Tensor Products of Galois Representations

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Outline

1. Introduction

- Let: K be a number field, $G_K = \mathrm{Gal}(\overline{K}/K)$,
  A/K an abelian variety over K, $d = \dim(A)$,
  $T_\ell(A) = \varprojlim A[\ell^n]$, the ℓ-adic Tate-module,
  $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, viewed as a $\mathbb{Q}_\ell[G_K]$-module,
  $\rho_{A/K,\ell} : G_K \to \mathrm{Aut}(T_\ell(A)) \subset \mathrm{Aut}(V_\ell(A)) \simeq \mathrm{GL}_{2d}(\mathbb{Q}_\ell)$,
  $\bar{\rho}_{A/K,\ell} : G_K \to \mathrm{Aut}(A[\ell]) \simeq \mathrm{GL}_{2d}(\mathbb{F}_\ell)$, the associated Galois representations.
- Faltings (1983): The homomorphism

$$\tau_{A/K,\ell} : \mathrm{End}_K(A) \otimes \mathbb{Q}_\ell \to \mathrm{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A))$$

  is an isomorphism. In particular, the intertwining number

$$(\rho_{A/K,\ell}, \rho_{A/K,\ell})_{G_K} := \dim_{\mathbb{Q}_\ell} \mathrm{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A))$$

  is independent of the choice of ℓ (and equals $\dim_{\mathbb{Q}} \mathrm{End}^0_K(A)$).

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- Zarhin (1977), (1985); Faltings (1984): For almost all ℓ

$$\overline{\tau}_{A,\ell} : \mathrm{End}_K(A) \otimes \mathbb{F}_\ell \to \mathrm{End}_{\mathbb{F}_\ell[G_K]}(A[\ell])$$

  is an isomorphism. Thus

$$(\overline{\rho}_{A/K,\ell}, \overline{\rho}_{A/K,\ell})_{G_K} := \dim_{\mathbb{F}_\ell} \mathrm{End}_{\mathbb{F}_\ell[G_K]}(A[\ell])$$

  does not depend on ℓ, if $\ell >> 0$.

- Question 1: Let B/K be another abelian variety. Are there analogous results for $\rho_{A,B,K,\ell} := \rho_{A/K,\ell} \otimes \rho_{B/K,\ell}$ and for $\overline{\rho}_{A,B,K,\ell} := \overline{\rho}_{A/K,\ell} \otimes \overline{\rho}_{B/K,\ell}$?

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- Remark: This question arises naturally when one studies the Hasse-Weil zeta function of a product of two curves, and is closely connected with the Tate Conjecture.
  For this, recall that if X/K is a smooth, projective variety, then the Tate Conjecture $T^r(X)$ states that the cycle map

  $$cyc_X^r : \mathfrak{A}^r(X) \otimes \mathbb{Q}_\ell \to (H^{2r}_{et}(\overline{X}, \mathbb{Q}_\ell)(r))^{G_K}$$

  is an isomorphism, where $\mathfrak{A}^r(X)$ denotes the group of cycles of codimension r on X, modulo homological equivalence.
  Indeed, $T^2(A \times B \times A^* \times B^*) \Rightarrow$ the analogue of Faltings' Theorem holds for $\rho_{A,B,K,\ell} = \rho_{A/K,\ell} \otimes \rho_{B/K,\ell}$.
  In this, the ring $\mathrm{End}_K(A)$ is replaced by a certain (abstract) ring of correspondences which contains $\mathrm{End}_K(A) \otimes \mathrm{End}_K(B)$.
  This leads to the following question:

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- Question 2: Let $\tau_{A,B,K,\ell} = \tau_{A/K,\ell} \otimes \tau_{B/K,\ell}$, so

  $$\tau_{A,B,K,\ell} : \mathrm{End}_K(A) \otimes \mathrm{End}_K(B) \otimes \mathbb{Q}_\ell \to \mathrm{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A,B)),$$

  where $V_\ell(A,B) = V_\ell(A) \otimes_{\mathbb{Q}_\ell} V_\ell(B)$.
  When is $\tau_{A,B,K,\ell}$ an isomorphism? In other words, when is

  $$(1)\ (\rho_{A,B,K,\ell}, \rho_{A,B,K,\ell})_{G_K} := \dim_{\mathbb{Q}_\ell}(\mathrm{End}_{\mathbb{Q}_\ell[G_K]}(V_\ell(A,B))) \stackrel{?}{=} \dim_{\mathbb{Q}} \mathrm{End}^0_K(A) \dim_{\mathbb{Q}} \mathrm{End}^0_K(B)?$$

  Similarly: when is

  $$(2)\qquad (\overline{\rho}_{A,B,\ell}, \overline{\rho}_{A,B,\ell})_{G_K} = \dim_{\mathbb{Q}} \mathrm{End}^0_K(A) \dim_{\mathbb{Q}} \mathrm{End}^0_K(B)?$$

  A first (naive) guess is that the following holds.

