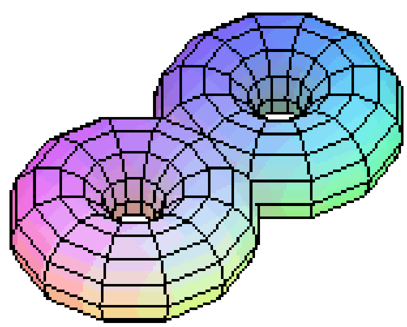
Hurwitz Spaces of Covers of an Elliptic Curve

1. Introduction

Riemann's Existence Theorem (RET) (1857):

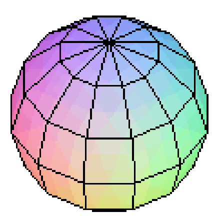
Every compact Riemann surface X has a non-constant meromorphic function, i.e. X admits a non-constant holo. map to the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

X



$\downarrow$ $\qquad$ $f \downarrow$

$\mathbb{C}_\infty$



Consequence: Every compact Riemann surface is a complex algebraic curve $X_{\mathbb{C}}$ defined by an equation

$$F(x, y) = 0, \quad \text{where } F \in \mathbb{C}[x, y],$$

and the (holomorphic) map $f : X \to \mathbb{C}_\infty$ corresponds to a morphism $f : X_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ of complex curves of the same degree (and conversely):

$$\begin{array}{ccc} X & & X_{\mathbb{C}} \\ f\downarrow & \leftrightarrow & \downarrow f \\ \mathbb{C}_\infty & & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

Properties of f:

1) $\deg(f) := \max_{y \in \mathbb{C}_\infty}(\#(f^{-1}(y))) < \infty$.

2) The set

$$R_f := \{y \in \mathbb{C}_\infty : \#(f^{-1}(y)) < \deg(f)\}$$

of ramification points of f is finite:

$$w := \#R_f < \infty.$$

Problem (Hurwitz, 1891) Fix integers N and w, and put $Y = \mathbb{C}_\infty$.

1) Investigate the totality $H(Y, N, w)$ of all covers $f : X \to Y$ with $\deg(f) = N$ and $\#R_f = w$.

2) Calculate the number $\#H(Y, N, R)$ of such covers with fixed ramification locus $R_f = R$.

Remarks: 1) A cover is a non-constant holomorphic map $f : X \to Y$. Throughout, we always consider equivalence classes of covers:

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi} & X_2 \\ & {}_{f_1}\searrow \quad \swarrow_{f_2} & \\ & Y & \end{array}$$

$$(X_1 \xrightarrow{f_1} Y) \sim (X_2 \xrightarrow{f_2} Y)$$
$$\Leftrightarrow \exists \phi : X_1 \xrightarrow{\sim} X_2 \quad \text{with } f_2 \circ \phi = f_1.$$

2) As Hurwitz observed, it is useful to refine the above problems by fixing the ramification type of the cover. For example, we might want to classify (or count) all simple covers:

A cover $f : X \to Y$ is called simple if

$$\#(f^{-1}(y)) \geq \deg(f) - 1, \quad \text{for all } y \in Y.$$

Theorem A (Hurwitz, 1891): If $Y = \mathbb{C}_\infty$, then

(a) $H(Y, N, w)$ is a "Riemannian space".

(b) $H^{simple}(Y, N, w)$ is a connected manifold of dimension w (provided that $w \geq 2N - 2$ and $2|w$).

(c) The discriminant map

$$\delta : H^{simple}(Y, N, w) \to Y^{(w)} \setminus \Delta_w$$

is finite and unramified. Thus, $\#H^{simple}(Y, N, R)$ depends only on $w = \#R$.

