# Subcovers of Curves and Moduli Spaces

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# Outline

# 1. Introduction

Many of Bob Accola's publications concern the topics

- Automorphisms and subcovers of curves.

More precisely, his main interest focused on the question of characterizing:

- the set of curves of genus $g$ with special automorphisms

inside the moduli space

$$M_g = \{\text{isomorphism classes } \langle X \rangle \text{ of curves of genus } g\}.$$

(Actually, Accola formulated his results in terms of the Teichmüller space $T_g$ in place of its quotient $M_g = \Gamma_g \backslash T_g$, but the description is the same.)

# 1. Introduction - 2

- In his later papers, he also considered questions about subcovers of curves which are not Galois.
- For example, in his joint paper with Emma Previato they discussed subspaces of $M_2$ defined by (non-Galois) elliptic subcovers in some detail.
- The purpose of this talk is to discuss some of his results and those of other authors which are related to those of Bob.

# 2. Automorphisms of curves

Throughout, $X$ denotes a projective complex curve (or compact Riemann surface) of genus $g \geq 2$. Recall:

- (Schwarz, 1879) $|\mathrm{Aut}(X)| < \infty$.

**Theorem 1.**
*Let* $g \geq 2$ *and put* $N(g) = \max_{X \in M_g} |\mathrm{Aut}(X)|$. *Then:*

(a) (Hurwitz, 1891) $N(g) \leq 84(g-1)$.

(b) (Accola, 1968; Maclachlan, 1969) $N(g) \geq 8(g+1)$.

**Theorem 2.**
(a) (Macbeath, 1961) $\exists_\infty g \geq 2$ *such that* $N(g) = 84(g-1)$.

(b) (Accola, 1968; Maclachlan, 1969) $\exists_\infty g \geq 2$ *such that* $N(g) = 8(g+1)$.

# 2. Automorphisms of curves – 2

**Question:** How can we characterize the set of curves with non-trivial automorphisms as a subset of $M_g$?

**Theorem 3** (Rauch, 1962). *If* $g_X = g \geq 4$, *then*

$$|\,\mathrm{Aut}(X)| \neq 1 \;\Leftrightarrow\; \langle X \rangle \in M_g^{sing}$$

**Remarks:** 1) There is a similar (but more complicated) description of $M_g^{sing}$ when $g = 2, 3.$

2) Since $M_g$ is normal and irreducible of dimension $3g - 3$, we see that $\dim M_g^{sing} \leq 3g - 1$.

**Question:** What is the exact dimension of $M_g^{sing}$? What are its irreducible components?

# 2. Automorphisms of curves – 3

This question is partially answered by:

**Theorem 4** (Kravetz, 1959; Baily, 1962; Kuribayashi, 1966). *Fix a finite group* $G$*, and integers* $g', r \geq 0$*. Then the set*

$$M_g(G; g', r) = \{\langle X \rangle \in M_g : G \leq \mathrm{Aut}(X), g_{X/G} = g', r = |Br(\pi_G)|\}$$

*is a closed subset of* $M_g$*. Here* $\pi_G : X \to X/G$ *denotes the quotient map, and* $Br(\pi_G) \subset X/G$ *the branch locus of* $\pi_G$*. Moreover,*

$$\dim M_g(G; g', r) = 3(g' - 1) + r,$$

*provided that* $M_g(G; g', r) \neq \emptyset$*.*

# 2. Automorphisms of curves – 4

**Remarks.** 1) Kuribayashi only treats the case that $G = \mathbb{Z}/p\mathbb{Z}$, where $p$ is prime. In this case $r = 2(g - pg')/(p-1) + 2$.

2) The hyperelliptic locus is $H_g = M_g(\mathbb{Z}/2\mathbb{Z}; 0; 2g+2)$, which has dimension $2g - 1$ by Theorem 4.

