbb{R}\lambda_i, \quad V_2 = \bigoplus_{i=1}^{g} \mathbb{R}\lambda_{g+i} \qquad (\mathbb{R}\text{-v.sp.})$$

Thus $V = V_1 \oplus V_2$. With respect to this decomposition

define $\chi_0 : \Lambda \to \mathbb{C}_1^\times$ by

$$(17) \qquad \chi_0(\lambda) = e\left(\tfrac{1}{2} E(\lambda_1, \lambda_2)\right),$$

where $\lambda = \lambda_1 + \lambda_2$ is the aforementioned decomposition.

Furthermore, let

$$(18) \qquad B = \left(H|_{V_2 \times V_2}\right) \otimes \mathbb{C},$$

and put

$$(19) \qquad \vartheta_0(v, T) = e\left(-\tfrac{i}{4} B(v, v)\right) \sum_{\lambda \in \Lambda} e\left(\tfrac{i}{2}(H-B)(v - \tfrac{1}{2}\lambda, \lambda)\right)$$

Then $\vartheta_0 \in \mathrm{Th}(H, \chi_0)$; cf. [B-L] and [M1], p.27.

step 2. $t_v^* \vartheta_0 \in L(H, \chi)$ for suitable $v \in V$.

Lemma 3.10. If $f \in \mathrm{Th}(H, \chi)$ then $w \in V$ then $t_w^* f \in \mathrm{Th}(H, \chi \cdot e(E(w, \cdot)))$, where

$$(20) \qquad (t_w^* f)(v) := e\left(\tfrac{i}{2} H(v, w)\right) f(v + w).$$

Thus, since $\chi\chi_0^{-1} : \Lambda \to \mathbb{C}_1^\times$ is a character and $E_{|\Lambda\times\Lambda}$ is non-degenerate, we can find $w = w_\chi$ s.th.

$$\chi = \chi_0\, e(E(w,\cdot))$$

and so

$$(21) \qquad \vartheta = t^*_{w_\chi}\vartheta_0 \quad \in Th(H,\chi).$$

step 3.

The Riemann-Roch theorem now follows from the following more precise result.

Theorem 3.11. Let $\vartheta \in Th(H,\chi)$ be as above, and let $\lambda : K(\underline{\delta}) \xrightarrow{\sim} K(L(H,\chi))$ be an iso. as in Lemma 3.8. Then

$$\{ t^*_{\lambda(g)}\vartheta \}_{g\in K_1(\underline{\delta})}$$

is a basis of $Th(H,\chi)$. Note that $\# K_1(\delta) = \sqrt{\# K(L)}$

Note. Once this basis has been set up properly, the proof is not difficult; cf. [M1], pp. 27-8, [GH], pp. 317-20.

Remark 3.12. By using the theory of theta groups, one can give a representation-theoretic interpretation of this basis and show that it is unique (once such an identification λ has been chosen). From this one can then, following Mumford, build up an algebraic theory of theta functions (cf. [M2] and [M3], III).

§4. Polarizations and moduli spaces

The proof of the R-R theorem of the previous section is closely related to the construction of *moduli spaces*: these are complex varieties which parametrize abelian varieties together with some additional structure such as a polarizations, we we now define.

Definition. A *polarization*[*)] of an abelian variety X is a homomorphism

$$\phi : X \to \hat{X} = \mathrm{Pic}^0(X)$$

which is of the form

$$\phi = \phi_L$$

for some *ample* line bundle $L \in \mathrm{Pic}(X)$.

Remark 4.1. 1) Since $K(L) \overset{\mathrm{Ker}(\phi_L)}{=}$ is finite and $\dim X = \dim \hat{X}$, it follows that ϕ_L is surjective.

2) If $L = L(H, X)$ and $L' = L(H', X')$ are two line bundles on X, then

$$(1) \qquad \phi_L = \phi_{L'} \Leftrightarrow L' \otimes L \in \mathrm{Pic}^0(X) \Leftrightarrow H = H'.$$

Thus, the set of polarizations $\mathrm{Pol}(X) = \{\phi_L\}$ can be canonically identified with the set

$$\mathrm{Herm}^+(V,\Lambda) = \{H \in \mathrm{Herm}(V,\Lambda) : H \text{ is pos def.}\};$$

via $H \mapsto \phi_{L(H,X)}$.

