Equivariant Atkin-Lehner Theory

Introduction

Atkin-Lehner Theory:

Atkin-Lehner (1970), Miyake (1971), Li (1975): theory of newforms (+ $\mathbb{T}$-algebra action) $\Rightarrow$ a canonical basis for $S_k(\Gamma_1(N))$ and hence also for $S_k(\Gamma(N))$.

However: The group

$$G = SL_2(\mathbb{Z}/N\mathbb{Z})$$

acts on the space $S_k(\Gamma(N))$, but newforms are not compatible with the group action!

Problem: (Equivariant A-L) Describe a (canonical) basis of the G-isotypic components of $S_k(\Gamma(N))$ in terms of oldforms/newforms.

Remark: This a variant of Hecke's Problem: construct an explicit basis of the G-isotypic components of $S_2(\Gamma(p))$. (Hecke, 1928)

Applications: 1) Study $S_k(\Gamma(N))$ as an $\mathbb{M}$-module, where

$$\mathbb{M} \subset \mathrm{End}_{\mathbb{C}}(S_k(\Gamma(N))$$

is the algebra of all modular operators: $\mathbb{M} = \langle \mathbb{T}, G \rangle$.

How large is $\mathbb{M}$?

2) In particular, for $k = 2$, how large is $\mathbb{M}$ compared to $\mathbb{E} := \mathrm{End}^o_{\bar{\mathbb{Q}}}(J_{X(N)})$? Is $\mathbb{M} = \mathbb{E}$?

3) What are the $\bar{\mathbb{Q}}$-isogeny factors of $J_{X(N)}$?

4) Calculate the rank

$$\mathrm{rank}(\mathrm{NS}(Z_{N,1}))$$

of the Neron Severi group of the modular diagonal quotient surface $Z_{N,1} = \Delta\backslash(X(N) \times X(N))$.

5) Study modular forms, particularly $\mathbb{T}\otimes\mathbb{T}$-eigenforms, on $Z_{N,1}$. (D. Carlton).

6) Computational: a canonical basis of $S_k(\Gamma(N))$ can be derived from one of $S_k(\Gamma(N))$ and $S_k(\Gamma_1(N))$ by twisting: $f \mapsto f_\chi$.

1. Fundamental Newforms

-joint work with Satya Mohit

Fix: k, N and put $V = S_k(\Gamma(N))$.

Recall: Atkin-Lehner Theory $\Rightarrow$

$$V = V^{\text{new}} \oplus V^{\text{old}} \tag{1}$$

such that: V^{new} has a basis of $\mathbb{T}$-eigenforms

$V^{\text{old}} = (V^{\text{new}})^{\perp}$ comes from lower level.

Caution: The Atkin-Lehner Theory for $\Gamma(N)$ is transported from that of $\Gamma_1(N^2)$ via $\beta_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$:

$$\beta_N^{-1}\Gamma(N)\beta_N = \Gamma_N \geq \Gamma_1(N^2),$$

where

$$\Gamma_N = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : a \equiv d \equiv 1 \ (N),\ c \equiv 0 \ (N^2)\}.$$

Thus, the A-L level for $\Gamma(N)$ is N^2, not N.

Example: $N = p, k = 2 \Rightarrow$

$$V^{\text{old}} = S_2(\Gamma_1(N)) + S_2(\Gamma^1(N)).$$

Basic Difficulty: $G = SL_2(\mathbb{Z}/N\mathbb{Z})$ acts on V, but (1) is not a decomposition of G-modules, due to the following twisting phenomenon:

Twisting Phenomenon: If

$$f(z) = \sum a_n q_N^n \in V, \text{ where } q_N = e^{2\pi i z/N},$$

and χ is a Dirichlet character mod N, then its χ-twist

$$f_\chi = \sum \chi(n) a_n q_N^n \in V,$$

and: 1) f_χ is often in V^{new}, even if $f \in V^{\text{old}}$;
2) twisting can be done by group elements:

$$f_\chi = f|_k R_\chi, \text{ where } R_\chi = \frac{1}{g(\bar\chi)} \sum \bar\chi(n) T^{nN/M};$$

here $M = cond(\chi), T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, $g(\chi) =$ Gauss sum. $\rightarrow$ variant of Shimura(1973), Atkin-Li(1978)

Definition. A normalized newform $f \in V^{\text{new}}$ is called fundamental if f_χ is again a newform, for all characters χ mod N.

