Subcovers of Curves and Moduli Spaces

Ernst Kani
Queen's University
Kingston, Ontario, Canada

Geometry at the Frontier III – Lange Fest
Universidad De La Frontera, Temuco, Chile
13 November 2018

Outline

1. Introduction

Much of Herbert Lange's early work concerns the following topic:

- Describe the set of curves $X/\mathbb{C}$ which admit a non-rational subcover.

Thus, he was interested in studying:

- the set $M_g(g', n)$ of curves X of genus g with a subcover $f : X \to Y$ of degree n to some curve Y of genus $g' \geq 1$

inside the moduli space of curves genus g, i.e.,

$$M_g = \{\text{isomorphism classes } \langle X \rangle \text{ of curves of genus } g\}.$$

This was the subject both of his dissertion (the case $g = 2$) and of his habilitation (for arbitrary $g \geq 2$), and of at least 5 publications.

1. Introduction - 2

- **Theorem 1 (Lange, 1977)** (a) If $g > g' \geq 1$ and $n \geq 2$, then the set $M_g(g', n)$ is a closed subset of M_g.
  (b) The subscheme $M_g(g', n)$ is equidimensional of dimension

  $$\dim M_g(g'n) = g - (n-2)(g'-1),$$

  provided that either $g' \geq 2$ and

  $$\frac{g+1}{g'+1} \leq n \leq \frac{g+1}{g'-1} \quad \text{or} \quad g' = 1 \text{ and } n > \frac{g+1}{2}.$$

  Moreover, $M_g(g', n) = \emptyset$ in all other cases (except possibly the case $n = \frac{g+1}{2}$ when $g' = 1$).

1. Introduction - 3

- The beautiful results of Lange naturally lead to further questions about about the geometric structure of the subschemes $M_g(g', n)$. For example:
- Questions: 1) How many irreducible components does $M_g(g', n)$ have? When is $M_g(g', n)$ irreducible?

  2) What is the "geometric type" of each irreducible component? When are they all rational or of general type?

  3) Is $M_g(g', n)$ connected? (Accola/Previato for $g = 2$)

  4) What can be said about the intersection of $M_g(g', n)$ with one or more $M_g(g'', n')$'s?
- These and other questions will be investigated in the case $g = 2$.

2. Automorphisms of curves

Throughout, X denotes a projective complex curve (or compact Riemann surface) of genus $g \geq 2$. Recall:

- (Schwarz, 1879) $|\operatorname{Aut}(X)| < \infty$.

Theorem 1.
Let $g \geq 2$ *and put* $N(g) = \max_{X \in M_g} |\operatorname{Aut}(X)|$. *Then:*

(a) (Hurwitz, 1891) $N(g) \leq 84(g-1)$.

(b) (Accola, 1968; Maclachlan, 1969) $N(g) \geq 8(g+1)$.

Theorem 2.
(a) (Macbeath, 1961) $\exists_\infty g \geq 2$ *such that* $N(g) = 84(g-1)$.

(b) (Accola, 1968; Maclachlan, 1969) $\exists_\infty g \geq 2$ *such that* $N(g) = 8(g+1)$.

2. Automorphisms of curves – 2

Question: How can we characterize the set of curves with non-trivial automorphisms as a subset of M_g?

Theorem 3 (Rauch, 1962). *If* $g_X = g \geq 4$, *then*

$$|\operatorname{Aut}(X)| \neq 1 \;\Leftrightarrow\; \langle X \rangle \in M_g^{sing}$$

Remarks: 1) There is a similar (but more complicated) description of M_g^{sing} when $g = 2, 3.$

2) Since M_g is normal and irreducible of dimension $3g - 3$, we see that $\dim M_g^{sing} \leq 3g - 1$.

Question: What is the exact dimension of M_g^{sing}? What are its irreducible components?