By Lemma 3.7, each H (or ϕ_L) has a *canonical* sequence $\underline{\delta} = (\delta_1, \ldots, \delta_g)$ with $\delta_1 | \ldots | \delta_g$ attached to it; we have

$$(2) \qquad \delta_1 \cdots \delta_g = \sqrt{\det E|_{\Lambda\times\Lambda}} = \sqrt{\#K(L)}.$$

This sequence is called the *type* of the polarization. Note that Lemma 3.8 gives an intrinsic characterization of this seq. in terms of $K(L) = \mathrm{Ker}(\phi_L)$.

Definition. A *polarized abelian variety* of *type* $\underline{\delta} = (\delta_1, \ldots, \delta_g)$ is a pair (X, ϕ) (or, equiv.,

---

[*)] This definition differs from that of [R], but is as in [M1]-[M2] etc.

a pair (X,H)) where X is an abelian variety and $\phi \in Pol(X)$ (resp. $H \in Herm^+(V,\Lambda)$). Two such pairs (X,ϕ) and (X',ϕ') are <u>isomorphic</u> if $\exists$ iso. $h : X \xrightarrow{\sim} X'$ s.th. $h^\# \phi' = \phi$, where $h^\#\phi' = \phi_{h^*L'}$, if $\phi' = \phi_{L'}$, $L' \in Pic(X')$ (or: $h^* H' = H$).

The <u>moduli</u> <u>space</u> of <u>polarized</u> <u>abelian</u> <u>varieties</u> of dimension g and type $\underline{\delta} = (\delta_1, \dots, \delta_g)$ is:

$$\mathcal{A}_g^{(\delta)} = \text{set of isomorphism classes of polarized ab. var's } (X,\phi) \text{ of type } \underline{\delta} \text{ and } \dim X = g.$$

By the lemmas of the previous section we see that we have a natural <u>surjective</u> map

$$\pi_\delta : \mathfrak{h}_g \twoheadrightarrow \mathcal{A}_g^{(\delta)}$$

via

$$\pi_\delta(T) = (\mathbb{C}^g/\Lambda_{\delta,T},\ H_T)$$

where $\Lambda_{\delta,T} = \mathbb{Z}^g \Delta_{\underline{\delta}} + \mathbb{Z}^g T \subset \mathbb{C}^g$ (lattice)

$H_T \in Herm^+(\mathbb{C}^g, \Lambda_{\delta,T})$ is def^d by:

$$H_T(z,w) = z^t Y^{-1} \overline{w}, \quad \text{where } T = X + iY.$$

Note that π_δ is not a bijection since in the above Lemmas 3.7-3.9 etc the basis $\{\lambda_1, \dots, \lambda_{2g}\}$ cannot be uniquely specified.

This indeterminancy may be removed by considering the action of (certain groups like) the <u>symplectic</u> <u>group</u> $Sp_{2g}(\mathbb{Z})$ acting on $\mathfrak{h}_g$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})/_{\pm I}$$

acts on $\mathfrak{h}_g$ via

check!

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}(T) = (AT+B)(CT+D)^{-1} \in \mathfrak{h}_g$$

More generally, consider the group

as in Lemma 3.7

$$\underline{\Gamma_\delta} = \{ A \in Gl(2g, \mathbb{Z})/\pm I : A^t J_{\underline{\delta}} A = J_\delta \}$$

which also acts on $\mathfrak{h}_g$ (in the same way).

Prop. 4.1 The surjection $\pi_{\underline{\delta}}$ is $\Gamma_{\underline{\delta}}$-equivariant and induces a bijection

$$\Gamma_\delta \backslash \mathfrak{h}_g \xrightarrow{\sim} A_g^{(\delta)} \quad ;$$

thus, $A_g^{(\delta)}$ may be endowed with the structure of a complex space. In particular, if $\underline{\delta} = (1, \dots, 1)$ then $\Gamma_\delta = Sp_{2g}(\mathbb{Z}) / \pm I$ and so we have

$$A_g = A_g^{(1..1)} \leftarrow Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{h}_g \ .$$

Remark. 1) A_g is called the moduli space of principally polarized abelian varieties.

2) It is more difficult to show that A_g and $A_g^{(\delta)}$ are quasi-projective. This is done by "evaluating the can. basis" (found in the previous section) at 0 ; cf. [H2], [M3] and [M4] for details.

§5. Symmetric theta divisors

It seems natural to call an effective divisor $D \geq 0$ on X a theta divisor if the pullback

$$(1) \qquad \pi^* D = (\vartheta)$$

is the divisor of zeros of some (normalized) theta function on V. However, such a definition is superfluous since we have, by our identification of theta-divisors as sections of line bundles that:

Prop. 5.1 Every effective divisor $D \geq 0$ on X is of the form (1) for a suitable ϑ.