Notation: a) $\mathcal{F} = \{\text{fundamental newforms}\}$,
$\mathcal{F}^{\text{CM}} = \{f \in \mathcal{F} : f_\chi = f, \text{for some } \chi \neq 1\}$,
$V^{\text{fund}} = \sum_{f \in \mathcal{F}} \mathbb{C} f \subset V^{\text{new}}$.

b) For any subset $S \subset V$, let $V_G(S) = G$-module generated by S, and write

$$V^{G\text{-old}} = V_G(V^{\text{old}}) \supset V^{\text{old}},$$
$$V^{G\text{-new}} = (V^{G\text{-old}})^\perp \subset V^{\text{new}}.$$

Remark: It turns out that a newform $f \in V$ is fundamental $\Leftrightarrow$ f is p-primitive in the sense of Atkin-Li, for all primes $p|N$.

Theorem 1: We have $V^{G\text{-new}} = V^{\text{fund}}$, so $V^{G\text{-new}}$ and $V^{G\text{-old}}$ are $\mathbb{M}$-modules, where $\mathbb{M} = \langle \mathbb{T}, G \rangle$, and we have the $\mathbb{M}$-module decomposition

$$V = V^{G\text{-old}} \oplus V^{G\text{-new}}.$$

Corollary. If $f \in \mathcal{F}$, then

$$V_G(f) = \sum \mathbb{C} f_\chi,$$

so $V_G(f)$ has a basis consisting of all twists of f, and hence is an $\mathbb{M}$-module. In particular, if $f \notin \mathcal{F}^{\text{CM}}$, then

$$\dim V_G(f) = \phi(N).$$

Theorem 2: If $f \in \mathcal{F} \setminus \mathcal{F}^{\text{CM}}$, then $V_G(f)$ is an irreducible, symmetric $\mathbb{M}$-module, and we have:

$$V_G(f) \simeq V_G(f') \Leftrightarrow V_G(f) = V_G(f') \Leftrightarrow f' = f_\chi.$$

Remarks: 1) Since $\mathbb{M}$ has an involution $*$, we can define the contragedient W^* of an $\mathbb{M}$-module W, and W is called symmetric if $W^* \simeq W$.

2) $f \in \mathcal{F}^{\text{CM}} \Rightarrow V_G(f)^* \not\simeq V_G(f)$.

3) For $N = p$, Theorem 2 is true for an arbitrary (non-CM) normalized newform $f \in V^{\text{new}}$, and so we get the following multiplicity 1 decomposition:

$$V = \bigoplus_{f \in \bar{\mathcal{N}}} V_G(f).$$

4) $V_G(f)$ is frequently irreducible as a G-module, but not always. If $N = p$, then have a classification. (This uses the knowledge of the irreducible representations of $G = SL_2(\mathbb{Z}/p\mathbb{Z})$.)

Proof (of Irreducibility). Main Observation:

$f \in \mathcal{F} \Rightarrow R_\chi$ acts bijectively on $V_G(f)$
$\Rightarrow V_G(f)_{|B}$ = direct sum of irreducible, pairwise non-isomorphic B-modules which are induced from $U \times D$. [Here B = Borel subgroup, U =unipotent subgroup, D=diagonal subgroup of G.]
This decomposition is incompatible with the $\mathbb{T}$-module decomposition $\Rightarrow$ irreducible.

Remark. Such induced modules were considered (for $SL_2(\mathbb{F}_q)$) by Gelfand-Graev, who called them fundamental representations. In representation theory, they are also called cuspidal representations.

2. Example: $V = S_2(\Gamma(p))$

Dimension Formulae:

$$\begin{aligned}
\dim V &= g &&= \tfrac{1}{24}(p+2)(p-3)(p-5)\\
\dim V^{\text{new}} &= g - 2g_1 &&= \tfrac{1}{24}(p-5)(p^2-3p+8)\\
\dim V^{G\text{-new}} &= &&= \tfrac{1}{48}(p-1)(p^2-2p-17)+b\\
\dim V^{G\text{-old}} &= \tfrac{p+1}{2}g_1 + \tfrac{p-1}{2}g_0 &&= \tfrac{1}{48}(p+1)(p^2-10p+33)-b\\
\dim V^{\text{old}} &= 2g_1 &&= \tfrac{1}{12}(p-5)(p-7),
\end{aligned}$$

where $g_i = g(X_i(p))$, and $b = \frac{p-1}{2}a$ with $a = \frac{p+1}{12} - g_0$, $0 \leq a \leq \frac{7}{6}$.