2. Automorphisms of curves – 3

This question is partially answered by:

Theorem 4 (Kravetz, 1959; Baily, 1962; Kuribayashi, 1966). *Fix a finite group G, and integers $g', r \geq 0$. Then the set*

$$M_g(G; g', r) = \{\langle X \rangle \in M_g : G \leq \mathrm{Aut}(X), g_{X/G} = g', r = |Br(\pi_G)|\}$$

is a closed subset of M_g. Here $\pi_G : X \to X/G$ denotes the quotient map, and $Br(\pi_G) \subset X/G$ the branch locus of π_G. Moreover,

$$\dim M_g(G; g', r) = 3(g' - 1) + r,$$

provided that $M_g(G; g', r) \neq \emptyset$.

2. Automorphisms of curves – 4

Remarks. 1) Kuribayashi only treats the case that $G = \mathbb{Z}/p\mathbb{Z}$, where p is prime. In this case $r = 2(g - pg')/(p-1) + 2$.

2) The hyperelliptic locus is $H_g = M_g(\mathbb{Z}/2\mathbb{Z}; 0; 2g+2)$, which has dimension $2g-1$ by Theorem 4.

Accola: called the sets $M_g(G; g', r)$ G-loci and studied these:

- for $G = \mathbb{Z}/2\mathbb{Z}$ and g arbitrary;
- for $G = (\mathbb{Z}/2\mathbb{Z})^t$, $t \leq 4$ and $g = 3$ or $g = 5$;
- for $G = \mathbb{Z}/3\mathbb{Z}$ and $g = 2$ or $g \geq 5$.

Accola's aim: to find relations between the G-loci and the θ-loci. The latter are defined by the vanishing (to a certain order) of the theta-null values.
$\Leftrightarrow$ the existence of special $\frac{1}{2}$-canonical divisors.

2. Automorphisms of curves – 5

Remark. One consequence of Bob's work was that he disproved the so-called $g-2$ conjecture (for $g=5$) and offered an alternative.

Key Tools: 1) A theorem of Castelnuovo (1893) on special divisors.

2) A relation between the genera of intermediate covers of $\pi_G : X \to X/G$ when G is non-cyclic. (Accola, 1970, 1971).

Remark. Accola's relations were generalized by me in 1985. After seeing this result, Bob suggested to Mike Rosen and me that these relations should hold on the level of Jacobians, and this turned out to be true; cf. K-Rosen, 1989. We asked him to be a joint author, but unfortunately he declined.

3. Subcovers of curves

Definition. A *subcover* of X is a finite morphism $f : X \to X'$, where X' is a curve of genus $g' \geq 0$. Its genus is $g' = g_{X'}$. Two subcovers $f_i : X \to X'_i$ are *equivalent* if $f_1 = \varphi \circ f_2$, for some isomorphism $\varphi : X'_2 \xrightarrow{\sim} X'_1$.

Remark. There is a 1-1 correspondence between the set of equivalence classes of subcovers of X and the set of subfields of the function field $F = \mathcal{M}(X)$ which properly contain $\mathbb{C}$.

A partial generalization of Schwarz's theorem is:

Theorem 5. (a) (de Franchis, 1913) *There are only finitely many (equivalence classes of) subcovers of X of genus $g' \geq 2$.*

(b) (Tamme, 1972) *There are only finitely many equivalence classes of subcovers of X of genus $g' \geq 1$ and of bounded degree.*

Remark. It can be shown that the number $N(X)$ of equivalence classes of subcovers of genus ≥ 2 satisfies $N(X) < 3^{g^2}$ (K., 1986).

3. Subcovers of curves - 2

The G-loci of M_g were generalized by H. Lange as follows. Put:

$$M_g(g', n) = \{\langle X \rangle \in M_g : \exists f : X \to X', g_{X'} = g', \deg(f) = n\}.$$

Theorem 6 (Lange, 1977). *Let $g \geq 2$ and $n \geq 2$.*
(a) *If either $g' = 1$ or $g > g' > 1$ and $n \leq \frac{g-1}{g'-1}$, then $M_g(g', n)$ is a closed equidimensional subset of M_g of dimension*

$$\dim M_g(g', n) = 2g - 2 - (2n - 3)(g' - 1).$$

Moreover, $M_g(g', n) = \emptyset$ when $n > \frac{g-1}{g'-1}$.