For the purposes of these notes, let us therefore make the following (not universally accepted) definition.

Definition. A theta divisor is an effective divisor $D \geq 0$ on X such that its associated

line bundle $L = L(D)$ induces a principal polarization, i.e. an isomorphism $\phi_L : X \xrightarrow{\sim} \hat{X}$.
For a principal polarization $\phi : X \xrightarrow{\sim} \hat{X}$ (or H), let

$$\boldsymbol{\Theta}_H = \boldsymbol{\Theta}_\phi = \{ D : D \text{ theta divisor} \}.$$

Remark 5.2 1) By Riemann-Roch, a divisor $D \geq 0$ is a theta-divisor $\Leftrightarrow$ $h^0(X, L(D)) = 1$.
Thus:

2) $\boldsymbol{\Theta}_H \neq \phi$, $\forall$ polarizations $\phi : X \xrightarrow{\sim} \hat{X}$.

3) If H, H' are two princ. pol., and $\Theta_\cdot \subset \boldsymbol{\Theta}_{H}$, then:

$$(2) \quad \Theta_1 = T_x^* \Theta_2, \text{ for some } x \in X \quad \Leftrightarrow \quad L(\Theta_1) \simeq L(T_x^* L(\Theta_2)), \text{ for some } x \in X$$

equality as divisors

$$\Leftrightarrow \quad \phi_{\Theta_1} = \phi_{\Theta_2}.$$

4) If $\Theta_0 \in \boldsymbol{\Theta}_H$, then $\boldsymbol{\Theta}_H = \{ T_x^* \Theta_0 : x \in X \}$.

Of particular interest are symmetric theta divisors: these are thetadivisors Θ satisfying

$$(3) \qquad \Theta^- := i^* \Theta = \Theta \qquad \text{(equality as divisors!)}$$

where $i = i_X : X \to X$ denotes the minus map: $i(x) = -x$. Notation: $\boldsymbol{\Theta}_H^{sym} = \{ \ \in \boldsymbol{\Theta}_H : \Theta^- = \Theta \}$.

Example 5.3. Riemann's ϑ-function (cf. equation (3.8) = eqn (8) of §3). Since $\vartheta(z, T)$ a principal polarization (on $X = \mathbb{C}^g / (\mathbb{Z}^g + T\mathbb{Z}^g)$), the divisor Θ_T defined by

$$(4) \qquad \pi^* \Theta_T = (\vartheta(\cdot, T))$$

is a theta divisor. Furthermore, since $\vartheta(z, T)$ is visible an even function, i.e.

$$(5) \qquad \vartheta(-z, T) = \vartheta(z, T)$$

it follows that $(\vartheta(\cdot, T))$ and hence Θ_T is symmetric.

Prop. 5.4. For each principal polarization H on X we have:

$$(6) \qquad \# \boldsymbol{\Theta}_H^{sym} = 2^{2g} = \# X[2]$$

Moreover, if $_0 \in \boldsymbol{\Theta}_H^{sym}$, then

$$(7) \qquad \boldsymbol{\Theta}_H^{sym} = \{ T_x^* \Theta_0 : x \in X[2] \},$$

where, as usual, $X[2] = \{x \in X : 2x = 0\}$.

Pf. Suppose first that $\Theta_0 \in \Theta_H^{sym}$. Then

$$i^* T_x^* \Theta_0 = T_{-x}^* i^* \Theta_0 = T_{-x}^* \Theta_0 ,$$

so $\Theta = T_x^* \Theta_0 \in \Theta_H$ is symmetric $\Leftrightarrow T_x^* \Theta_0 = T_{-x}^* \Theta_0$ $\Leftrightarrow T_{2x}^* \Theta_0 = \Theta_0 \Leftrightarrow 2x = 0 \Leftrightarrow x \in X[2]$. This proves (7). Thus, to prove (6), it is enough to show that $\Theta_H^{sym} \neq \phi$.

First proof: Given H (and X), we can a suitable $T \in \mathfrak{h}_g$ s.th. the period matrix of $\Omega = (\mathbb{Z}^g + T\mathbb{Z}^g)$. By example 5.3, $\Theta_T \in \Theta_H^{sym}$ is then a symmetric Θ-divisor.

Second proof: Let $L = L(H, \chi_0)$ be a line bundle (for some χ_0). Then $i^* L \cong L(i^* H, i^* \chi_0) = L(H, i^* \chi_0)$. Since $(i^* \chi_0)\chi_0^{-1}$ is a character and E is non-deg., we can find $w \in V$ s.th.

$$(i^* \chi_0) \chi_0^{-1} = e(E(w, \cdot)).$$

Put $\chi = \chi_0 e(E(\frac{1}{2} w, \cdot))$; then $i^* \chi = \chi$ and so $i^* L(H, \chi) \simeq L(H, \chi)$. Then, if Θ is the divisor of a section of $L(H,\chi)$ we have $i^* L(\Theta) \simeq L(\Theta)$, so $i^* \Theta = \Theta$.