Observation (Hurwitz): RET $\Rightarrow$ the calculation of $\#H(Y, N, R)$ is a purely group-theoretic problem, albeit one that is "highly complicated" (Hurwitz):

$$H(Y, N, R) \xrightarrow{\sim} \mathrm{Hom}'(\pi_1(Y \setminus R), S_N)/S_N.$$

Hurwitz (1891/1901) found a "satisfactory solution" for calculating $n_{N,w} := \#H^{simple}(\mathbb{C}_\infty, N, R)$:

$$\begin{aligned} n_{2,w} &= 1 \\ n_{3,w} &= \tfrac{1}{3!}(3^{w-1} - 3), \\ n_{4,w} &= \tfrac{1}{4!}(2^{w-2} - 4)(3^{w-1} - 3), \text{ etc.} \end{aligned}$$

Question 1: Is there an intrinsic description of the topology and/or complex structure of the Hurwitz spaces $H(Y, N, w)$?

Recall: 1) The points of $H(Y, N, w)$ correspond to covers $f : X \to Y$ of degree N with w ramification points.

2) The topology of $H^{simple}(Y, N, w)$ is induced by the disciminant map

$$\delta : H^{simple}(Y, N, w) \to Y^{(w)} \setminus \Delta_w.$$

Thus: a neighbourhood of a cover $f \in H(Y, N, w)$ consists (roughly) of those covers whose ramification loci are close to that of f.

Question 2: Generalizations of Hurwitz spaces?

a) Construct $H(Y, N, w)$ for other Riemann surfaces/complex curves Y;

b) Study rationality conditions: over which ground fields $K \subset \mathbb{C}$ are the covers defined?

2. Intrinsic Description of Hurwitz Spaces

Key Observation (Grothendieck, 1960): A topological (complex) space H is uniquely characterized by the set of maps $\mathrm{Hom}(T, H)$, as T runs over all topological (complex) spaces.

In other words: As a topological space, H is determined by the functor

$$F_H : \underline{Top} \to \underline{Sets}$$

which is given by $F_H(T) = \mathrm{Hom}_{top}(T, H)$. (Similarly for complex spaces.)

Problem: For each complex space T, describe the holomorphic maps

$$T \to H = H^{simple}(Y, N, w).$$

Fulton (1969): Consider families of covers, i.e. covers of curve families$/T$:

$$f : \mathcal{X} \to Y_T = \mathbb{P}^1_T = \mathbb{P}^1 \times T.$$

Thus: For each $t \in T$, the fibre $f_t : \mathcal{X}_t \to (\mathbb{P}^1_T)_t = \mathbb{P}^1$ of f at t is a cover (of curves) in the previous sense, i.e. $f_t \in H(\ldots)$.

Expect: 1) For each family of covers $f : \mathcal{X} \to Y_T = \mathbb{P}^1_T$, the assignment $t \mapsto f_t$ defines (naturally) a holomorphic map $[f] : T \to H$.

2) Each holomorphic map $g : T \to H$ arises uniquely in this way, i.e. $g = [f]$, for a unique family of covers $f : \mathcal{X} \to Y_T$ (up to isomorphism).

Reformulation: Let

$$H^{simple}(Y_T/T, N, w)$$
$$= (\text{set of families of simple covers over } T \text{ with } f_t \in H^{simple}(Y, N, w), \forall t)/\sim.$$

It is easy to see that the assignment

$$T \mapsto H^{simple}(Y_T/T, N, w)$$

defines a functor

$$\mathcal{H}_{N,w} : \underline{\mathbb{C} - spaces} \to \underline{Sets},$$

and that

$$\text{Expectation} \Leftrightarrow \mathcal{H}_{N,w} \simeq F_H$$
$$\overset{def}{\Leftrightarrow} H \text{ represents the functor } \mathcal{H}_{N,w}.$$

Theorem B (Fulton, 1969): If $N \geq 3$, then the Hurwitz space $H^{simple}(Y, N, w)$ (as defined by Hurwitz) represents the above functor $\mathcal{H}_{N,w}$.

