Tensor Products of Galois Representations

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1. Introduction

Motivation 1: The Hasse-Weil ζ-Function of Products

Recall: The zeta function $\zeta_{X/K}(s)$ of a smooth, projective variety X over a number field K is given (up to finitely many Euler factors) by the zeta function

$$\zeta_{\mathcal{X}}(s) := \prod_{x\in|\mathcal{X}|} (1 - N(x)^{-s})^{-1},$$

of any projective model $\mathcal{X}/\mathfrak{O}_K$ of X/K. (This product converges for $\Re(s) > \dim X + 1$.) Write: $\zeta_{X/K}(s) \sim \zeta_{\mathcal{X}}(s)$.

Example: If $X = \mathrm{Spec}(K)$, then $\zeta_{X/K}(s) \sim \zeta_{\mathrm{Spec}(\mathfrak{O}_K)}(s) = \zeta_K(s)$, the Dedekind ζ-function of K.

Main Principle: $\zeta_{X/K}(s)$ "encodes the arithmetic" of X: this is expressed in terms of specific conjectures (see below).

Fact (Grothendieck/Artin/Serre): We have the factorization

$$\zeta_{X/K}(s) \sim \prod_{m=0}^{2d} L_m(X,s)^{(-1)^m} = \frac{L_0(X,s)\cdots L_{2d}(X,s)}{L_1(X,s)\cdots L_{2d-1}(X,s)}$$

where $d = \dim X$ and each $L_m(X,s) = L(\rho_m, s)$ is the L-function associated to a suitable (rational) compatible system $\rho_m = \{\rho_{m,\ell}\}_\ell$ of ℓ-adic Galois representations

$$\rho_{m,\ell} = \rho_{X,m,\ell} : G_K = \mathrm{Gal}(\overline{K}/K) \;\to\; \mathrm{Aut}_{\mathbb{Q}_\ell}(V_{X,m,\ell}).$$

(Explicitly: $V_{X,m,\ell} = H^m_{et}(\overline{X}, \mathbb{Q}_\ell)$.)

Example 1: If X/K is a curve, then

$$\zeta_{X/K}(s) \sim \zeta_K(s)\zeta_K(s-1)L_1(X,s)^{-1}.$$

Here $L_1(X,s) = L(\rho_{J_{X/K}}, s)$ is the L-function associated to the system $\rho_{J_{X/K}} = \{\rho_{J_{X/K,\ell}}\}_\ell$ of Galois representations

$$\rho_{J_{X/K,\ell}} : G_K \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(J_X))$$

afforded by the Tate space of the Jacobian J_X/K of X:

$$V_\ell(J_X) = T_\ell(J_X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \text{ where } T_\ell(J_X) = \varprojlim J_X[\ell^n].$$

Example 2: If $X = X_1 \times X_2$ is a product surface, then (by the Künneth formula) we have

$$\zeta_{X/K}(s) \sim \frac{\zeta_{X_1}(s)\zeta_{X_2}(s)\zeta_{X_1}(s-1)\zeta_{X_2}(s-1)}{\zeta_K(s)\zeta_K(s-1)^2\zeta_K(s-2)} L(\rho_1 \otimes \rho_2, s).$$

Here $\rho_1 \otimes \rho_2 = \{\rho_{J_{X_1}/K,\ell} \otimes \rho_{J_{X_2}/K,\ell}\}_\ell$ is the system of Galois representations afforded by the ℓ-adic tensor product modules

$$V_\ell(J_{X_1}, J_{X_2}) := V_\ell(J_{X_1}) \otimes_{\mathbb{Q}_\ell} V_\ell(J_{X_1}).$$

Thus, the "new contribution" of the ζ-function $\zeta_X(s)$ is the part coming from the L-function $L(\rho_1 \otimes \rho_2, s)$ associated to the tensor product representation $\rho_1 \otimes \rho_2$.