**Accola:** called the sets $M_g(G; g', r)$ G-loci and studied these:

- for $G = \mathbb{Z}/2\mathbb{Z}$ and $g$ arbitrary;
- for $G = (\mathbb{Z}/2\mathbb{Z})^t$, $t \leq 4$ and $g = 3$ or $g = 5$;
- for $G = \mathbb{Z}/3\mathbb{Z}$ and $g = 2$ or $g \geq 5$.

**Accola's aim:** to find relations between the G-loci and the $\theta$-loci. The latter are defined by the vanishing (to a certain order) of the theta-null values.

$\Leftrightarrow$ the existence of special $\frac{1}{2}$-canonical divisors.

# 2. Automorphisms of curves – 5

**Remark.** One consequence of Bob's work was that he disproved the so-called $g-2$ conjecture (for $g=5$) and offered an alternative.

**Key Tools:** 1) A theorem of Castelnuovo (1893) on special divisors.

2) A relation between the genera of intermediate covers of $\pi_G : X \to X/G$ when $G$ is non-cyclic. (Accola, 1970, 1971).

**Remark.** Accola's relations were generalized by me in 1985. After seeing this result, Bob suggested to Mike Rosen and me that these relations should hold on the level of Jacobians, and this turned out to be true; cf. K-Rosen, 1989. We asked him to be a joint author, but unfortunately he declined.

# 3. Subcovers of curves

**Definition.** A *subcover* of $X$ is a finite morphism $f : X \to X'$, where $X'$ is a curve of genus $g' \geq 0$. Its genus is $g' = g_{X'}$. Two subcovers $f_i : X \to X'_i$ are *equivalent* if $f_1 = \varphi \circ f_2$, for some isomorphism $\varphi : X'_2 \xrightarrow{\sim} X'_1$.

**Remark.** There is a 1-1 correspondence between the set of equivalence classes of subcovers of $X$ and the set of subfields of the function field $F = \mathcal{M}(X)$ which properly contain $\mathbb{C}$.

A partial generalization of Schwarz's theorem is:

**Theorem 5.** (a) (de Franchis, 1913) *There are only finitely many (equivalence classes of) subcovers of $X$ of genus $g' \geq 2$.*

(b) (Tamme, 1972) *There are only finitely many equivalence classes of subcovers of $X$ of genus $g' \geq 1$ and of bounded degree.*

**Remark.** It can be shown that the number $N(X)$ of equivalence classes of subcovers of genus $\geq 2$ satisfies $N(X) < 3^{g^2}$ (K., 1986).

# 3. Subcovers of curves - 2

The $G$-loci of $M_g$ were generalized by H. Lange as follows. Put:

$$M_g(g', n) = \{\langle X \rangle \in M_g : \exists f : X \to X', g_{X'} = g', \deg(f) = n\}.$$

**Theorem 6** (Lange, 1977). *Let* $g \geq 2$ *and* $n \geq 2$*.*
(a) *If either* $g' = 1$ *or* $g > g' > 1$ *and* $n \leq \frac{g-1}{g'-1}$*, then* $M_g(g', n)$ *is a closed equidimensional subset of* $M_g$ *of dimension*

$$\dim M_g(g', n) = 2g - 2 - (2n - 3)(g' - 1).$$

*Moreover,* $M_g(g', n) = \emptyset$ *when* $n > \frac{g-1}{g'-1}$.

(b) *The set* $M_g(0, n)$ *is a constructible subset of* $M_g$*, and* $\bigcup_{k=2}^{n} M_g(0, k)$ *is a closed subset of* $M_g$*. Moreover,*

$$\dim M_g(0, n) \leq 2g - 2 + (2n - 3).$$

# 3. Subcovers of curves - 3

**Remarks.** 1) It is clear that the G-loci are contained in Lange's loci. Indeed, $M_g(G; g', r) \subset M_g(g', |G|)$.

2) Lange also considers the sets

$$M_g(g', n, r) = \{\langle X \rangle \in M_g : \exists f : X \to X', g_{X'} = g', \\ \deg(f) = n, |Br(f)| = r\}.$$

He proves that they are constructible subsets of $M_g$ of dimension

$$\dim M_g(g', n, r) = r + 3(g' - 1),$$

whenever $M_g(g', n, r) \neq \emptyset$.