This theorem generalizes to the algebraic setting by replacing complex spaces by schemes:

Theorem C (Fulton, 1969): If $N \geq 3$, then the functor

$$\mathcal{H}_{N,w} : \underline{Sch} \to \underline{Sets}$$

is representable by a scheme $H_{N,w}/\mathbb{Z}$ of finite type. In particular, for any field K we have

$$H_{N,w}(K) = H^{simple}(\mathbb{P}^1/K, N, w).$$

In addition, the restriction of the discriminant map to $H_{N,w} \otimes \mathbb{Z}[1/N!] \subset H_{N,w}$,

$$\delta : H_{N,w} \otimes \mathbb{Z}[1/N!] \to (\mathbb{P}^1_{\mathbb{Z}[1/N!]})^{(w)} \setminus \Delta_w,$$

is finite and etale.

Remark: Little seems to be known about the geometric structure of $H_{N,w}$.

Aim: Study analogues of these results in the case that $Y = E$ is an elliptic curve (and $w = 2$).

Remark: In recent years, there have are an abundance of results and applications of Hurwitz spaces:

1) Inverse Galois theory: Fried, Völklein, . . .

- Fried, Völklein, Harbater, Debes, Wevers,. . . : studied moduli spaces of other types of covers $/\mathbb{P}^1$.

2) Moduli problems of curves: Fulton, Mumford and Harris, . . .

- used $H_{N,w}$ to study the geometry of M_g, the moduli space of curves of genus g.

3) String theory: Gromov/Witten, Dijkgraaf,. . .

also: Cordes/Moore/Ramgoolan, Kontsevich, . . .

3. The Case $Y = E$ and $w = 2$

Reference: IEM Preprint No. 9 (2001), IEM Essen.
(See also www.mast.queensu.ca/~kani)
– to appear in: Collectanea Mathematica

Let E/K be an elliptic curve over a field K (char $\neq 2$). Fix $N \geq 2$ prime to char(K).

Note: If $(X \xrightarrow{f} E) \in H^{simple}(E/K, N, 2)$, then by the Riemann-Hurwitz relation

$$2g_X - 2 = N(2g_E - 2) + w = w = 2 \Rightarrow g_X = 2.$$

More generally: Study the set $H^{(2)}(E/K, N)$ of all genus 2 covers of degee N of E/K:

$$f : X \to E, \quad \deg(f) = N \text{ and } g_X = 2.$$

Similarly, study the set $H^{(2)}(E_T/T, N)$ of families of such covers:

$$f : \mathcal{X} \to E_T = E \times T, \quad f_t \in H^{(2)}(E_t/K(t), N).$$

As before, the assignment $T \mapsto H^{(2)}(E_T/T, N)$ defines a functor

$$\mathcal{H}^{(2)}_{E/K,N} : \underline{Sch} \to \underline{Sets}.$$

Theorem 1. If N is odd, then $\mathcal{H}^{(2)}_{E/K,N}$ is representable by a smooth, quasi-projective surface $H^{(2)}_{E/K,N}$ over K which has (over $\overline{K}$)

$$\sum_{d|N} \sigma(d) - \sigma(N)$$

irreducible components. Thus $H^{(2)}_{E/K,N}$ is irreducible if and only if N is prime.

Remarks: 1) The above result does not extend to the case that N is even. However, a slightly weaker result is true in that case: the functor $\mathcal{H}$ is coarsely representable by such a variety.

2) The reason that H breaks up into components is the following:

Each $X \xrightarrow{f} E$ factors as $X \xrightarrow{f'} E' \xrightarrow{u_f} E$, where $u_f : E' \to E$ is the max. unramified subcover of f.

Thus: $H^{(2)}_{E/K,N}$ is a union of components which are indexed by subgroups $G \leq E$ with $\#G|N$ (and $\#G \neq N$); explictly, $G = \mathrm{Ker}(\hat{u}_f)$.