Remark: In the case that $X_1 = X_2 = E$ is an elliptic curve, then $V_\ell(E,E) \simeq S^2(V_\ell(E)) \oplus \det(V_\ell(E))$, so $\rho_1 \otimes \rho_1$ is essentially the symmetric square representation.

Motivation 2: The Tate Conjectures

Conjecture 1 (Hasse/Weil): $\zeta_{X/K}(s)$ has a meromorphic continuation to the entire complex plane $\mathbb{C}$.

Refinement: Each $L_m(X,s)$ has a meromorphic continuation to $\mathbb{C}$. (Note: $L_m(X,s)$ converges for $\Re(s) > 1 + \frac{m}{2}$.)

Conjecture 2 (Tate): The order of the pole of $L_{2m}(X,s)$ at $s = m+1$ equals the rank of the group $\mathfrak{A}^m(X/K) = Z^m(X)/\equiv$ of codimension m-cycles on X/K modulo numerical equivalence:

$$-\mathrm{ord}_{s=m+1} L_{2m}(X,s) = \mathrm{rk}(\mathfrak{A}^m(X/K)).$$

Furthermore, for all primes ℓ,

$$(T^m(X)) \qquad \mathrm{rk}(\mathfrak{A}^m(X/K)) = \dim_{\mathbb{Q}_\ell}(H^{2m}_{et}(\overline{X},\mathbb{Q}_\ell)(m))^{G_K}.$$

Remarks: 1) Tate (1963) also has a conjectural interpetation of $\mathrm{ord}_{s=1} L_1(X,s)$ which generalizes the Birch/Swinnerton-Dyer Conjecture.

2) In the case that $X = A \times \hat{A}$, where A/K is an abelian variety and $\hat{A}$ is its dual, Conjecture $T^1(A \times \hat{A})$ is equivalent to the following statement which was proved by Faltings (1983).

Theorem 0 (Faltings): If A/K is an abelian variety, then

$$(1) \quad \mathrm{rk}(\mathrm{End}_K(A)) = \dim_{\mathbb{Q}_\ell} \mathrm{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A)), \quad \forall \text{ primes } \ell.$$

More precisely, we have a natural $\mathbb{Q}_\ell$-linear ring isomorphism

$$(2) \qquad \tau_{A/K,\ell} : \mathrm{End}_K(A) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A)).$$

2. Some Questions

Notation: If $\rho_i : G_K \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_i)$ are two ℓ-adic Galois representations, then put

$$(\rho_1, \rho_2)_{G_K} := \dim_{\mathbb{Q}_\ell} \mathrm{Hom}_{\mathbb{Q}_\ell[G_K]}(V_1, V_2).$$

This is often called the intertwining number of ρ_1 and ρ_2.

Example: If A/K is an abelian variety, then by Faltings

$$(\rho_{A/K,\ell}, \rho_{A/K,\ell})_{G_K} = \mathrm{rk}(\mathrm{End}_K(A)).$$

Notation: If A/K and B/K are two abelian varieties, let

$$V_\ell(A,B) = V_\ell(A) \otimes_{\mathbb{Q}_\ell} V_\ell(B)$$

be the tensor product of the Tate spaces and let

$$\rho_{A,B,K,\ell} := \rho_{A/K,\ell} \otimes \rho_{B/K,\ell} : G_K \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A,B))$$

be the associated tensor product representation.

Question 1: Is there a "Faltings Theorem" for $\rho_{A,B,K,\ell}$, i.e., is there an arithmetic interpretation of $(\rho_{A,B,K,\ell}, \rho_{A,B,K,\ell})_{G_K}$?

Observation: The Tate Conjecture $(T^2(A\times B\times \hat{A}\times \hat{B}))$ implies: $\exists$ a subgroup $\mathfrak{A}_{A,B,K} \le \mathfrak{A}^2((A\times B\times \hat{A}\times \hat{B})/K)$ such that

$$(\rho_{A,B,K,\ell}, \rho_{A,B,K,\ell})_{G_K} = \mathrm{rk}(\mathfrak{A}_{A,B,K}), \quad \forall \ell.$$

In particular, the left hand side is independent of ℓ.