(b) *The set $M_g(0, n)$ is a constructible subset of M_g, and $\bigcup_{k=2}^{n} M_g(0, k)$ is a closed subset of M_g. Moreover,*

$$\dim M_g(0, n) \leq 2g - 2 + (2n - 3).$$

3. Subcovers of curves - 3

Remarks. 1) It is clear that the G-loci are contained in Lange's loci. Indeed, $M_g(G; g', r) \subset M_g(g', |G|)$.

2) Lange also considers the sets

$$M_g(g', n, r) = \{\langle X \rangle \in M_g : \exists f : X \to X', g_{X'} = g', \\ \deg(f) = n, |Br(f)| = r\}.$$

He proves that they are constructible subsets of M_g of dimension

$$\dim M_g(g', n, r) = r + 3(g' - 1),$$

whenever $M_g(g', n, r) \neq \emptyset$.

3) Lange uses the methods of Mumford (Geometric Invariant Theory) in his proof of these facts. He also uses the above Theorem of de Franchis/Tamme.

4. Hurwitz spaces

Hurwitz spaces: these classify isomorphism classes of curve covers $f : Z \to Y$, where Y is fixed.

Their study often sheds light on M_g and on the subsets $M_g(g', n)$.

Theorem 7. (a) (Klein, 1882; Hurwitz, 1891) *The Hurwitz set*

$$H^s_{g,n} = \{\langle f\rangle : f : X \to \mathbb{P}^1 \text{ is simply branched}, g_X = g, \deg(f) = n\}$$

has a natural structure of a connected complex manifold of dimension

$$\dim H^s_{g,n} = 2g - 2 + 2n, \text{ if } n \geq 3.$$

(b) (Fulton, 1969) $H^s_{g,n}$ *represents the Hurwitz functor of isomorphism classes of simply branched covers of* $\mathbb{P}^1$ *of type* (g, n)*.*

4. Hurwitz spaces - 2

Remark. By Fulton's result we see immediately that the "forget map" $\langle f : X \to \mathbb{P}^1 \rangle \mapsto \langle X \rangle$ defines a morphism $\mu^s_{g,n} : H^s_{g,n} \to M_g$. Since $\mu^s_{g,n}$ is surjective for $n > 2g - 2$, it follows that M_g is irreducible. (This argument is due to Klein, 1882.)

Theorem 8 (Fried/Völklein, 1991). *Let G be a finite group and let $g \geq 0$ and $r \geq 2$ be integers. If the set*

$$\begin{aligned} H_g(G; r) = \{\langle f, \alpha \rangle : & f : X \to \mathbb{P}^1 \text{ is a } G\text{-cover with } g_X = g, \\ & |Br(f)| = r, \text{ and } \alpha : G \xrightarrow{\sim} \mathrm{Aut}(f)\} \end{aligned}$$

is nonempty, then it has a natural structure of a smooth complex analytic space which is equidimensional of dimension r.

4. Hurwitz spaces - 3

Remarks: 1) Fried/Völklein give a description of the components of $H_g(G;r)$ in terms of the action of the so-called braid group B_r.

2) They also consider Hurwitz spaces of covers $f : X \to \mathbb{P}^1$ with a fixed degree n, fixed $r = |Br(f)|$ and fixed monodromy group G_f. This is includes the Klein/Hurwitz case of simple covers (where $G_f = S_n$).

3) Wewers, 1997 pointed out in his thesis that $H_g(G;r)$ represents the associated Hurwitz functor of G-covers when $Z(G) = 1$. In general, however, it is just a coarse moduli scheme for this functor. But this suffices to see that the "forget map" induces a morphism

$$\mu_{G;g,r} : H_g(G;r) \to M_g$$

whose image is clearly the G-locus $M_g(G;0,r)$.

4. Hurwitz spaces - 4

Remark. By comparing dimensions, we see that the above map

$$\mu_{G;g,r} : H_g(G;r) \to M_g(G;0,r) \subset M_g$$

has relative dimension 3.