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- Hypothesis $H_{A,B,K}$: The following are equivalent:
  (i) Formula (1) holds for all primes ℓ;
  (i′) Formula (1) holds for one prime ℓ;
  (ii) $\mathrm{Hom}_{\overline{K}}(A, B) = 0$.
- Observation: While $H_{A,B,K}$ holds for some abelian varieties A/K and B/K, it is not true in general. There are (at least) two classes of counterexamples:
  (i) $A/\mathbb{Q}$ and $B/\mathbb{Q}$ are modular abelian varieties which have a common internal twist (in the sense of Ribet);
  (ii) A/K and B/K are CM elliptic curves which are defined over $\mathbb{Q}$ and K is a suitable real quadratic field.
  Thus, a better guess is the following:

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- Hypothesis $\overline{H}_{A,B}$: The hypothesis $H_{A,B,K}$ holds whenever K is large enough, i.e., whenever

$$\mathrm{End}_{\overline{K}}(A) = \mathrm{End}_K(A) \quad \text{and} \quad \mathrm{End}_{\overline{K}}(B) = \mathrm{End}_K(B).$$

- Observation: If $\overline{H}_{A,B}$ holds for A/K and B/K, and if (ii) holds, then for every finite extension L/K and prime ℓ we have an induced isomorphism

$$\tilde{\tau}_{A,B,L} : (\mathrm{End}_{\overline{K}}(A) \otimes \mathrm{End}_{\overline{K}}(B))^{G_L} \otimes \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{End}_{\mathbb{Q}_\ell[G_L]}(V_\ell(A,B)).$$

Thus, $H_{A,B,L}$ holds if and only if

$$(\mathrm{End}_{\overline{K}}(A) \otimes \mathrm{End}_{\overline{K}}(B))^{G_L} = \mathrm{End}_L(A) \otimes \mathrm{End}_L(B).$$

2. Main Results

- Theorem 1. If A and B are isogenous (over $\overline{\mathbb{Q}}$) to products of elliptic curves, then $\overline{H}_{A,B}$ holds.
- Definition: A *modular abelian variety* A/K is a quotient of the Jacobian variety $J_1(N)_K$ of the modular curve $X_1(N)/K$, for a suitable N.
- Theorem 2. If A and B are modular abelian varieties, then $\overline{H}_{A,B}$ holds.
- Remark: Both Theorem 1 and Theorem 2 are special cases of a more general theorem. For this, I introduce the class of abelian varieties of *generalized* GL_2*-type* (see below). These include:
  - products of elliptic curves

2. Main Results - 2

- K. Murty's abelian varieties of type (T) (1983)

- K. Ribet's abelian varieties $A/\mathbb{Q}$ of GL_2-type (1992); these include the Shimura quotients A_f, where $f \in S_2(\Gamma_1(N))^{new}$.

- Theorem 3. If A and B are abelian varieties of generalized GL_2-type, then $\overline{H}_{A,B}$ holds.

- Corollary 1: If A/K and B/K are abelian varieties of generalized GL_2-type, then $H_{A,B,K}$ holds $\Leftrightarrow$

$$(\mathrm{End}_{\overline{K}}(A) \otimes \mathrm{End}_{\overline{K}}(B))^{G_K} = \mathrm{End}_K(A) \otimes \mathrm{End}_K(B).$$

- Corollary 2: If A/K and B/K are abelian varieties of generalized GL_2-type whose $\overline{\mathbb{Q}}$-endomorphisms are defined over K, then $H_{A,B,K}$ holds.

2. Main Results - 3

- Theorem 4: If A/K and B/K are elliptic curves without CM, then the following conditions are equivalent:
  (i) Formula (1) holds for all primes ℓ;
  (i′) Formula (1) holds for one prime ℓ;
  (ii) $\mathrm{Hom}_{\overline{K}}(A, B) = 0$;
  (iii) Formula (2) holds for almost all primes ℓ.
- Remarks: 1) The equivalence of the first 3 conditions is a special case of Corollary 2 above.
  2) The proof of the equivalence of (i) and (iii) uses a result of Frey/Jarden (2002), together with a modification of the representation theoretic results of §5.