Subproblem: Can $\mathfrak{A}_{A,B,K}$ be interpreted in terms of endomorphisms of abelian varieties?

Remark: It is not difficult to see that we have a natural embedding

$$\mathrm{End}_K(A) \otimes \mathrm{End}_K(B) \hookrightarrow \mathfrak{A}_{A,B,K}.$$

This naturally raises the following question.

Question 2: When is $\mathrm{rk}(\mathrm{End}_K(A) \otimes \mathrm{End}_K(B)) = \mathrm{rk}(\mathfrak{A}_{A,B,K})$?

Combining Questions 1 and 2 leads to:

Question 3: Let $\tau_{A,B,K,\ell} = \tau_{A/K,\ell} \otimes_{\mathbb{Q}_\ell} \tau_{B/K,\ell}$, so $\tau_{A,B,K,\ell}$ can be viewed as a ring homomorphism

$$\tau_{A,B,K,\ell} : \mathrm{End}_K(A) \otimes \mathrm{End}_K(B) \otimes \mathbb{Q}_\ell \to \mathrm{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A,B))$$

via the identification

$$\mathrm{End}_{\mathbb{Q}_\ell}(V_\ell(A,B)) \simeq \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell(A)) \otimes_{\mathbb{Q}_\ell} \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell(B)).$$

When is $\tau_{A,B,K,\ell}$ an isomorphism? In other words, when is

(3) $\quad (\rho_{A,B,K,\ell}, \rho_{A,B,K,\ell})_{G_K} = \mathrm{rk}(\mathrm{End}_K(A))\,\mathrm{rk}(\mathrm{End}_K(B))$?

A first (naive) guess is that the following holds.

Hypothesis $H_{A,B,K}$: The following are equivalent:

(i) Formula (3) holds for all primes ℓ;

(i′) Formula (3) holds for one prime ℓ;

(ii) $\mathrm{Hom}_{\overline{K}}(A,B) = 0$.

Observation: While $H_{A,B,K}$ holds for some abelian varieties A/K and B/K, it is not true in general. There are (at least) two classes of counterexamples.

Counterexamples to Hypothesis $H_{A,B,K}$:

1) $A/\mathbb{Q}$ and $B/\mathbb{Q}$ are modular abelian varieties which have a common internal twist (in the sense of Ribet);

2) A/K and B/K are CM elliptic curves which are defined over $\mathbb{Q}$ and K is a suitable real quadratic field.

Remark: Note that $\mathrm{End}_K(A) \neq \mathrm{End}_{\overline{\mathbb{Q}}}(A)$ in both cases. Thus, a better guess is the following:

Hypothesis $\overline{H}_{A,B}$: The hypothesis $H_{A,B,K}$ holds whenever K is large enough, i.e., whenever

$$\mathrm{End}_{\overline{K}}(A) = \mathrm{End}_K(A) \quad \text{and} \quad \mathrm{End}_{\overline{K}}(B) = \mathrm{End}_K(B).$$

Observation: If $\overline{H}_{A,B}$ holds for A/K and B/K, and if (ii) holds (i.e., if $\mathrm{Hom}_{\overline{K}}(A,B) = 0$), then for every finite extension L/K and prime ℓ we have an induced isomorphism

$$\tilde{\tau}_{A,B,L} : (\mathrm{End}_{\overline{K}}(A) \otimes \mathrm{End}_{\overline{K}}(B))^{G_L} \otimes \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{End}_{\mathbb{Q}_\ell[G_L]}(V_\ell(A,B)).$$

From this it follows that

$$\mathfrak{A}_{A,B,K} \otimes \mathbb{Q} = (\mathrm{End}_{\overline{K}}(A) \otimes \mathrm{End}_{\overline{K}}(B))^{G_L} \otimes \mathbb{Q},$$

and so we obtain a solution of our subproblem in this case. In particular, $H_{A,B,L}$ holds if and only if

$$(\mathrm{End}_{\overline{K}}(A) \otimes \mathrm{End}_{\overline{K}}(B))^{G_L} \otimes \mathbb{Q} = \mathrm{End}_L(A) \otimes \mathrm{End}_L(B) \otimes \mathbb{Q}.$$

3. Main Results.

Theorem 1. If A and B are isogenous (over $\overline{\mathbb{Q}}$) to products of elliptic curves, then $\overline{H}_{A,B}$ holds.