To get a "Hurwitz space" that is closer to to $M_g(G;0,r)$, note that the the group $\mathrm{PGL}_2 = \mathrm{Aut}(\mathbb{P}^1)$ acts on $H_g(G;r)$ via $(\alpha, \langle f \rangle) \mapsto \langle \alpha \circ f \rangle$, and that $\mu_{G;g,r}$ is invariant under this action. Thus, $\mu_{G;g,r}$ factors over the quotient, and so we obtain an induced map

$$\overline{\mu}_{G;g,r} : \overline{H}_g(G;r) := \mathrm{PGL}_2 \backslash H_g(G;r) \to M_g(G;0,r)$$

which is generically finite. (Bertin, 1996 $\Rightarrow \overline{\mu}_{G;g,r}$ is finite.)

5. The case g = 2: Humbert surfaces

In their nice and very interesting joint paper (2006), Accola and Previato study many of the above loci in the case that $g = 2$. Here I want to add some further comments to their study.

One of these concerns the role of Humbert surfaces.

Humbert surfaces: these naturally live in the moduli space

$$A_2 = \{\langle A, \lambda_\theta \rangle\}$$

of isomorphism classes of principally polarized abelian surfaces.

Note. Via the Torelli map $\langle X \rangle \mapsto \langle J_X, \lambda_X \rangle$ we can (and will) view M_2 as a subset of A_2, i.e. $M_2 \subset A_2$.

Here, J_X is the Jacobian surface of X, and $\lambda_X = \phi_{\theta_X} : J_X \xrightarrow{\sim} \hat{J}_X$ is the polarizarion induced by the theta-divisor θ_X.

5. The case g = 2: Humbert surfaces - 2

Theorem 9 (Humbert, 1900). *For each positive integer* $n \equiv 0, 1\,(mod\,4)$, $\exists$ *an irreducible surface* $H_n \subset A_2$ *(now called a Humbert surface) such that:*

(a) $\mathrm{End}(A) \neq \mathbb{Z} \Leftrightarrow (A, \lambda) \in H_n$, for some n;

(b) $M_2 = A_2 \setminus H_1$;

(c) $\exists f : X \to E, g_E = 1, \Leftrightarrow \langle J_X, \lambda_X \rangle \in H_{N^2}$, for some $N \geq 2$.

Remarks: 1) Each Humbert surface H_n is a closed subset of A_2.

2) Part (c) had already been stated and proved by Biermann, 1883, but perhaps Humbert did not know this.

3) A cover $f : X \to E$ is called minimal if it doesn't factor over an isogeny of E. In K., 1994, property (c) was refined to:

(c′) $(J_X, \lambda_X) \in H_{N^2} \Leftrightarrow \exists f : X \to E, \deg(f) = N, f$ minimal.

5. The case g = 2: Humbert surfaces - 3

Corollary. *For any* $n \geq 2$ *we have*

$$M_2(1,n) = \bigcup_{1<N|n} H_{N^2} \cap M_2$$

Thus, $M_2(1,n)$ *is equidimensional of dimension 2, and has* $d(n) - 1$ *irreducible components, where* $d(n) = |\{d \geq 1 : d|n\}|$.

Proof. Each subcover $f : X \to E$ factors as $f = h \circ f_{min}$, where $f_{min} : X \to E'$ is minimal, so the formula follows from (c'). The other assertions follow from this formula and Theorem 9, together with the fact that $H_{N^2} \cap M_2 \neq \emptyset$ when $N > 1$.

Question ([AP]): Is $M_2(1,n)$ always connected?

Answer: YES! (See below.)

6. The case $g = 2$: Hurwitz spaces

In [AP], there is a lengthy discussion of the Hurwitz spaces which are related to subspaces of M_2. They discuss two approaches:

Approach 1: via group theory (Riemann's Existence Theorem).

In this, one uses Hurwitz theory to construct covers $\overline{f} : \mathbb{P}^1 \to \mathbb{P}^1$ which, after base change with E, yield the desired covers $f : X \to E$; cf. Kuhn, 1988; Frey, 1995; Shaska, 2001.