3) Lange uses the methods of Mumford (Geometric Invariant Theory) in his proof of these facts. He also uses the above Theorem of de Franchis/Tamme.

# 4. Hurwitz spaces

**Hurwitz spaces:** these classify isomorphism classes of curve covers $f : Z \to Y$, where $Y$ is fixed.

Their study often sheds light on $M_g$ and on the subsets $M_g(g', n)$.

**Theorem 7.** (a) (Klein, 1882; Hurwitz, 1891) *The Hurwitz set*

$$H^s_{g,n} = \{\langle f \rangle : f : X \to \mathbb{P}^1 \text{ is simply branched, } g_X = g, \deg(f) = n\}$$

*has a natural structure of a connected complex manifold of dimension*

$$\dim H^s_{g,n} = 2g - 2 + 2n, \text{ if } n \geq 3.$$

(b) (Fulton, 1969) $H^s_{g,n}$ *represents the Hurwitz functor of isomorphism classes of simply branched covers of* $\mathbb{P}^1$ *of type* $(g, n)$.

# 4. Hurwitz spaces - 2

**Remark.** By Fulton's result we see immediately that the "forget map" $\langle f : X \to \mathbb{P}^1 \rangle \mapsto \langle X \rangle$ defines a morphism $\mu^s_{g,n} : H^s_{g,n} \to M_g$. Since $\mu^s_{g,n}$ is surjective for $n > 2g - 2$, it follows that $M_g$ is irreducible. (This argument is due to Klein, 1882.)

**Theorem 8** (Fried/Völklein, 1991). *Let $G$ be a finite group and let $g \geq 0$ and $r \geq 2$ be integers. If the set*

$$H_g(G; r) = \{\langle f, \alpha \rangle : f : X \to \mathbb{P}^1 \text{ is a } G\text{-cover with } g_X = g, \\ |Br(f)| = r, \text{ and } \alpha : G \xrightarrow{\sim} \mathrm{Aut}(f)\}$$

*is nonempty, then it has a natural structure of a smooth complex analytic space which is equidimensional of dimension $r$.*

# 4. Hurwitz spaces - 3

**Remarks:** 1) Fried/Völklein give a description of the components of $H_g(G;r)$ in terms of the action of the so-called braid group $B_r$.

2) They also consider Hurwitz spaces of covers $f : X \to \mathbb{P}^1$ with a fixed degree $n$, fixed $r = |Br(f)|$ and fixed monodromy group $G_f$. This is includes the Klein/Hurwitz case of simple covers (where $G_f = S_n$).

3) Wewers, 1997 pointed out in his thesis that $H_g(G;r)$ represents the associated Hurwitz functor of $G$-covers when $Z(G) = 1$. In general, however, it is just a coarse moduli scheme for this functor. But this suffices to see that the "forget map" induces a morphism

$$\mu_{G;g,r} : H_g(G;r) \to M_g$$

whose image is clearly the G-locus $M_g(G;0,r)$.

# 4. Hurwitz spaces - 4

**Remark.** By comparing dimensions, we see that the above map

$$\mu_{G;g,r} : H_g(G;r) \to M_g(G;0,r) \subset M_g$$

has relative dimension 3.
To get a "Hurwitz space" that is closer to to $M_g(G;0,r)$, note that the the group $\mathrm{PGL}_2 = \mathrm{Aut}(\mathbb{P}^1)$ acts on $H_g(G;r)$ via $(\alpha, \langle f \rangle) \mapsto \langle \alpha \circ f \rangle$, and that $\mu_{G;g,r}$ is invariant under this action. Thus, $\mu_{G;g,r}$ factors over the quotient, and so we obtain an induced map

$$\overline{\mu}_{G;g,r} : \overline{H}_g(G;r) := \mathrm{PGL}_2 \backslash H_g(G;r) \to M_g(G;0,r)$$

which is generically finite. (Bertin, 1996 $\Rightarrow \overline{\mu}_{G;g,r}$ is finite.)