The G-Generation of V:

$$\begin{aligned}
f \in \mathcal{N}_0 &:= \mathcal{N}(\Gamma_0(p)) &&\Rightarrow \dim V_G(f|\beta_p) = p\\
f \in \mathcal{N}_1 &:= \mathcal{N}(\Gamma_1(p)) \setminus \mathcal{N}(\Gamma_0(p)) &&\Rightarrow \dim V_G(f|\beta_p) = p+1\\
f \in \mathcal{N}_2 &:= \mathcal{N}(\Gamma_0(p^2)) \setminus (\mathcal{N}^* \cup \mathcal{N}_3) &&\Rightarrow \dim V_G(f|\beta_p) = p-1\\
f \in \mathcal{N}_3 &:= \text{CM-forms in } \mathcal{N}(\Gamma_0(p^2)) &&\Rightarrow \dim V_G(f|\beta_p) = \tfrac{p-1}{2},
\end{aligned}$$

where

$$\mathcal{N}^* = \bigcup_{\chi} \mathcal{N}(\Gamma_0(p,\chi^2))|R_{\chi^{-1}}.$$

If we let $\overline{\mathcal{N}_i} = \mathcal{N}_i/\sim$ (identifying quadratic twists), then

$$V = \left(\bigoplus_{f \in \mathcal{N}_0 \cup \overline{\mathcal{N}}_1} V_G(f|\beta_p) \right) \oplus \left(\bigoplus_{f \in \overline{\mathcal{N}}_2 \cup \mathcal{N}_3} V_G(f|\beta_p) \right).$$

Furthermore,

$$\begin{aligned}
\#\mathcal{N}_0 &= g_0(p) \\
\#\mathcal{N}_1 &= g_1(p) - g_0(p) \\
\#\mathcal{N}_2 &= g_0(p^2) - g_1(p) - 2g_0(p) - h(p) \\
\#\mathcal{N}_3 &= h(p),
\end{aligned}$$

where $h(p) = \begin{cases} h(\mathbb{Q}(\sqrt{-p})) & \text{if } p \equiv 3 \pmod 4 \\ 0 & \text{if } p \equiv 1 \pmod 4 \end{cases}.$

3. Geometric Interpretation ($k = 2$)

Recall: The Shimura Construction:

$\mathbb{T}$-eigenform $f \ \mapsto \ A_f \subset J(N)$ abelian subvariety

Note: $\dim A_f = [K_f : \mathbb{Q}]$, where $K_f = \mathbb{Q}(\{a_n(f)\})$.

Put: $A_{f,G} = \sum_{g \in G} g(A_f) \subset J(N)$.

Observations: 1) $A_{f,G}$ is defined over $\mathbb{Q}$.

2) $T_{\mathbb{C}}^*(A_{f,G}) = \sum_\sigma V_G(f^\sigma) = \sum_{\Gamma_f \backslash G_{\mathbb{Q}}} V_G(f^\sigma)$, where $\Gamma_f = \{\sigma \in G_{\mathbb{Q}} : f^\sigma = f_\chi, \text{for some } \chi\} \geq G_f := \mathrm{Gal}(\bar{\mathbb{Q}}/K_f)$.

Theorem 3: If $f \in \mathcal{F} \setminus \mathcal{F}^{\mathrm{CM}}$, then

$$\dim A_{f,G} = \phi(N)[Z_f : \mathbb{Q}] = \phi(N)[G_f : \Gamma_f],$$

where $Z_f = \mathrm{Fix}(\Gamma_f) \subset K_f$.

Furthermore, if $\mathbb{M}_f \subset \mathrm{End}^0_{\mathbb{Q}}(A_{f,G})$ denotes the projection of $\mathbb{M}$ onto $A_{f,G}$, then

a) $Z(\mathbb{M}_f) = Z_f$,

b) $\dim_{\mathbb{Q}} \mathbb{M}_f = \phi(N)^2[Z_f : \mathbb{Q}]$.