Approach 2: The "modular approach".

Here one shows that a certain Hurwitz functor $\mathcal{H}_N$ is coarsely representable by an open subset of the modular diagonal quotient surface

$$Z_N = G_N \backslash (X(N) \times X(N)),$$

where $X(N) = \Gamma(N) \backslash \mathfrak{H}$ is the usual modular curve of level N and $G_N := \Gamma(1)/(\pm\Gamma(N))$ acts in a twisted diagonal fashion on the product surface $X(N) \times X(N)$.

6. The case g = 2: Hurwitz spaces - 2

Remark. The geometry of Z_N and other modular diagonal quotient surfaces was determined by Hermann, 1991 and by K.-Schanz, 1998.

Observation. If $f : X \to E$ is a minimal cover, then we can choose 0_E such that f is " normalized" (i.e., the divisor $f_* W_X$ on E has a certain shape, where W_X is the Weierstrass divisor on X).

Theorem 10 (K., 2003; Frey-K., 2009) *If $N \geq 3$, then the functor $\mathcal{H}_N$ which classifies equivalence classes of normalized (minimal) covers $f : X \to E$ of degree N is coarsely represented by an open subset U_N of the modular diagonal quotient surface Z_N.*

Remark. The proof of the above theorem rests on two key ideas.

1) The modular description of Z_N.

2) The "basic construction" of Frey-K., 1991.

6. The case g = 2: Hurwitz spaces - 3

The modular description: The surface Z_N is a coarse moduli scheme for the functor $\mathcal{Z}_N$ which classifies isomorphism classes of triples (E, E', ψ), where E and E' are elliptic curves and $\psi : E[N] \xrightarrow{\sim} E'[N]$ is an anti-isometry (wrt. the Weil-pairings).

The basic construction: If (E, E', ψ) is a triple as above, then the abelian surface $J_\psi := (E \times E')/Graph(\psi)$ has a canonical principal polarization λ_ψ (which is induced from the product polarization on $E \times E'$).

Moreover, if the theta-divisor X_ψ of λ_ψ is irreducible, then we have a (minimal) cover $f_\psi : X_\psi \to E$ of degree N, and every (minimal) cover $f : X \to E$ of degree N arises this way.

6. The case g $= 2$: Hurwitz spaces - 4

Theorem 11. *The rule* $(E, E', \psi) \mapsto (J_\psi, \lambda_\psi)$ *defines a finite morphism*

$$\beta_N : Z_N \rightarrow A_2$$

whose image is the Humbert surface H_{N^2}*. Moreover, the normalization of* H_{N^2} *is the symmetric modular diagonal quotient surface*

$$Z_N^{sym} := \langle\tau\rangle \backslash Z_N,$$

where $\tau \in \mathrm{Aut}(Z_N)$ *is induced from the map that interchanges the factors of* $X(N) \times X(N)$.

Remark. The geometry of the surfaces Z_N^{sym} (for N prime) was investigated by Hermann, 1992.

7. The refined Humbert invariant

We next want to study 1-dimensional subvarieties of M_2 and of A_2. These arise when we consider intersections of Humbert surfaces, or when we study G-loci. They are defined by considering a refinement of the Humbert invariant.

Definition. Let A be an abelian surface with a principal polarization λ given by $\theta \in \mathrm{Div}(A)$, i.e., $\lambda = \phi_\theta$. Put

$$\mathrm{NS}(A) := \mathrm{Div}(A)/\equiv \quad \text{and} \quad \mathrm{NS}(A, \lambda) = \mathrm{NS}(A)/\mathbb{Z}\theta.$$

The refined Humbert invariant is defined by

$$q_{(A,\lambda)}(D) = (D.\theta)^2 - 2(D.D), \quad \forall D \in \mathrm{NS}(A),$$

where $(\,.\,)$ denotes the intersection pairing on the Neron-Severi group $NS(A)$. It is easy to see that $q_{(A,\lambda)}$ defines a positive-definite quadratic form on $\mathrm{NS}(A, \lambda)$; cf. K., 1994.

