Mazur's Question and Modular Diagonal Quotient Surfaces

Introduction

Let E/K be an elliptic curve over a number field K,
N an odd prime,
$\bar{\rho}_{E/K,N} : G_K \to \mathrm{Aut}(E[N]) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$
its associated Galois representation modulo N.

Question: To what extent is the isogeny class of E/K determined by the isomorphism class of $\bar{\rho}_{E/K,N}$?

Mazur (1978): $\exists$? E and $E'/\mathbb{Q}$ with $E \not\sim E'$ such that $\bar{\rho}_{E/K,N} \simeq \bar{\rho}_{E'/K,N}$ for some $N \geq 7$?

Kraus-Oesterlé (1992): Yes! (for $N = 7$).

Frey + group ($\sim$ 1993): Computer search: lots of examples for $N = 7, 11$.

Halberstadt-Kraus (1997): $\exists$ ∞'ly many examples for $N = 7$.

K.-Rizzo (1999): $\exists$ ∞'ly many families of examples for $N = 11$.

Note: Faltings' Theorem (=Mordell Conjecture) $\Rightarrow$
$\mathbb{S}_{N,E}(K) \stackrel{\text{def}}{=} \{E'/K : \bar{\rho}_{E'/K,N} \simeq \bar{\rho}_{E/K,N}, \}/\simeq$
is finite, for all $N \geq 7$.

Conjecture 1 (Frey, 1988): $\exists$ a constant $M_{E,K}$ s. th.
$\mathbb{S}'_{N,E}(K) := \{E' \in \mathbb{S}_{N,E}(K) : E' \not\sim E\} = \emptyset$,
for $N \geq M_{E,K}$.

Theorem 0 (Frey, 1996): For $K = \mathbb{Q}$, Conjecture 1 is equivalent to the Asymptotic Fermat Conjecture:

(AFC) For every $a, b, c \in \mathbb{Z}, abc \neq 0$, the set

$$F_{a,b,c} = \bigcup_{n \geq 4} \{(x_n, y_n, z_n) \in \mathbb{Z}^3 : ax_n^n + by_n^n = cz_n^n, \\ (x_n, y_n, z_n) = 1\}$$

is finite. [Wiles(1993): $\#F_{1,1,1} = 10$.]

Conjecture 2 (Darmon, 1994): $\exists$ constant M_K s. th.

$$\mathbb{S}'_N(K) := \bigcup_{E/K} \mathbb{S}'_{N,E}(K) = \emptyset, \quad \forall N \geq M_K.$$

Conjecture 3 (Darmon, 1994): $\exists$ constant M s. th.

$$\#(\mathbb{S}'_N(K)/\text{twists}) < \infty, \quad \forall N \geq M.$$

Conjecture 3′: Conjecture 3 is true for $M = 23$, provided we restrict attention to prime N's.

Refinement: Let's write $E \approx E'$ if there exists a cyclic isogeny $f : E \to E'$ with $\deg(f) \leq 27$ (and $\deg(f) \neq 22, 23, 26$).

Conjecture 4: For any number field K, the set

$$\mathbb{S}_N^*(K) := \{(E, E') : E \in \mathbb{S}_{N,E'}(K), E \not\approx E'\}/\simeq$$

is finite modulo twists, for all primes $N \geq 23$.

Remark. Clearly, Conjecture 4 $\Rightarrow$ Conjecture 3′ (because $\mathbb{S}_N^*(K) \supset \mathbb{S}'_N(K)$).

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Aims of this talk:

1) To explain the above conjectures (and counterexamples) in terms of the underlying geometrical objects (modular curves and surfaces).

2) To study related conjectures concerning the geometry of these objects.

1. Geometric Interpretations

(a) Special Case: E/K fixed

Notation: For E/K and $N \geq 1$, let

$$\tilde{\mathbb{S}}_{N,E}(K) = \{(E', \psi) : E' \in \mathbb{S}_{N,E}(K)\}/\simeq$$

where $\psi : \bar{\rho}_{E/K,N} \xrightarrow{\sim} \bar{\rho}_{E'/K,N}$.