Definition: A cover $f : X \to E$ is called minimal if $\deg(u_f) = 1$.

Theorem 2. For every $N \geq 3$ (prime to $\mathrm{char}(K)$), the functor $\mathcal{H}^{(min)}_{E/K,N}$ which classifies minimal genus 2 covers is representable by a smooth, irreducible quasi-projective surface $H^{(min)}_{E/K,N}$ over K.

More precisely, we have

$$H^{(min)}_{E/K,N} \otimes_K \overline{K} \simeq E \times H_{E/K,N}$$

where $H_{E/K,N} \subset X(N)$ is an open subvariety (curve) of the modular curve $X(N)$ of (full) level N.

Remarks: 1) If $K = \mathbb{C}$, then $X(N) = \Gamma(N)\backslash\mathfrak{H}^*$, which is a Galois cover of $X(1) \simeq \mathbb{P}^1$ of degree

$$\overline{sl}(N) := |\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}|.$$

2) The reason that E appears as a factor of $\mathcal{H}^{(min)}_{E/K,N}$ is due to the fact that the group $E(K)$ acts on E and hence on $H^{(2)}_{E/K,N}$ etc. via translation: $f \mapsto T_x \circ f$.

Thus: introduce and study normalized covers.

Definition: A cover $f : X \to E$ with $g_X = 2$ is called normalized if it is minimal and if

$$f(W) \subset E[2] \text{ and } \#(f^{-1}(0_E) \cap W) = \begin{cases} 3 & N \text{ odd} \\ 0 & \text{else} \end{cases}$$

where $W = \mathrm{Fix}(\sigma_X)$ denotes the set of 6 Weierstrass points of X. (Here: σ_X is the hyperelliptic involution of X.)

Notes: 1) If $f : X \to E$ is minimal, then $\exists!\, y \in E(K)$ such that $T_y \circ f : X \to E$ is normalized.

2) If f is normalized, then $f \circ \sigma_X = [-1]_E \circ f$. Thus $\mathrm{Disc}(f)$ is symmetric with respect to $[-1]_E$, i.e. $[-1]_E^* \mathrm{Disc}(f) = \mathrm{Disc}(f)$.

Example: Let

$$\begin{aligned} E: \quad y^2 &= (x-a)(x-b)(x-c), \quad abc \neq 0 \\ X: \quad s^2 &= (t^2-a)(t^2-b)(t^2-c). \end{aligned}$$

Then the cover $f : X \to E$, given by $f^*x = t^2$, $f^*y = s$, is normalized and of degree 2.

Theorem 3. For every $N \geq 3$ (as above), the functor $\mathcal{H}_{E/K,N}$ which classifies normalized genus 2 covers is representable by a smooth, irreducible affine curve $H_{E/K,N}/K$ such that $H_{E/K,N} \otimes \overline{K} \subset X(N)$.

Theorem 4: Let

$$D_{E/K,N} = X(N)_{/\overline{K}} \setminus (H_{E/K,N} \otimes \overline{K})$$

denote the degeneracy locus. Then

$$\#D_{E/K,N} \le \tfrac{1}{12N}(5N+6)\overline{sl}(N),$$

and equality holds if and only if $\mathrm{char}(K) \nmid N!$.

– reinterpretation of results of Crelle J. 485 (1997) + J. No. Th. 64 (1997).

Theorem 5: The assignment $(X \xrightarrow{f} E) \mapsto \mathrm{Disc}(f)$ is represented by a quasi-finite morphism

$$\delta = \delta_{E/K,N} : H_{E/K,N} \to \mathbb{P}^1_K \simeq (E^{(2)})^{sym}.$$

Furthermore, if $\mathrm{char}(K) \nmid N!$, then δ is finite and unramified outside of $\pi_E(E[2]) \subset \mathbb{P}^1$.