7. The refined Humbert invariant - 2

Remark. If $\bar{D} \in \mathrm{NS}(A, \lambda)$ is primitive (i.e., if $\mathrm{NS}(A, \lambda)/\mathbb{Z}\bar{D}$ is torsionfree), then it was shown in K., 1994 that $n := q_{(A,\lambda)}(\bar{D})$ is the classical Humbert invariant of A (which Humbert defined via the period matrix of A).

Notation: If $q : \mathbb{Z}^r \to \mathbb{Z}$ is an integral, positive-definite quadratic form in r variables, then we put

$$H(q) := \{(A, \lambda) \in A_2 : q_{(A,\lambda)} \to q\}.$$

Here, the symbol $q_{(A,\lambda)} \to q$ means that $q_{(A,\lambda)}$ primitively represents q, i.e., there exists an injective homomorphism $h : \mathbb{Z}^r \hookrightarrow \mathrm{NS}(A, \lambda)$ such that $q_{(A,\lambda)} \circ h = q$ and such that $\mathrm{NS}(A, \lambda)/h(\mathbb{Z}^r)$ is torsionfree.

Remark. It follows from the above remark that $H_n = H(nx^2)$.

8. Generalized Humbert varieties

Theorem 12. *If q is a positive quadratic form in r variables, then the generalized Humbert variety $H(q)$ is a closed subset of A_2 of dimension*

$$\dim H(q) \;=\; 3 - r,$$

provided that $H(q) \neq \emptyset$. If this is the case and if q' is another positive quadratic form, then

$$H(q) \;=\; H(q') \quad \Leftrightarrow \quad q \;\sim_{\mathrm{GL}_r}\; q'.$$

Question: When is $H(q) \neq \emptyset$?

Remark. This question can be answered completely for binary quadratic forms $q = [a, b, c]$, i.e., for

$$q(x, y) \;=\; ax^2 + bxy + cy^2.$$

8. Generalized Humbert varieties - 2

Notation. If $n, m, d \geq 1$ are integers with $(n, d) = 1$, then let

$$T(n, m, d) \;=\; \{q = [a, b, c] \in \mathbb{Z}^3 : \text{conditions (i)-(iii) below hold}\}$$

(i) $\mathrm{disc}(q) := b^2 - 4ac = -16m^2d$;
(ii) $q \to (mn)^2$;
(iii) $q \equiv 0, 1 \pmod 4$.

Theorem 13. *Let q be an integral binary quadratic form such that $q \to N^2$, for some $N \geq 1$. Then*

$$\begin{aligned} H(q) \neq \emptyset \quad &\Leftrightarrow \quad H(q) \text{ is an irreducible curve} \\ &\Leftrightarrow \quad q \in T(N/m, m, d), \text{ for some } m|N, d \geq 1 \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{with } (N/m, d) = 1. \end{aligned}$$

Remark. The hypothesis $q \to N^2$ implies that $H(q) \subset H_{N^2}$. Thus, the above theorem classifies the 1-dimensional $H(q)$' s which lie on the Humbert surface H_{N^2}.

8. Generalized Humbert varieties - 3

Corollary: *If* $m \equiv 0, 1 \,(mod\, 4)$ *and* $m, N > 1$, *then*

$$H_m \cap H_{N^2} \cap M_2 \neq \emptyset.$$

In particular, $M_2(1, n)$ *is connected, for all* $n \geq 2$.

Proof. Wlog $m > 1$. Consider $q = [N^2, 2\varepsilon N, m] \in T(1, N, \frac{m-\varepsilon}{4})$, where $\varepsilon = \mathrm{rem}(m, 4)$. Then $H(q) \neq \emptyset$ by Theorem 13. Since $q \to N^2$ and $q \to m$, we see that $H(q) \subset H_m \cap H_{N^2}$. Moreover, since $q(x, y) = (Nx + \varepsilon y)^2 + (m - \varepsilon^2)y^2 > 1$ (when $N, m > 1$), we see that $q \not\to 1$. Thus $H(q) \not\subset H_1 = A_2 \setminus M_2$, and hence $H(q) \cap M_2 \neq \emptyset$.