# 5. The case g = 2: Humbert surfaces

In their nice and very interesting joint paper (2006), Accola and Previato study many of the above loci in the case that $g = 2$. Here I want to add some further comments to their study.

One of these concerns the role of Humbert surfaces.

**Humbert surfaces:** these naturally live in the moduli space

$$A_2 = \{\langle A, \lambda_\theta \rangle\}$$

of isomorphism classes of principally polarized abelian surfaces.

**Note.** Via the Torelli map $\langle X \rangle \mapsto \langle J_X, \lambda_X \rangle$ we can (and will) view $M_2$ as a subset of $A_2$, i.e. $M_2 \subset A_2$.

Here, $J_X$ is the Jacobian surface of $X$, and $\lambda_X = \phi_{\theta_X} : J_X \xrightarrow{\sim} \hat{J}_X$ is the polarizarion induced by the theta-divisor $\theta_X$.

# 5. The case g = 2: Humbert surfaces - 2

**Theorem 9** (Humbert, 1900). *For each positive integer $n \equiv 0, 1\,(mod\,4)$, $\exists$ an irreducible surface $H_n \subset A_2$ (now called a Humbert surface) such that:*

(a) $\mathrm{End}(A) \neq \mathbb{Z} \Leftrightarrow (A, \lambda) \in H_n$, for some $n$;

(b) $M_2 = A_2 \setminus H_1$;

(c) $\exists f : X \to E, g_E = 1, \Leftrightarrow \langle J_X, \lambda_X \rangle \in H_{N^2}$, for some $N \geq 2$.

**Remarks:** 1) Each Humbert surface $H_n$ is a closed subset of $A_2$.

2) Part (c) had already been stated and proved by Biermann, 1883, but perhaps Humbert did not know this.

3) A cover $f : X \to E$ is called minimal if it doesn't factor over an isogeny of $E$. In K., 1994, property (c) was refined to:

(c′) $(J_X, \lambda_X) \in H_{N^2} \Leftrightarrow \exists f : X \to E, \deg(f) = N, f$ minimal.

# 5. The case g = 2: Humbert surfaces - 3

**Corollary.** *For any* $n \geq 2$ *we have*

$$M_2(1,n) = \bigcup_{1<N|n} H_{N^2} \cap M_2$$

*Thus,* $M_2(1,n)$ *is equidimensional of dimension 2, and has* $d(n)-1$ *irreducible components, where* $d(n) = |\{d \geq 1 : d|n\}|$.

*Proof.* Each subcover $f : X \to E$ factors as $f = h \circ f_{min}$, where $f_{min} : X \to E'$ is minimal, so the formula follows from (c'). The other assertions follow from this formula and Theorem 9, together with the fact that $H_{N^2} \cap M_2 \neq \emptyset$ when $N > 1$.

**Question ([AP]):** Is $M_2(1,n)$ always connected?

**Answer:** YES! (See below.)

# 6. The case g = 2: Hurwitz spaces

In [AP], there is a lengthy discussion of the Hurwitz spaces which are related to subspaces of $M_2$. They discuss two approaches:

**Approach 1:** via group theory (Riemann's Existence Theorem).

In this, one uses Hurwitz theory to construct covers $\overline{f} : \mathbb{P}^1 \to \mathbb{P}^1$ which, after base change with $E$, yield the desired covers $f : X \to E$; cf. Kuhn, 1988; Frey, 1995; Shaska, 2001.

**Approach 2:** The "modular approach".

Here one shows that a certain Hurwitz functor $\mathcal{H}_N$ is coarsely representable by an open subset of the modular diagonal quotient surface

$$Z_N = G_N \backslash (X(N) \times X(N)),$$

where $X(N) = \Gamma(N) \backslash \mathfrak{H}$ is the usual modular curve of level $N$ and $G_N := \Gamma(1)/(\pm\Gamma(N))$ acts in a twisted diagonal fashion on the product surface $X(N) \times X(N)$.