3. Analysis of Condition (1)

- Notation: If V is a $\mathbb{Q}_\ell[G_K]$-module, let $\overline{V} = V \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ denote the associated $\overline{\mathbb{Q}}_\ell[G_K]$-module. Here $\overline{\mathbb{Q}}_\ell$ denotes an algebraic closure of $\mathbb{Q}_\ell$.
- Lemma 1: $\tau_{A,B,K,\ell}$ is an isomorphism (i.e., condition (1) holds for A, B, K, ℓ) if and only if the following two conditions hold:
(i) (Irreducibility) If $V \subset \overline{V}_\ell(A)$ and $W \subset \overline{V}_\ell(B)$ are irreducible $\overline{\mathbb{Q}}_\ell[G_K]$-submodules, then $V \otimes W$ is also irreducible.
(ii) (Multiplicity 1) If $V_i \subset \overline{V}_\ell(A)$ and $W_i \subset \overline{V}_\ell(B)$ are irreducible $\overline{\mathbb{Q}}_\ell[G_K]$-submodules (for $i = 1, 2$), then

$$V_1 \otimes W_1 \simeq V_2 \otimes W_2 \quad \Leftrightarrow \quad V_1 \simeq V_2 \text{ and } W_1 \simeq W_2.$$

3. Analysis of Condition (1) - 2

- Counterexamples to $H_{A,B,K}$:
  1) Let $E_i/\mathbb{Q}$ be two elliptic curves with CM by F_i, where $F_1 \neq F_2$. If $K = (F_1F_2)^+$, then $H_{E_1,E_2,K}$ does not hold.
  Here $\mathrm{Hom}_{\overline{\mathbb{Q}}}(E_1, E_2) = 0$, but (1) does not hold (for any ℓ) because $\dim_{\mathbb{Q}} \mathrm{End}_K(E_i) = 1$ and
  $(\rho_{E_1,E_2,K,\ell}, \rho_{E_1,E_2,K,\ell})_{G_K} = 2 \neq 1.$
  (Here $V_\ell(E_1, E_2)$ is reducible, so Property (i) fails.)

3. Analysis of Condition (1) - 3

- Counterexamples to $H_{A,B,K}$:
  2) Let $E_i/\mathbb{Q}$ be two modular (non-CM) elliptic curves with associated newforms $f_i \in S_2(\Gamma_0(N_i))$, and assume that E_1 and E_2 are not $\overline{\mathbb{Q}}$-isogenous.
  Moreover, let χ be a Dirichlet character of order $m > 2$, and let g_i be the newform associated to the twist $(f_i)_\chi$ of f_i by χ.
  If $A_i = A_{g_i}/\mathbb{Q}$ is the Shimura quotient associated to g_i, then $H_{A_1,A_2,\mathbb{Q}}$ does not hold.
  Indeed, $A_i \otimes \overline{\mathbb{Q}} \sim E_i^{\phi(m)} \otimes \overline{\mathbb{Q}}$, so $\mathrm{Hom}_{\overline{\mathbb{Q}}}(A_1, A_2) = 0$, but (1) does not hold. (Here Property (i) holds, but (ii) fails because A_1 and A_2 have "simultaneous inner twists").

4. Abelian Varieties of Generalized GL_2-type

- Definition: A $\overline{\mathbb{Q}}_\ell[G_K]$-module V has *restricted* GL_2*-type* if $V = \oplus V_i$ is a direct sum of two-dimensional $\overline{\mathbb{Q}}_\ell[G_K]$-modules V_i such that each V_i is of one of the following two types:
  (I) V_i is irreducible and
  $$\det V_i = \chi_\ell,$$
  where χ_ℓ is the cyclotomic ℓ-adic character on G_K.
  (II) $V_i \simeq \overline{V}_\ell(E_i)$, for some CM elliptic curve E_i/K.
- Definition: An abelian variety A/K has *generalized* GL_2*-type* if there is a finite extension L/K such that
  (i) $\mathrm{End}^0_L(A) = \mathrm{End}^0_{\overline{\mathbb{Q}}}(A)$;
  (ii) $\overline{V}_\ell(A)$ has restricted GL_2-type as a G_L-module, $\forall \ell$.

4. Abelian Varieties of Generalized GL_2-type - 2

- Remark: The class $(\mathrm{genGL}_2)_K$ of abelian varieties A/K of generalized GL_2-type is closed under products. Moreover, if $A \in (\mathrm{genGL}_2)_K$ and if $B \subset A$, then $B, A/B \in (\mathrm{genGL}_2)_K$.
- Lemma 2: If $A \in (\mathrm{genGL}_2)_K$, then there is a decomposition $A \sim A^{nCM} \times A^{CM}$ such that for any L/K with (i) we have that
(a) $A^{CM} \otimes L \sim$ product of CM elliptic curves E_i/L, and $\overline{V}_\ell(A^{CM})$ is a direct sum of 1-dimensional G_L-modules;
(b) Each G_L-irreducible component V of $\overline{V}_\ell(A^{nCM})$ has dimension 2 and is *strongly irreducible*, i.e. $V_{|U}$ is irreducible, $\forall$ open $U \leq G_L$. Moreover, $\overline{V}_\ell(A^{nCM})$ *has no internal twists,* i.e., if V_i are two irreducible submodules of $\overline{V}_\ell(A^{nCM})$, then

$$V_1 \simeq V_2 \otimes \chi, \text{ for some } \chi \in \mathrm{Hom}(G_L, \overline{\mathbb{Q}}_\ell^\times) \quad \Rightarrow \quad \chi = 1.$$