Definition: A *modular abelian variety* A/K is an abelian variety which is isogenous to a quotient of the Jacobian variety $J_1(N)_K$ of the modular curve $X_1(N)_K$, for a suitable N.

Theorem 2. If A and B are modular abelian varieties, then $\overline{H}_{A,B}$ holds.

Remark: Both Theorem 1 and Theorem 2 are special cases of a more general theorem. For this, I introduce the class of abelian varieties of *generalized* GL_2*-type* (see below). These include:

- products of elliptic curves
- K. Murty's abelian varieties of type (T) (1983)
- K. Ribet's abelian varieties $A/\mathbb{Q}$ of GL_2-type (1992); these include the Shimura quotients A_f, where $f \in S_2(\Gamma_1(N))^{new}$.

Theorem 3. If A and B are abelian varieties of generalized GL_2-type, then $\overline{H}_{A,B}$ holds.

Corollary 1: If A/K and B/K are abelian varieties of generalized GL_2-type, then $H_{A,B,K}$ holds $\Leftrightarrow$

$$(\mathrm{End}_{\overline{K}}(A) \otimes \mathrm{End}_{\overline{K}}(B))^{G_K} = \mathrm{End}_K(A) \otimes \mathrm{End}_K(B).$$

Corollary 2: If A/K and B/K are abelian varieties of generalized GL_2-type whose $\overline{\mathbb{Q}}$-endomorphisms are defined over K, then $H_{A,B,K}$ holds.

Example: If A/K and B/K are isogenous to products of elliptic curves without CM, then $H_{A,B,K}$ holds by Corollary 2. In other words, the following conditions are equivalent:

(i) Formula (3) holds for all primes ℓ;

(i′) Formula (3) holds for one prime ℓ;

(ii) $\mathrm{Hom}_{\overline{K}}(A, B) = 0$;

Remark: Recall that formula (3) was the following identity:

$$(\rho_{A,B,K,\ell}, \rho_{A,B,K,\ell})_{G_K} = \mathrm{rk}(\mathrm{End}_K(A))\,\mathrm{rk}(\mathrm{End}_K(B)).$$

4. Analysis of Condition (3).

Notation: If V is a $\mathbb{Q}_\ell[G_K]$-module, let $\overline{V} = V \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ denote the associated $\overline{\mathbb{Q}}_\ell[G_K]$-module. Here $\overline{\mathbb{Q}}_\ell$ denotes an algebraic closure of $\mathbb{Q}_\ell$.

Lemma 1: $\tau_{A,B,K,\ell}$ is an isomorphism (i.e., condition (3) holds for A, B, K, ℓ) if and only if the following two conditions hold:

(1) (Irreducibility) If $V \subset \overline{V}_\ell(A)$ and $W \subset \overline{V}_\ell(B)$ are irreducible $\overline{\mathbb{Q}}_\ell[G_K]$-submodules, then $V \otimes W$ is also irreducible.