# 6. The case g = 2: Hurwitz spaces - 2

**Remark.** The geometry of $Z_N$ and other modular diagonal quotient surfaces was determined by Hermann, 1991 and by K.-Schanz, 1998.

**Observation.** If $f : X \to E$ is a minimal cover, then we can choose $0_E$ such that $f$ is "normalized" (i.e., the divisor $f_* W_X$ on $E$ has a certain shape, where $W_X$ is the Weierstrass divisor on $X$).

**Theorem 10** (K., 2003; Frey-K., 2009) *If $N \geq 3$, then the functor $\mathcal{H}_N$ which classifies equivalence classes of normalized (minimal) covers $f : X \to E$ of degree $N$ is coarsely represented by an open subset $U_N$ of the modular diagonal quotient surface $Z_N$.*

**Remark.** The proof of the above theorem rests on two key ideas.

1) The modular description of $Z_N$.

2) The "basic construction" of Frey-K., 1991.

# 6. The case g = 2: Hurwitz spaces - 3

**The modular description:** The surface $Z_N$ is a coarse moduli scheme for the functor $\mathcal{Z}_N$ which classifies isomorphism classes of triples $(E, E', \psi)$, where $E$ and $E'$ are elliptic curves and $\psi : E[N] \overset{\sim}{\to} E'[N]$ is an anti-isometry (wrt. the Weil-pairings).

**The basic construction:** If $(E, E', \psi)$ is a triple as above, then the abelian surface $J_\psi := (E \times E')/Graph(\psi)$ has a canonical principal polarization $\lambda_\psi$ (which is induced from the product polarization on $E \times E'$).

Moreover, if the theta-divisor $X_\psi$ of $\lambda_\psi$ is irreducible, then we have a (minimal) cover $f_\psi : X_\psi \to E$ of degree $N$, and every (minimal) cover $f : X \to E$ of degree $N$ arises this way.

# 6. The case g = 2: Hurwitz spaces - 4

**Theorem 11.** *The rule* $(E, E', \psi) \mapsto (J_\psi, \lambda_\psi)$ *defines a finite morphism*

$$\beta_N : Z_N \to A_2$$

*whose image is the Humbert surface* $H_{N^2}$. *Moreover, the normalization of* $H_{N^2}$ *is the symmetric modular diagonal quotient surface*

$$Z_N^{sym} := \langle\tau\rangle \backslash Z_N,$$

*where* $\tau \in \mathrm{Aut}(Z_N)$ *is induced from the map that interchanges the factors of* $X(N) \times X(N)$.

**Remark.** The geometry of the surfaces $Z_N^{sym}$ (for $N$ prime) was investigated by Hermann, 1992.

# 7. The refined Humbert invariant

We next want to study 1-dimensional subvarieties of $M_2$ and of $A_2$. These arise when we consider intersections of Humbert surfaces, or when we study G-loci. They are defined by considering a refinement of the Humbert invariant.

**Definition.** Let $A$ be an abelian surface with a principal polarization $\lambda$ given by $\theta \in \mathrm{Div}(A)$, i.e., $\lambda = \phi_\theta$. Put

$$\mathrm{NS}(A) := \mathrm{Div}(A)/\equiv \quad \text{and} \quad \mathrm{NS}(A,\lambda) = \mathrm{NS}(A)/\mathbb{Z}\theta.$$

The refined Humbert invariant is defined by

$$q_{(A,\lambda)}(D) = (D.\theta)^2 - 2(D.D), \quad \forall D \in \mathrm{NS}(A),$$

where $(\,.\,)$ denotes the intersection pairing on the Neron-Severi group $NS(A)$. It is easy to see that $q_{(A,\lambda)}$ defines a positive-definite quadratic form on $\mathrm{NS}(A,\lambda)$; cf. K., 1994.

# 7. The refined Humbert invariant - 2

**Remark.** If $\bar{D} \in \mathrm{NS}(A, \lambda)$ is primitive (i.e., if $\mathrm{NS}(A, \lambda)/\mathbb{Z}\bar{D}$ is torsionfree), then it was shown in K., 1994 that $n := q_{(A,\lambda)}(\bar{D})$ is the classical Humbert invariant of $A$ (which Humbert defined via the period matrix of $A$).