5. Representation Theory: non-CM Case

- Let: $k = \overline{\mathbb{Q}}_\ell$ and $G = G_K$. Here we study $k[G]$-modules V satisfying the following property:

  (3) $\qquad V$ is strongly irreducible of dimension 2.

  (Recall: this means that $V_{|U}$ is irreducible, $\forall$ open $U \le G$.)

- Theorem 5 (Irreducibility Criterion): If V, W satisfy (3), then $V \otimes W$ is irreducible $\Leftrightarrow$

  (4) $\qquad V \not\simeq W \otimes \chi, \quad$ for all $\chi \in \mathrm{Hom}(G, k^\times)$.

- Remark: By using Schur's Lemma, this follows easily from a result of D. Ramakrishnan (2000) on adjoint representations.

5. Representation Theory: non-CM Case - 2

- Theorem 6 (Cancellation Criterion): If V_i, W_i satisfy (3) for $i = 1, 2$, and if

(5) $$V_i \otimes W_j \text{ is irreducible, for all } i, j \in 1, 2,$$

then $V_1 \otimes W_1 \simeq V_2 \otimes W_2 \Leftrightarrow \exists \chi \in \mathrm{Hom}(G, k^\times)$ such that

(6) $$V_1 \simeq V_2 \otimes \chi \quad \text{and} \quad W_1 \simeq W_2 \otimes \chi^{-1}.$$

- Remarks: 1) In view of Lemmas 1 and 2, Theorems 5 and 6 imply Theorem 3 in the non-CM case (i.e, when $A \sim A^{nCM}$.) 2) The proof of Theorem 6 uses the following identity (which was also used in Ramakrishnan's proof):

$$\wedge^2(V \otimes W) \simeq (S^2V \otimes \wedge^2 W) \oplus (\wedge^2 V \otimes S^2 W).$$

(As usual, S^2V denotes the symmetric square of V.)

6. Representation Theory: CM Case

- Recall: If E/K is a CM elliptic curve with $F := \mathrm{End}_K^0(E) \neq \mathbb{Q}$, then $F \subset K$ and F is an imaginary quadratic field. Moreover,

$$\overline{V}_\ell(E) \simeq \psi_1 \oplus \psi_2, \quad \text{with } \psi_i \in \mathrm{Hom}(G_K, \overline{\mathbb{Q}}_\ell^\times).$$

  In addition, $\psi_1\psi_2 = \chi_\ell$.

- Lemma 3: Let E_i/K be an elliptic curve with CM by $F_i \subset K$, and let $\overline{V}_\ell(E_i) = \psi_{i1} \oplus \psi_{i2}$, where $i = 1, 2$. Assume that $F_1 \neq F_2$. If p is a prime which splits completely in K, then

$$\mathbb{Q}(\psi_{1i}\psi_{2j}(\sigma_{\mathfrak{P}})) \simeq F_1F_2, \quad \forall i, j = 1, 2,$$

  where $\sigma_{\mathfrak{P}} \in G_K$ is a Frobenius element at $\mathfrak{P} \mid p$.

6. Representation Theory: CM Case - 2

- Remarks: 1) Using Lemma 3, it follows easily that Property (ii) holds if $A = A^{CM}$ and $B = B^{CM}$. Since Property (i) is trivial, we thus see that Theorem 3 holds in this case. Combining this with the results of §5, this proves Theorem 3 because it is easy to verify Properties (i) and (ii) for the "mixed terms" $V_i \otimes \psi_j$.
  2) By using a more general version of the Irreducibility Criterion (Theorem 5) and the results of Ribet (1980), one can also show:
- Theorem 7: If $A/\mathbb{Q}$ and $B/\mathbb{Q}$ are modular abelian varieties with $\mathrm{Hom}_{\overline{\mathbb{Q}}}(A, B) = 0$, then Property (i) holds, i.e.,

$$V \otimes W \text{ is } G_{\mathbb{Q}}\text{-irred., if } V \subset \overline{V}_\ell(A), W \subset \overline{V}_\ell(B) \text{ are } G_{\mathbb{Q}}\text{-irred.}$$