Remark: Ribet(1980) calls $\mathrm{Gal}(K_f/Z_f)$ the group of *inner twists*. Using his results (and Shimura's), one can show:

Theorem 4: If f is a non-CM $\mathbb{T}$-eigenform, then $A_{f,G}$ is a (complete) isogeny factor of $J(N)_{/\bar{\mathbb{Q}}}$ and

$$A_{f,G} \sim B^n,$$

for some simple abelian variety $B/\bar{\mathbb{Q}}$.

Furthermore, if $f \in \mathcal{F}$, then Z_f is the centre of $\mathbb{E}_f := \mathrm{End}^0_{\bar{\mathbb{Q}}}(A_{f,G})$, i.e. $Z_f = Z(\mathbb{E}_f)$ and

$$\dim_{\mathbb{Q}} \mathbb{E}_f = \phi(N)^2[Z_f : \mathbb{Q}] = \dim_{\mathbb{Q}} \mathbb{M}_f.$$

Note: The above assertion is *false* for $f \in \mathcal{F}^{\mathrm{CM}}$:

$$f \in \mathcal{F}^{\mathrm{CM}} \overset{\text{Shimura}}{\Rightarrow} A_f \sim E^n,\ E\text{: CM elliptic curve}$$
$$\Rightarrow A_{f,G} \sim E^m,$$

where $m = \left(\frac{p-1}{2}\right) h(p)$ (if $N = p$). Thus

$$\mathbb{E}_f = \mathrm{End}^0_{\bar{\mathbb{Q}}}(A_{f,g}) = M_m(K),$$

where $K = \mathbb{Q}(\sqrt{-p})$, but

$$\mathbb{M}_f = \bigoplus_{i=1}^{h} M_{\frac{p-1}{2}}(K),$$

since the $V_G(f^\sigma)$'s are $\mathbb{M}$-irreducible and pairwise non-isomorphic.

Application 1: An Isogeny Relation:

$$J(p) \sim J_0(p)^p \times (J_1(p)/J_0(p))^{\frac{p+1}{2}} \times J_f^{p-1} \times J_{CM}^{\frac{p-1}{2}}.$$

Here $J_f \subset J_0(p^2)$ is the abelian subvariety whose cotangent space "is"

$$T^*_{\mathbb{C}}(J_f) \simeq N^*_{J_f} T^*_{\mathbb{C}}(J_f) = \sum_{f \in \mathcal{N}_2} \mathbb{C} f$$

($\Rightarrow \dim J_f = \frac{1}{2} \# \mathcal{N}_2$), and $J_{CM} \sim E^h$, where E is an elliptic curve with $\mathrm{End}^0(E) = \mathbb{Q}(\sqrt{-p})$.

Note: If $A \le J_X$ is an abelian subvariety (here X is any curve), then the polarization induces a surjection $N_A : J_X \to A$ and hence an injection

$$N_A^* : T^*_{\mathbb{C}}(A) \to T^*_{\mathbb{C}}(J_X) \overset{can}{\simeq} H^0(X, \Omega^1_X).$$

Application 2: Comparison of Algebras:

Recall: $\mathbb{M} = \langle \mathbb{T}, G \rangle \subset \mathbb{E} = \mathrm{End}^0_{\mathbb{Q}}(J(p))$. Then:

$$\begin{aligned}
\dim \mathbb{T} &= g = \tfrac{p-1}{2}(g_0(p^2) - g_0(p)) + g_1(p) \\
\dim \mathbb{M} &= (p-1)g + (p+1)g_1(p) - g_0(p) \\
\dim \mathbb{E} &= \dim \mathbb{M} + \tfrac{1}{2}(p-1)^2 h(h-1) \\
\dim C_G(\mathbb{M}) &= \tfrac{1}{24}(p-1)(p-5) + \tfrac{1}{2} y + h \\
\dim C_G(\mathbb{E}) &= \dim C_G(\mathbb{M}) + 2h(h-1),
\end{aligned}$$

where $y = g_0(p) - (-1)^{\frac{p-1}{2}} \frac{1}{2}\left(1 + \left(\frac{2}{p}\right)\right)$.