Theorem 6: If $\mathrm{char}(K) \nmid N!$, then

$$\deg(\delta_{E/K,N}) = \tfrac{1}{6}(N-1)\overline{sl}(N).$$

Remarks: 1) This degree can be viewed as a measure of non-rigidity of coverings ($\to$ Völklein).

2) H. Völklein proved Theorem 6 for $N = 3, 5, 7$ by using group theory (and a computer).

4. Some applications

(a) Rationality Questions (K a number field)

Since $g_{X(N)} \geq 2$ for $N \geq 7$, we have by Faltings' theorem (= Mordell's Conjecture):

Corollary 1: $\#\mathcal{H}_{E/K,N}(K) < \infty$, if $N \geq 7$.

Question: Is $\mathcal{H}_{E/K,N}(K) = \emptyset$, for $N >> 0$?

This is false (even for N prime), for there exist curves X/K with ∞'ly many $f_N : X \to E$ for which $N = \deg(f_N)$ is prime.

Conjecture (*) For each E/K there exist only finitely many genus 2 curves X/K which have a (minimal) morphism $f : X \to E$ of degree $N \geq 7$.

Remark: ABC conj. $\Rightarrow$ Asym. Fermat $\Rightarrow$ Conj. (*).

Moreover, the converse: Conj. (*) $\Rightarrow$ Asym. Fermat is "almost true": it implies a slightly weaker version of Frey's Conjecture 5 (which by Frey and Wiles is equivalent to the Asymptotic Fermat Conjecture (for $K = \mathbb{Q}$).)

(b) Moduli

Question: For which curves Y/K does there exist a (minimal) morphism $f : Y \to E$ of degree N?

Corollary 2: For every N there exists a morphism

$$\mu_N : H_{E(N)/X'(N),N} \to M_2$$

to the moduli space of curves of genus 2. Moreover:

a) $\mathrm{Im}(\mu_N) =$ Humbert surface with Inv. $\Delta = N^2$;

b) $\deg(\mu_N) = 2\overline{sl}(N)$; more precisely,

$\mathrm{Im}(\mu_N) \sim Z^{sym}_{N,-1} := (X(N) \times X(N))/\langle \Delta_{N,-1}, \tau \rangle$,

where $\tau(x, y) = (y, x)$ and

$$\Delta_{N,-1} = \{(g, \alpha_{-1}(g)) : g \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}\},$$

where $\alpha_{-1}(g) = Q_{-1} g Q_{-1}^{-1}$ with $Q_{-1} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

In particular, the normalization (and compactification) of the Humbert surface $\mathrm{Im}(\mu_N)$ is the symmetric Diagonal Quotient Surface $Z^{sym}_{N,-1}$.

(c) Counting Covers: $(K = \overline{K})$

Corollary 3: If $N \geq 2$ and $\mathrm{char}(K) \nmid N$, then for every $R \subset E$ with $\#R = 2$ we have

$$c_N := \sum_{f \in H^s(E/K,N,R)} \frac{1}{|\mathrm{Aut}(f)|} = \tfrac{1}{3}(N\sigma_3(N) - N^2\sigma_1(N)),$$

where $\sigma_k(N) = \sum_{d|N} d^k$. Thus, if $\mathrm{char}(K) = 0$, then $F_2(q) := \sum c_N q^N$ is a quasi-modular form of weight 6; explicitly we have

$$(1) \qquad F_2(q) = \frac{1}{51840}(10E_2^3 - 6E_2E_4 - 4E_6),$$

where $E_k = 1 + b_k \sum_{n\geq 1} \sigma_{k-1}(n)q^n$ with $b_2 = -24, b_4 = 240$ and $b_6 = -504$.

Remarks: 1) The identity (1) was first proven by R. Dijkgraaf (1995) by using the methods of mirror symmetry ($\rightarrow$ B. Mazur).