Theorem 1. If $N \geq 3$, then there is an affine smooth curve $X_{E/K,N}$ over K which parametrizes the sets $\{\tilde{\mathbb{S}}_{N,E}(K')\}_{K'/K}$, i.e. there are natural bijections

$$\tilde{\mathbb{S}}_{N,E}(K') \xrightarrow{\sim} X_{E/K,N}(K'), \quad \forall K'/K.$$

Moreover, $X_{E/K,N}$ consists of $\phi(N)$ irreducible components $\{X_{E/K,N,\varepsilon}\}_{\varepsilon \in (\mathbb{Z}/N\mathbb{Z})^\times}$, and we have

$$X_{E/K,N,\varepsilon} \otimes \mathbb{C} \simeq X'(N) = \Gamma(N)\backslash\mathfrak{H},$$

where $\Gamma(N) = \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$.

Remarks 1) This result is well-known and can be proved by the methods of Katz/Mazur's book.

2) The curve $X_{E/K,N,\varepsilon}$ classifies pairs (E', ψ) with $\det(\psi) = \varepsilon$, i.e. $\psi : E[N] \xrightarrow{\sim} E'[N]$ satisfies

$$e_N^{E'}(\psi(P), \psi(Q)) = e_N^E(P, Q)^\varepsilon, \quad \forall P, Q \in E[N].$$

(b) General Case

Notation: For $K/\mathbb{Q}$ and $N \geq 1$, let

$$\tilde{\mathbb{S}}_N(K) = \{(E, E', \psi) : E' \in \mathbb{S}_{N,E}(K)\}/\simeq.$$

Theorem 2. If $N \geq 3$, then there is an affine normal surface Z_N over $\mathbb{Q}$ which coarsely parametrizes the sets $\{\tilde{\mathbb{S}}_N(K)\}_{K/\mathbb{Q}}$, i.e. there are maps

$$\tilde{\mathbb{S}}_N(K) \to Z_N(K), \quad \forall K/\mathbb{Q},$$

which are injective modulo (simutaneous) twists.

Moreover, Z_N consists of $\phi(N)$ irreducible components $\{Z_{N,\varepsilon}\}_{\varepsilon \in (\mathbb{Z}/N\mathbb{Z})^\times}$, and we have

$$Z_{N,\varepsilon} \otimes \mathbb{C} \simeq \tilde{\Delta}_{N,\varepsilon} \backslash (\mathfrak{H} \times \mathfrak{H}),$$

where $\Gamma(N) \times \Gamma(N) \leq \tilde{\Delta}_{N,\varepsilon} \leq \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$,

Remarks 1) This can be proved by an extension of the methods of Katz/Mazur's book.

2) The irreducible surface $Z_{N,\varepsilon}$ (coarsely) classifies triples (E, E', ψ) with $\det(\psi) = \varepsilon$.

3) In general, $Z_{N,\varepsilon} \otimes \mathbb{C} \not\simeq Z_{N,\varepsilon'} \otimes \mathbb{C}$ unless $\varepsilon \equiv \varepsilon' x^2 \pmod{N}$, for some $x \in \mathbb{Z}$.

4) Clearly, $Z_{N,\varepsilon} \otimes \mathbb{C}$ is the (finite) quotient

$$\Phi : X'(N) \times X'(N) \to Z_{N,\varepsilon} \otimes \mathbb{C}$$

by the group $\Delta_{N,\varepsilon} = \tilde{\Delta}_{N,\varepsilon}/(\Gamma(N) \times \Gamma(N))$. Explicitly, $\Delta_{N,\varepsilon}$ is the "twisted" diagonal subgroup

$$\Delta_{N,\varepsilon} = \{(g, \alpha_\varepsilon(g)) : g \in G_N\} \leq G_N \times G_N$$

of $G_N = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Here $\alpha_\varepsilon \in \mathrm{Aut}(G_N)$ is defined by $\alpha_\varepsilon : g \mapsto Q_\varepsilon g Q_\varepsilon^{-1}$, where $Q_\varepsilon = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Thus, $Z_{N,\varepsilon} \otimes \mathbb{C}$ (and $Z_{N,\varepsilon}$) may be called a modular diagonal quotient surface.