4. Numerical Examples

$N = 7$: Here

$$\begin{aligned} g &= 3, \\ g_0 &= g_1 = 0, \\ \dim V^{G\text{-old}} &= \tfrac{11+1}{2} g_1 + \tfrac{11-1}{2} g_0 = 0, \\ \dim V^{G\text{-new}} &= g - \dim V^{G\text{-old}} = 3; \\ g_0(7^2) &= \tfrac{1}{12}(7-1)(7-5) + g_0 = 1, \\ \#\mathcal{N}_2 &= g_0(7^2) - g_1 - 2g_0 - h(p) = 0, \\ \dim J_f &= \tfrac{1}{2}\#\mathcal{N}_2 = 0. \end{aligned}$$

Thus, the above isogeny relation becomes

$$J(7) \sim E^3,$$

where $E = J_{\mathrm{CM}}$ is the CM-elliptic curve with $\mathrm{End}^0(E) = \mathbb{Q}(\sqrt{-7})$.

$N = 11$: In this case we have:

$$\begin{aligned} g &= 26, \\ g_0 &= g_1 = 1, \\ \dim V^{G\text{-old}} &= \tfrac{11+1}{2} g_1 + \tfrac{11-1}{2} g_0 = 11, \\ \dim V^{G\text{-new}} &= g - \dim V^{G\text{-old}} = 15; \\ g_0(11^2) &= \tfrac{1}{12}(11-1)(11-5) + g_0 = 6, \\ \#\mathcal{N}_2 &= g_0(11^2) - g_1 - 2g_0 - h(p) = 2, \\ \dim J_f &= \tfrac{1}{2}\#\mathcal{N}_2 = 1. \end{aligned}$$

Here the isogeny relation becomes:

$$J(11) \sim E_1^{11} \times E_2^{10} \times E_3^{5}$$

where $E_1 = X_0(11)$, $E_2 = J_f$ and $E_3 = J_{\text{CM}}$. This relation is (essentially) due to Hecke(1928); cf. also Ligozat(1976).

$N = 13$**:** In this case we have:

$$\begin{aligned}
g &= 50, \\
g_0 &= 0, \\
g_1 &= \tfrac{1}{24}(13-5)(13-7) = 2 \\
\dim V^{G\text{-old}} &= \tfrac{13+1}{2} g_1 + \tfrac{13-1}{2} g_0 = 14, \\
\dim V^{G\text{-new}} &= g - \dim V^{G\text{-old}} = 36; \\
g_0(13^2) &= \tfrac{1}{12}(13-1)(13-5) + g_0 = 8, \\
\#\mathcal{N}_2 &= g_0(11^2) - g_1 - 2g_0 - h(p) = 6, \\
\dim J_f &= \tfrac{1}{2}\#\mathcal{N}_2 = 3.
\end{aligned}$$

Here one has the isogeny relation:

$$J(11) \sim J_1(13)^7 \times J_f^{12},$$

where $\dim J_f = 3$ and $\dim J_1(13) = 2$.

5. Application to $Z_{N,1}$

Situation: If $G \leq \mathrm{Aut}(X)$ acts on a curve X,
$\Rightarrow$ G acts *diagonally* on the surface $Y := X \times X$.

Then:

$$\begin{aligned} \mathrm{rk}(NS(Y)) &= 2 + \dim \mathrm{End}^0(J_X) \\ \mathrm{rk}(NS(G\backslash Y)) &= 2 + \dim C_G(\mathrm{End}^0(J_X)), \end{aligned}$$

where $C_G(\mathbb{E}) = \{\alpha \in \mathbb{E} : g\alpha = \alpha g\}$ denotes the centralizer of G in $\mathbb{E} = \mathrm{End}^0(J_X)$.

Now: if $X = X(N)$, then the quotient

$$Z_{N,1} = G\backslash(X \times X)$$

is the modular diagonal quotient surface of determinant 1, and so, by Application 1 above we have

Theorem 5: If $N = p$ is a prime, then

$$\begin{aligned} \mathrm{rk}(NS(Z_{N,1})) &= 2 + \dim C_G(\mathbb{E}) \\ &= 2 + \dim C_G(\mathbb{M}) + 2h(h-1) \\ &= 2 + \tfrac{1}{24}(p-1)(p-5) + \tfrac{1}{2}y + h. \end{aligned}$$

In particular, $NS(Z_{N,1}) \otimes \mathbb{Q}$ is generated by *modular correspondences* $\Leftrightarrow$ either $p \equiv 1\ (4)$ or $p \equiv 3\ (4)$ and $h(p) = 1$.