Remark 5.5. 1) The divisors in Θ_H^{sym} are often called theta characteristics (of the polarization H). Note if let $\mathcal{A}_g^{sym}$ denote the moduli space which classifies iso. classes (X, Θ), where Θ is a symmetric Θ-divisor (theta char.), then Prop. 5.4 states that the map

$$\pi : \mathcal{A}_g^{sym} \longrightarrow \mathcal{A}_g$$
$$(X, \Theta) \longmapsto (X, \phi_{L(\Theta)})$$

is a surjective cover of degree 2^{2g}.

2) Note that Θ-characteristics are not totally homogeneous, for one can distinguish between odd and even characteristics (cf. [M2]).

Example 5.6. Recall that the Jacobian $X = J_C$ of a curve C comes equipped with a symm. theta divisor Θ_T defined by the theta function $\vartheta_T(z)$ attached to (a choice of) the period lattice $\Omega = \mathbb{Z}^g + T\mathbb{Z}^g$ of C. (Note that Θ_T is symmetric!)

On the other hand, by fixing a base point $P_0 \in C$, the Abel-Jacobi map

$$A_{P_0} : C^{(g-1)} \to J_C$$
$$D \mapsto cl(D-(g-1)P_0)$$

defines a divisor $W^{g-1} = W^{g-1}_{P_0}$ (which depends on the choice of P_0). Since both divisors have the same Chern class, we see that

$$T_x^*(W^{g-1}_{P_0}) = \Theta_T$$

for some $x \in J_C$. While x will depend on the choice of T (i.e. the theta characteristic), we do have:

$$(8) \qquad \begin{aligned} 2x &= cl((2g-2)P_0 - \overset{\text{can. class}}{\omega_C}) \\ &= -A_{P_0}(\omega_C). \end{aligned}$$

(cf. [GH], p 340).

§6. Hermitian structures on line bundles and Arakelov theory

In [F], Faltings defined a canonical, pos. def. hermitian metric on $L = L(\Theta)$, where Θ is a symmetric Θ-divisor (attached to a principal polarization) of an ab var. X, as follows:

1) If $\Theta_0 = (\vartheta)$ is the divisor of a theta function $\vartheta = \vartheta(z,T)$, then put

$$(1) \qquad \|1\|_{L(\Theta_0)}(z) = \sqrt[4]{\det(Y)}\; |e(\tfrac{i}{4} H_T(z,z))\, \vartheta(z,T)|.$$

2) More generally, if $\Theta = T_x^*\Theta_0$, $x \in X[2]$ is another symm. Θ-divisor, then define by translation:

$$(2) \qquad \|1\|_{L(\Theta)}(z) = \|\vartheta_0\|(z-x).$$

As Faltings remarks, these metrics can be characterized by the property that its curvature is translation-invariant. In fact, these metrics are a special case of a theorem of Moret-Bailly ([MB1], cf. also [MB2])

Theorem 6.1. (Moret-Bailly): There is a unique way of attaching to each pair (X, L), where X is an abelian variety and L a line bundle on X, a set $\pi(X, L)$ of pos. def. hermitian C^∞-metrics on L such that:

(1) If $u : L_1 \xrightarrow{\sim} L_2$ is an isomorphism, then $u(\pi(X, L_1)) = \pi(X, L_2)$.

(2) $\pi(X, X \times \mathbb{C})$ = set of constant metrics

(3) $\pi(X, L_1) \otimes \pi(X, L_2) \subset \pi(X, L_1 \otimes L_2)$.

(4) If $f : X_1 \to X_2$ is a morphism, $L_2 \in \mathrm{Pic}(X_2)$ then $f^* \pi(X_2, L_2) \subset \pi(X_1, f^* L_2)$.

Moreover, each $\pi(X, L) \neq \phi$, and if $\rho \in \pi(X, L)$ then $\pi(X, L) = \{\lambda \rho : \lambda \in \mathbb{R}_+\}$. Furthermore, $\pi(X, L) = \{\rho :$ its curvature K_ρ is translation invariant$\}$.

Pf. [MB1], pp. 50-52; cf. also [MB2], p. 48 ff.

Rmk. This characterization is analogous to Néron's char. of ht functions.

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