(2) (Multiplicity 1) If $V_i \subset \overline{V}_\ell(A)$ and $W_i \subset \overline{V}_\ell(B)$ are irreducible $\overline{\mathbb{Q}}_\ell[G_K]$-submodules (for $i = 1, 2$), then

$$V_1 \otimes W_1 \simeq V_2 \otimes W_2 \quad \Leftrightarrow \quad V_1 \simeq V_2 \text{ and } W_1 \simeq W_2.$$

Counterexamples to $H_{A,B,K}$: 1) Let $E_i/\mathbb{Q}$ be two elliptic curves with CM by F_i, where $F_1 \neq F_2$. If $K = (F_1F_2)^+$, then $H_{E_1,E_2,K}$ does not hold. Here $\mathrm{Hom}_{\overline{\mathbb{Q}}}(E_1, E_2) = 0$, but (3) does not hold (for any ℓ) because $\dim_{\mathbb{Q}} \mathrm{End}_K(E_i) = 1$ and $(\rho_{E_1,E_2,K,\ell}, \rho_{E_1,E_2,K,\ell})_{G_K} = 2 \neq 1$. (Here Property (1) fails.)

2) Let $E_i/\mathbb{Q}$ be two modular (non-CM) elliptic curves with associated newforms $f_i \in S_2(\Gamma_0(N_i))$, and assume that E_1 and E_2 are not $\overline{\mathbb{Q}}$-isogenous. Moreover, let χ be a Dirichlet character of order $m > 2$, and let g_i be the newform associated to the twist $(f_i)_\chi$ of f_i by χ. If $A_i = A_{g_i}/\mathbb{Q}$ is the Shimura quotient associated to g_i, then $H_{A_1,A_2,\mathbb{Q}}$ does not hold.

Indeed, $A_i \otimes \overline{\mathbb{Q}} \sim E_i^{\phi(m)} \otimes \overline{\mathbb{Q}}$, so $\mathrm{Hom}_{\overline{\mathbb{Q}}}(A_1, A_2) = 0$, but (3) does not hold. (Here Property (1) holds, but (2) fails because A_1 and A_2 have "simultaneous inner twists").

5. Abelian Varieties of Generalized GL_2-type

Definition: A $\overline{\mathbb{Q}}_\ell[G_K]$-module V has *restricted* GL_2*-type* if $V = \oplus V_i$ is a direct sum of two-dimensional $\overline{\mathbb{Q}}_\ell[G_K]$-modules V_i such that each V_i is of one of the following two types:

(I) V_i is irreducible and $\det V_i = \chi_\ell$, where χ_ℓ is the cyclotomic ℓ-adic character on G_K.

(II) $V_i \simeq \overline{V}_\ell(E_i)$, for some CM elliptic curve E_i/K.

Definition: An abelian variety A/K has *generalized* GL_2*-type* if there is a finite extension L/K such that

(i) $\mathrm{End}_L^0(A) = \mathrm{End}_{\overline{\mathbb{Q}}}^0(A)$;

(ii) $\overline{V}_\ell(A)$ has restricted GL_2-type as a G_L-module, $\forall \ell$.

Remark: The class $(\mathrm{genGL}_2)_K$ of abelian varieties A/K of generalized GL_2-type is closed under products. Moreover, if $A \in (\mathrm{genGL}_2)_K$ and if $B \leq A$, then $B, A/B \in (\mathrm{genGL}_2)_K$.

Lemma 2: If $A \in (\mathrm{genGL}_2)_K$, then there is a decomposition $A \sim A^{nCM} \times A^{CM}$ such that for any L/K with (i) we have that

(a) $A^{CM} \otimes L \sim$ product of CM elliptic curves E_i/L, and $\overline{V}_\ell(A^{CM})$ is a direct sum of 1-dimensional G_L-modules;

(b) Each G_L-irreducible component V of $\overline{V}_\ell(A^{nCM})$ has dimension 2 and is *strongly irreducible*, i.e. $V_{|U}$ is irreducible, $\forall$ open $U \leq G_L$. Moreover, $\overline{V}_\ell(A^{nCM})$ *has no internal twists,* i.e., if V_i are two irreducible submodules of $\overline{V}_\ell(A^{nCM})$, then

$$V_1 \simeq V_2 \otimes \chi, \text{ for some } \chi \in \mathrm{Hom}(G_L, \overline{\mathbb{Q}}_\ell^\times) \quad \Rightarrow \quad \chi = 1.$$

6. Representation Theory: non-CM Case.

Let: $k = \overline{\mathbb{Q}}_\ell$ and $G = G_K$. Here we study $k[G]$-modules V with:

(4) $\qquad V$ is strongly irreducible of dimension 2.