2) Theorem 6 $\Rightarrow$ Corollary 3 by using the identities

$$\sum_{n|N} \sigma_1(n)\mathrm{sl}(N/n) = \sigma_3(N),$$

$$\sum_{n|N} n\sigma_1(n)\mathrm{sl}(N/n) = N^2\sigma_1(N).$$

(d) Curves with minimal degeneration:

Let $H = H_{E/K,N} \subset X = X(N)$ be the moduli space,
$f : Y_N \to E_H = E \times H$ the universal cover,
$p : \overline{Y}_N \to X$ the minimal model of Y_N over X,
$h_{\overline{Y}_N/X} = \deg_X(p_*\omega^0_{Y_N/X})$ its modular height.

Corollary 4: The curve $\overline{Y}_N/X(N)$ is semi-stable and has bad reduction at $X \setminus H$. Furthermore, its Jacobian $J = J_N$ has bad reduction at $X(N)_\infty := X(N) \setminus X'(N)$, and its modular height is

$$h_{\overline{Y}_N/X} = h_{J/X} = \tfrac{1}{2}(2g_{X(N)} - 2 + \#X(N)_\infty).$$

In particular, for $N = 3$ one thus obtains a semi-stable family $p : \overline{Y}_3 \to \mathbb{P}^1$ whose Jacobian has precisely 4 places of bad reduction.

Remarks: 1) By a theorem of Faltings it follows (in $\mathrm{char} = 0$) that for any such curve we have the inequality

$$h_{\overline{Y}_N/X} = h_{J/X} \leq \tfrac{1}{2}(2g_{X(N)} - 2 + \#X(N)_\infty).$$

2) In a recent preprint E. Viehweg and K. Zuo study the structure of families of abelian varieties with such "minimal degeneration".

5. The Basic Construction

Reference: Frey/K., Curves of genus 2 covering elliptic curves … (Texel Conference, 1989)

Given:

$$\begin{array}{c} X \\ f\downarrow \\ E \end{array} \rightsquigarrow \quad \begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ E & & E^{\perp} \end{array} \quad \rightsquigarrow \psi : E[N] \xrightarrow{\sim} E^{\perp}[N].$$

(via the duality theory of J_X.)

Conversely: given anti-isometry $\psi : E[N] \to E'[N]$, one can recover a (normalized) genus 2 cover

$$f_\psi : X_\psi \to E.$$

However: the curve X_ψ may be reducible! $\Rightarrow H_{E/K,N} \subset X_{E/K,N,-1}$.

Note: 1) The moduli space $X_{E/K,N,-1}$ classifies pairs (E', ψ), where $\psi : E[N] \to E'[N]$ is an anti-isometry.

2) This construction also works for families! (Cf. IEM Preprint, op. cit.): $\Rightarrow$ Theorem 3 $\Rightarrow$ Theorem 2 $\Rightarrow$ Theorem 1.

6. Proof of Theorem 6 (Overview)

Remark: The proof of Theorem 6 uses the methods of Arithmetic Algebraic Geometry.

More precisely, it uses:

– a study of degenerations of the universal cover

$$f_{univ} : X_H \to E \times H;$$

In other words:

1) study the degeneration of the minimal model $M(X_H)$ of X_H; this uses the modular height of relative curves.

2) study whether or not f_{univ} extends to a cover

$$f : M(X_H) \to E \times X(N).$$

– intersection theory on $M(X_H)$.

7. Study of Degenerations

Let $H = H_{E/\overline{K},N}$ denote the moduli space,
$f_H : Y_H \to E_H = E \times_{\overline{K}} H$ the universal cover,
$X = X(N) \supset H$ the natural compactification,
$\overline{Y}/X$ the minimal model of the generic fibre of Y_H.

Facts. 1) The fibres of $\overline{Y}/X$ are semi-stable.

2) f_H extends to a morphism $f = f_X : \overline{Y} \to E_X$ which is finite if and only if $\mathrm{char}(K) \nmid N!$.