5) The (affine) surface $Z_{N,\varepsilon} \otimes \mathbb{C}$ has a natural compactification

$$\bar{Z}_{N,\varepsilon} \otimes \mathbb{C} := \Delta_{N,\varepsilon} \backslash (X(N) \times X(N)),$$

where $X(N)_{/\mathbb{C}} = \Gamma(N) \backslash \mathfrak{H}^\star$ is the usual modular curve of level N.

6) Let $X(N)_{/\mathbb{Q}}$ denote Shimura's canonical model of $X(N)$. Then the above quotient map descends uniquely to a morphism

$$\bar{\Phi}_{\mathbb{Q}} : X(N)_{/\mathbb{Q}} \times X(N)_{/\mathbb{Q}} \to \bar{Z}_{N,\varepsilon} \supset Z_{N,\varepsilon}.$$

2. The Geometry of $Z_{N,\varepsilon}$

Notation. Let $\tilde{Z}_{N,\varepsilon}$ denote the minimal desingularization of the surface $\bar{Z}_{N,\varepsilon} \otimes \mathbb{C}$ (which has a finite number of (isolated) cyclic quotient singularities.)

Theorem 3 (C.F. Hermann; K.-Schanz) The rough classification type of $\tilde{Z}_{N,\varepsilon}$ is completely determined by its geometric genus $p_g = p_g(\tilde{Z}_{N,\varepsilon})$; in particular, its Kodaira dimension is

$$\kappa(\tilde{Z}_{N,\varepsilon}) = \min(p_g - 1, 2).$$

Remark. In his paper, C.F. Hermann calls the surface $\bar{Z}_{N,\varepsilon} \otimes \mathbb{C}$ a "degenerate Hilbert modular surface" of discriminant $\Delta = N^2$.

Corollary. $\tilde{Z}_{N,\varepsilon}$ is of general type $\forall \varepsilon \Leftrightarrow N \geq 13$.

Remark. We also have:

- The surface $\tilde{Z}_{7,1}$ is rational $\longrightarrow$ Halberstadt-Kraus.
- The surface $\tilde{Z}_{11,1}$ is elliptic $\longrightarrow$ K.-Rizzo.

Since such surfaces have (in general) many rational points, one can expect many counterexamples to Mazur's question for these values of N.

3. Hecke Curves on Z_N and Conjecture 4

Need: a geometric interpretation of the condition "$E \sim E'$" or "$E \approx E'$".

Recall: The modular curve $X_0'(n) = X_0(n) \setminus \{\text{cusps}\}$ (coarsely) parametrizes the sets

$$\mathcal{X}_0(n)(K) = \{(E, E', f)_{/K} : E \xrightarrow{f} E' \text{ cyclic}, \deg(f) = n\}.$$