(Recall: this means that $V_{|U}$ is irreducible, $\forall$ open $U \le G$.)

Theorem 5 (Irreducibility Criterion): If V, W satisfy (4), then $V \otimes W$ is irreducible $\Leftrightarrow$

(5) $\qquad V \not\simeq W \otimes \chi, \quad$ for all $\chi \in \mathrm{Hom}(G, k^\times)$.

Remark: By using Schur's Lemma, this follows easily from a result of D. Ramakrishnan (2000) on adjoint representations.

Theorem 6 (Cancellation Criterion): If V_i, W_i satisfy (4) for $i = 1, 2$, and if

(6) $\qquad V_i \otimes W_j$ is irreducible, for all $i, j \in 1, 2$,

then $V_1 \otimes W_1 \simeq V_2 \otimes W_2 \Leftrightarrow \exists \chi \in \mathrm{Hom}(G, k^\times)$ such that

(7) $\qquad V_1 \simeq V_2 \otimes \chi \quad$ and $\quad W_1 \simeq W_2 \otimes \chi^{-1}$.

Remarks: 1) In view of Lemmas 1 and 2, Theorems 5 and 6 imply Theorem 3 in the non-CM case (i.e, when $A \sim A^{nCM}$.)

2) The proof of Theorem 6 uses the following identity (which was also used in Ramakrishnan's proof):

$$\wedge^2(V \otimes W) \simeq (S^2V \otimes \wedge^2 W) \oplus (\wedge^2 V \otimes S^2 W).$$

(As usual, S^2V denotes the symmetric square of V.)

7. Representation Theory: CM Case.

Recall: If E/K is a CM elliptic curve with $F := \mathrm{End}^0_K(E) \neq \mathbb{Q}$, then $F \subset K$ and F is an imaginary quadratic field. Moreover,

$$\overline{V}_\ell(E) \simeq \psi_1 \oplus \psi_2, \quad \text{with } \psi_i \in \mathrm{Hom}(G_K, \overline{\mathbb{Q}}_\ell^\times).$$

In addition, $\psi_1\psi_2 = \chi_\ell$.

Lemma 3: Let E_i/K be an elliptic curve with CM by $F_i \subset K$, and let $\overline{V}_\ell(E_i) = \psi_{i1} \oplus \psi_{i2}$, where $i = 1, 2$. Assume that $F_1 \neq F_2$. If p is a prime which splits completely in K, then

$$\mathbb{Q}(\psi_{1i}\psi_{2j}(\sigma_{\mathfrak{P}})) \simeq F_1F_2, \quad \forall i, j = 1, 2,$$

where $\sigma_{\mathfrak{P}} \in G_K$ is a Frobenius element at $\mathfrak{P} \mid p$.

Remarks: 1) Using Lemma 3, it follows easily that Property (2) holds if $A = A^{CM}$ and $B = B^{CM}$. Since Property (1) is trivial, we thus see that Theorem 3 holds in this case. Combining this with the results of §6, this proves Theorem 3 because it is easy to verify Properties (1) and (2) for the "mixed terms" $V_i \otimes \psi_j$.

2) By using a more general version of the Irreducibility Criterion (Theorem 5) and the results of Ribet (1980), one can also show:

Theorem 7: If $A/\mathbb{Q}$ and $B/\mathbb{Q}$ are modular abelian varieties with $\mathrm{Hom}_{\overline{\mathbb{Q}}}(A, B) = 0$, then Property (1) holds, i.e.,

$$V \otimes W \text{ is } G_\mathbb{Q}\text{-irred., if } V \subset \overline{V}_\ell(A), W \subset \overline{V}_\ell(B) \text{ are } G_\mathbb{Q}\text{-irred.}$$