**Notation:** If $q : \mathbb{Z}^r \to \mathbb{Z}$ is an integral, positive-definite quadratic form in $r$ variables, then we put

$$H(q) := \{(A, \lambda) \in A_2 : q_{(A,\lambda)} \to q\}.$$

Here, the symbol $q_{(A,\lambda)} \to q$ means that $q_{(A,\lambda)}$ primitively represents $q$, i.e., there exists an injective homomorphism $h : \mathbb{Z}^r \hookrightarrow \mathrm{NS}(A, \lambda)$ such that $q_{(A,\lambda)} \circ h = q$ and such that $\mathrm{NS}(A, \lambda)/h(\mathbb{Z}^r)$ is torsionfree.

**Remark.** It follows from the above remark that $H_n = H(nx^2)$.

# 8. Generalized Humbert varieties

**Theorem 12.** *If $q$ is a positive quadratic form in $r$ variables, then the generalized Humbert variety $H(q)$ is a closed subset of $A_2$ of dimension*

$$\dim H(q) \;=\; 3 - r,$$

*provided that $H(q) \neq \emptyset$. If this is the case and if $q'$ is another positive quadratic form, then*

$$H(q) \;=\; H(q') \quad \Leftrightarrow \quad q \;\sim_{\mathrm{GL}_r}\; q'.$$

**Question:** When is $H(q) \neq \emptyset$?

**Remark.** This question can be answered completely for binary quadratic forms $q = [a, b, c]$, i.e., for

$$q(x, y) \;=\; ax^2 + bxy + cy^2.$$

# 8. Generalized Humbert varieties - 2

**Notation.** If $n, m, d \geq 1$ are integers with $(n, d) = 1$, then let

$$T(n, m, d) \;=\; \{q = [a, b, c] \in \mathbb{Z}^3 : \text{conditions (i)-(iii) below hold}\}$$

(i) $\mathrm{disc}(q) := b^2 - 4ac = -16m^2 d$;
(ii) $q \to (mn)^2$;
(iii) $q \equiv 0, 1 \pmod 4$.

**Theorem 13.** *Let $q$ be an integral binary quadratic form such that $q \to N^2$, for some $N \geq 1$. Then*

$$\begin{aligned} H(q) \neq \emptyset \quad &\Leftrightarrow \quad H(q) \text{ is an irreducible curve} \\ &\Leftrightarrow \quad q \in T(N/m, m, d), \text{ for some } m | N, d \geq 1 \\ &\qquad\qquad \text{with } (N/m, d) = 1. \end{aligned}$$

**Remark.** The hypothesis $q \to N^2$ implies that $H(q) \subset H_{N^2}$. Thus, the above theorem classifies the 1-dimensional $H(q)$' s which lie on the Humbert surface $H_{N^2}$.

# 8. Generalized Humbert varieties - 3

**Corollary:** *If* $m \equiv 0, 1\,(mod\,4)$ *and* $m, N > 1$, *then*

$$H_m \cap H_{N^2} \cap M_2 \neq \emptyset.$$

*In particular,* $M_2(1, n)$ *is connected, for all* $n \geq 2$.

*Proof.* Wlog $m > 1$. Consider $q = [N^2, 2\varepsilon N, m] \in T(1, N, \frac{m-\varepsilon}{4})$, where $\varepsilon = \mathrm{rem}(m, 4)$. Then $H(q) \neq \emptyset$ by Theorem 13. Since $q \to N^2$ and $q \to m$, we see that $H(q) \subset H_m \cap H_{N^2}$. Moreover, since $q(x, y) = (Nx + \varepsilon y)^2 + (m - \varepsilon^2)y^2 > 1$ (when $N, m > 1$), we see that $q \not\to 1$. Thus $H(q) \not\subset H_1 = A_2 \setminus M_2$, and hence $H(q) \cap M_2 \neq \emptyset$.