Theorem 7: Suppose $\mathrm{char}(K) \nmid N!$. Then:

(a) The fibres $\overline{Y}_x$ of $\overline{Y}/X$ are stable curves with at most one singularity.

(b) $\overline{Y}_x$ is singular if and only if $x \in D_{E/\overline{K},N} = X_\infty \,\dot\cup\, X_1$, where X_∞ is the set of cusps of X. (Note that $\#X_\infty = \overline{sl}(N)/N$.)

(c) If $x \in X_\infty$, then $\overline{Y}_x$ is an irreducible curve whose normalization is a curve of genus 1.

(d) If $x \in X_1$, then $\overline{Y}_x = E_{x,1} \cup E_{x,2}$ is the union of two curves of genus 1 which meet transversely in a unique point P_x.

8. Calculation of Intersection Numbers

Let $F = \kappa(X)$ denote the function field of $X = X(N)$,
$f_F : Y_F \to E_F$ the generic cover over F,
$D_F = \mathrm{Diff}(f_F)$ the different divisor of f_F,
$W_{C_F} \in \mathrm{Div}(Y_F)$ the hyperelliptic divisor of Y_F,
D and W their respective closures in $\overline{Y}$,
$\omega^0_{\overline{Y}/X}$ the relative dualizing sheaf of $p_{\overline{Y}} : \overline{Y} \to X$.

Theorem 8: The modular height of $\overline{Y}/X$ is

$$h_{\overline{Y}/X} := \deg((p_{\overline{Y}})_*(\omega^0_{\overline{Y}/X})) = \frac{1}{12}\overline{sl}(N),$$

and the self-intersection number of $\omega^0_{\overline{Y}/X}$ is

$$(\omega^0_{\overline{Y}/X})^2 = \frac{7}{5}\#X_1 + \frac{1}{5}\#X_\infty = \frac{1}{12N}(7N-6)\overline{sl}(N).$$

Remark: The proof uses Theorem 4, the Noether formula and Mumford's formula (which holds if $g = 2$):

$$h = \omega^2 + \delta_0 + \delta_1 \quad \text{and} \quad 5\omega^2 = \delta_0 + 7\delta_1,$$

where $h = h_{\overline{Y}/X}, \omega = \omega^0_{\overline{Y}/X}$), and δ_0 (respect. δ_1) is the number of singular points of all fibres which do not (respect. do) disconnect the fibre.

Theorem 9: (a) D is an irreducible curve on $\overline{Y}$ which represents the dualizing sheaf: $\omega^0_{\overline{Y}/X} \sim D$.

(b) If $q_1 = pr_1 \circ f_{|D} : D \to E$ and $q_2 = pr_2 \circ f_{|D} : D \to X$, then $\pi_E \circ q_1 = \overline{\delta}_{E,N} \circ q_2$, where $\overline{\delta} : X \to \mathbb{P}^1$ is the unique extension of $\delta : H \to \mathbb{P}^1$. Thus

$$\deg(\overline{\delta}) = \deg(q_1) = (\omega^0_{\overline{Y}/X} . f^*(P \times X)).$$

(c) We have $6D \sim 2W + f^*(E \times A)$, for some $A \in \mathrm{Div}(X)$, and hence

$$\deg(q_1) = \frac{N}{6}\deg(A) = \frac{N}{36}(9(\omega^0_{\overline{Y}/X})^2 - W^2).$$

(d) The self-intersection number of W is

$$W^2 = \frac{6}{7}\#X_1 - \frac{9}{7}\omega^2 = -\frac{3}{4N}(N-2)\overline{sl}(N).$$

Remark: To compute W^2, consider the pullback W^* of W to (the desingularization of) $\overline{Y} \times_X X(2N)$, and observe that $W^* = W_1 + \ldots + W_6 + B$, where the W_i's are 6 disjoint sections and B is a fibral divisor supported on the fibres over $X(2N)_\infty$.