Thus, if n, k satisfy $(nk, N) = 1$, then the rule

$$(E, E', f) \mapsto (E, E', (kf)_{|E[N]})$$

induces morphisms

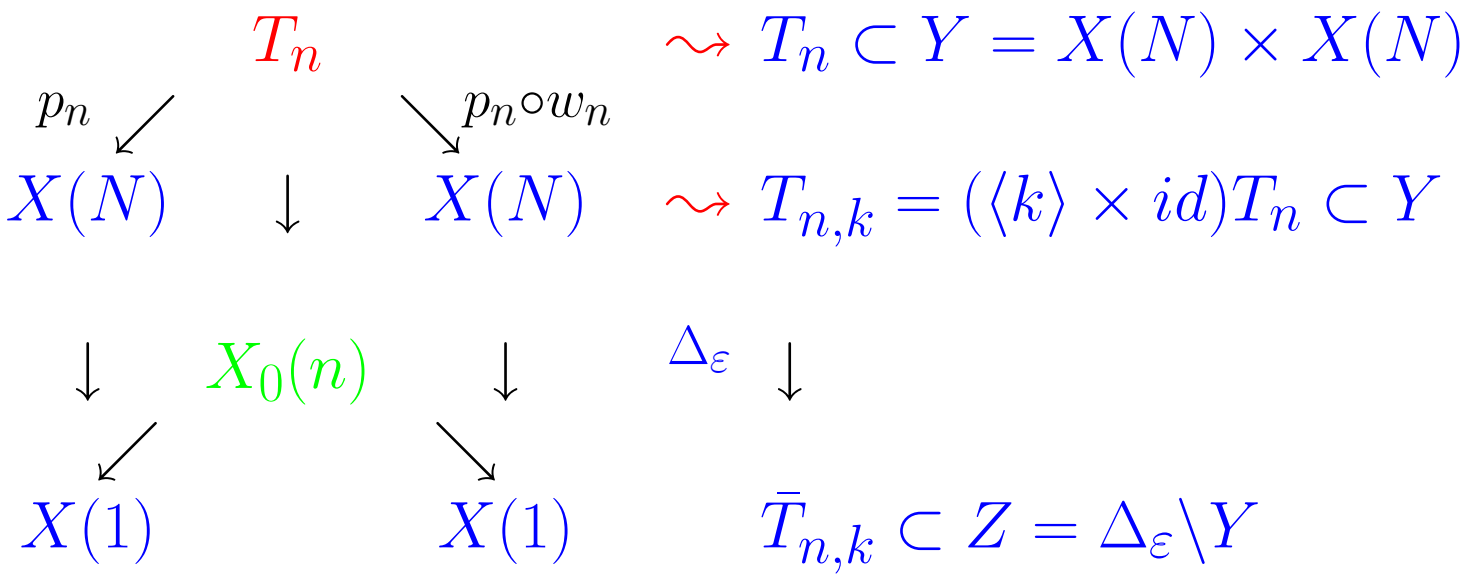
$$\tau_{n,k} : X_0'(n) \to Z_N \text{ and } \bar{\tau}_{n,k} : X_0(n) \to \bar{Z}_N.$$

We call the image $\bar{T}_{n,k} = \text{Im}(\bar{\tau}_{n,k}) \subset \bar{Z}_{N,\varepsilon}$ a Hecke curve; here $\varepsilon \equiv nk^2 (\text{mod } N)$.

Remarks. 1) The map $\bar{\tau}_{n,k} : X_0(n) \to \bar{T}_{n,k}$ is finite and birational. Thus $g(\bar{T}_{n,k}) = g(X_0(n))$ and hence

$$g(\bar{T}_{n,k}) \leq 1 \quad \Leftrightarrow \quad n \leq 27, n \neq 22, 23, 26.$$

2) As the name suggests, the $T_{n,k}$'s are closely connected to the (irreducible) Hecke correspondences $T_n = T(1, n)$ on $X(N)$:



Conjecture 5: If $N \geq 23$ is prime, then every curve C on $\bar{Z}_{N,\varepsilon}$ of genus $g(C) \leq 1$ is modular, i.e. $C = \bar{T}_{n,k}$, for some n, k.

Remark. Conj. 4 $\Rightarrow$ Conj. 5

$\Leftarrow$ via Lang's Conjecture

Lang's Conjecture: If Z is a surface of general type and

$$Z_{exc} = \bigcup_{\substack{C \subset Z \\ g(C) \leq 1}} C,$$

then **a)** Z_{exc} consists of finitely many curves;
b) the open variety $Z \setminus Z_{exc}$ is Mordellic.

Remark. Conjecture 5 $\Rightarrow$ Lang's Conjecture, part a) for $\bar{Z}_{N,\varepsilon}$.