# 8. Generalized Humbert varieties - 4

**Application:** *Irreducible components of* $H_m \cap H_{N^2}$.

It follows immediately from the definitions that

$$H_m \cap H_n = \bigcup_{q \to m,n} H(q),$$

where the union is taken over the equivalence classes of all integral, positive definite binary quadratic forms $q$ which represent both $n$ and $m$ primitively.

The above forms $q$ can be computed by using the reduction theory of binary quadratic forms (together with Theorem 13). For example,

$$\begin{aligned} H_5 \cap H_4 &= H[1,0,4] \cup H[4,0,5] \cup H[4,4,5], \\ H_5 \cap H_9 &= H[4,0,5] \cup H[5,2,9] \cup H[5,4,8], \end{aligned}$$

# 8. Generalized Humbert varieties - 5

**Remark.** The proof of Theorem 13 rests on the fact that the $H(q)$'s can be obtained as the images of certain modular curves $T_{N,A}$ lying on the modular diagonal quotient surface $Z_N$. More precisely, there are 3 steps involved:

- For each primitive matrix $A \in M_2(\mathbb{Z})$, there is an explicit irreducible curve $T_{A,N}$ on $Z_N$ (which is induced by a modular correspondence on $X(N) \times X(N)$).
- If $A = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ is as above and $d = \det(A)$, then

  $$\beta_N(T_{A,N}) \;=\; H(q_{A,N}),$$

  where

  $$q_{A,N} \;=\; [N^2, 2m(x-w), m^2(\mathrm{tr}(A)^2 - 4yz)/N^2],$$

  so $q_{A,N} \in T(n, N/m, d)$, with $n = \gcd(\mathrm{tr}(A), y, z, N)$.
- Every $q \in T(n, N/n, d)$ is equivalent to $q_{A,N}$, for some $A$.

# 8. Generalized Humbert varieties - 6

In some cases, the birational structure of $H(q)$ is known:

**Theorem 14** (K., 2016) *If $q \in T(N,1,d)$, then the normalization $\tilde{H}(q)$ of $H(q)$ is the Fricke curve*

$$X_0^+(d) = X_0(d)/\langle w_d \rangle, \quad \textit{where} \ \ w_d = \left(\begin{smallmatrix} 0 & -1 \\ d & 0 \end{smallmatrix}\right)$$

*is the Fricke involution, except when $q$ is a (so-called) ambiguous form. In the latter case*

$$\tilde{H}(q) \simeq X_0(d)^+/\langle \alpha \rangle,$$

*for some (explicit) Atkin-Lehner involution $\alpha$.*

# 9. The G-loci for g = 2

We now come back to study the G-loci in $M_2$.

**Theorem 15.** *The G-loci in $M_2$ of dimension $\geq 1$ are all rational varieties. Explicitly, they are:*

$$\begin{aligned} M_2(C_2; 0, 6) &= M_2 \\ M_2(C_2; 1, 2) &= H_2 \\ M_2(V_4; 0, 5) &= H_2 \\ M_2(D_4; 0, 6) &= H'[4, 0, 4] \\ M_2(D_6; 0, 6) &= H'[4, 4, 4] \end{aligned}$$

*where $C_2 = \mathbb{Z}/2\mathbb{Z}, V_4 = C_2 \times C_2$, and $D_n$ is the dihedral group of order $2n$. Moreover, $H'(q) = H(q) \cap M_2$.*

# 9. The G-loci for g = 2 - 2

Remark. The curves belonging to $H_2$, $H'[4,0,4]$ and to $H'[4,4,4]$ have the following explicit equations:

(a) $y^2 = x(x-1)(x-\alpha)(x-\beta)(x-\alpha\beta)$ (Jacobi, 1832)

(b) $y^2 = x(1-x^2)(1-\kappa^2x^2)$ (Legendre, 1832)

(c) $y^2 = x^6 + ax^3 + 1$ (Bolza, 1888)

These families are also discussed in [AP], who also give the associated period matrices for these curves.