8. Generalized Humbert varieties - 4

Application: *Irreducible components of* $H_m \cap H_{N^2}$.

It follows immediately from the definitions that

$$H_m \cap H_n = \bigcup_{q \to m,n} H(q),$$

where the union is taken over the equivalence classes of all integral, positive definite binary quadratic forms q which represent both n and m primitively.

The above forms q can be computed by using the reduction theory of binary quadratic forms (together with Theorem 13). For example,

$$\begin{aligned} H_5 \cap H_4 &= H[1,0,4] \cup H[4,0,5] \cup H[4,4,5], \\ H_5 \cap H_9 &= H[4,0,5] \cup H[5,2,9] \cup H[5,4,8], \end{aligned}$$

8. Generalized Humbert varieties - 5

Remark. The proof of Theorem 13 rests on the fact that the $H(q)$'s can be obtained as the images of certain modular curves $T_{N,A}$ lying on the modular diagonal quotient surface Z_N. More precisely, there are 3 steps involved:

- For each primitive matrix $A \in M_2(\mathbb{Z})$, there is an explicit irreducible curve $T_{A,N}$ on Z_N (which is induced by a modular correspondence on $X(N) \times X(N)$).
- If $A = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ is as above and $d = \det(A)$, then

  $$\beta_N(T_{A,N}) \;=\; H(q_{A,N}),$$

  where

  $$q_{A,N} \;=\; [N^2, 2m(x-w), m^2(\mathrm{tr}(A)^2 - 4yz)/N^2],$$

  so $q_{A,N} \in T(n, N/m, d)$, with $n = \gcd(\mathrm{tr}(A), y, z, N)$.
- Every $q \in T(n, N/n, d)$ is equivalent to $q_{A,N}$, for some A.

8. Generalized Humbert varieties - 6

In some cases, the birational structure of $H(q)$ is known:

Theorem 14 (K., 2016) *If $q \in T(N, 1, d)$, then the normalization $\tilde{H}(q)$ of $H(q)$ is the Fricke curve*

$$X_0^+(d) = X_0(d)/\langle w_d \rangle, \quad \textit{where} \;\; w_d = \left(\begin{smallmatrix} 0 & -1 \\ d & 0 \end{smallmatrix}\right)$$

is the Fricke involution, except when q is a (so-called) ambiguous form. In the latter case

$$\tilde{H}(q) \simeq X_0(d)^+/\langle \alpha \rangle,$$

for some (explicit) Atkin-Lehner involution α.

9. The G-loci for g = 2

We now come back to study the G-loci in M_2.

Theorem 15. *The G-loci in M_2 of dimension ≥ 1 are all rational varieties. Explicitly, they are:*

$$\begin{aligned} M_2(C_2; 0, 6) &= M_2 \\ M_2(C_2; 1, 2) &= H_2 \\ M_2(V_4; 0, 5) &= H_2 \\ M_2(D_4; 0, 6) &= H'[4, 0, 4] \\ M_2(D_6; 0, 6) &= H'[4, 4, 4] \end{aligned}$$

where $C_2 = \mathbb{Z}/2\mathbb{Z}$, $V_4 = C_2 \times C_2$, and D_n is the dihedral group of order $2n$. Moreover, $H'(q) = H(q) \cap M_2$.

9. The G-loci for g = 2 - 2

Remark. The curves belonging to H_2, $H'[4,0,4]$ and to $H'[4,4,4]$ have the following explicit equations:

(a) $y^2 = x(x-1)(x-\alpha)(x-\beta)(x-\alpha\beta)$ (Jacobi, 1832)

(b) $y^2 = x(1-x^2)(1-\kappa^2x^2)$ (Legendre, 1832)

(c) $y^2 = x^6 + ax^3 + 1$ (Bolza, 1888)

These families are also discussed in [AP], who also give the associated period matrices for these curves.