4. Evidence for Conjecture 5

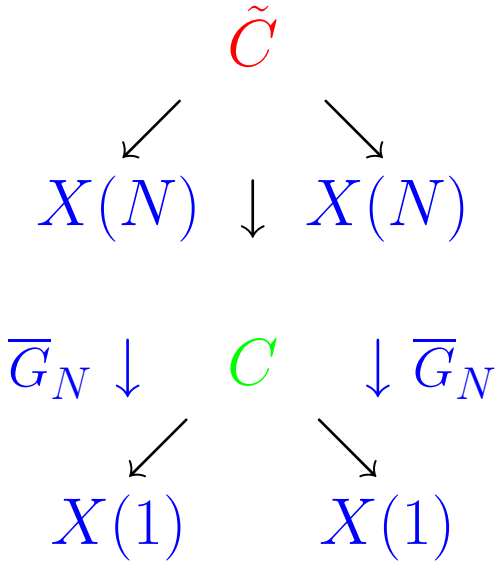
a) G_N-equivariant curves:

Let $\overline{G}_N = G_N/\{\pm 1\} = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$.

Proposition 1. If $N \geq 23$ is prime, then

(a) $H < \overline{G}_N \Rightarrow g(H\backslash X(N)) \geq 2$.

(b) Every curve C on $Z_{N,\varepsilon}$ with $g(C) \leq 1$ lifts to a $\Delta_{N,\varepsilon}$-equivariant curve $\tilde{C}$ on $X(N) \times X(N)$:



However: $\exists$ ∞'ly many $\Delta_{N,\varepsilon}$-equivariant curves C on $Z_{N,\varepsilon}$ with sufficiently large genus $g(C) >> 0$.

b) Minimal models:

Conjecture 6: (Hermann, 1991) If $N \geq 7$, then the minimal model $\tilde{Z}^{min}_{N,\varepsilon}$ of $\tilde{Z}_{N,\varepsilon}$ is obtained by blowing down "known curves".

Remarks. 1) Conj. 5 $\Rightarrow$ Conjecture 6 (for $N \geq 23$).

2) Conjecture 6 $\Leftrightarrow$ explicit formula for $P_2(\tilde{Z}^{min})$

$\Leftrightarrow$ explicit formula for $K^2_{\tilde{Z}^{min}}$.

In particular: Conject. 6 $\Rightarrow$ $K^2_{\tilde{Z}^{min}} - K^2_{\tilde{Z}} \leq 6$.

(Note: Vanishing thms $\Rightarrow$ $K^2_{\tilde{Z}^{min}} - K^2_{\tilde{Z}} \leq f(N)$, where $f(N)$ is a quadratic polynomial in N.)

3) Conjecture 6 is a natural analogue of a Conjecture of Hirzebruch/Zagier for Hilbert modular surfaces; this latter conjecture was proven by C.F. Hermann in 1987 in many cases. His method also yields:

Theorem 4 (Hermann) If $N \equiv 7 \pmod 8$ is prime and $\varepsilon \equiv -1 \pmod N$, then Conjecture 6 is true.

Remark. Conjecture 6 is true for $N \leq 13$.

5. The Néron-Severi Group of $Z_{N,\varepsilon}$

Key Open Question: If $N = p \geq 23$, is every curve $C \subset \bar{Z}_{N,\varepsilon}$ of genus ≤ 1 a modular curve, i.e. of the form $C = T_{n,k}$?

Simpler Question: Up to (algebraic) equivalence, are all the curves/divisors on $\bar{Z}_{N,\varepsilon}$ modular?

Notation: Let $NS^0(\bar{Z}_{N,\varepsilon}) = NS(\bar{Z}_{N,\varepsilon}) \otimes \mathbb{Q}$, where $NS(\bar{Z}_{N,\varepsilon})$ denotes the Neron-Severi group of $\bar{Z}_{N,\varepsilon}$, i.e.

$$NS(\bar{Z}_{N,\varepsilon}) = \mathrm{Div}(\bar{Z}_{N,\varepsilon})/(\text{algebraic equivalence}).$$

In addition, we write

$$\overline{NS}^0(Z_{N,\varepsilon}) = NS^0(Z_{N,\varepsilon})/\langle cl(X_P)), cl(X'_P)\rangle,$$

where $X_P = \overline{\Phi}_{\mathbb{Q}}(X \times P)$ and $X'_P = \overline{\Phi}_{\mathbb{Q}}(P \times X)$ for $X = X(N)_{/\mathbb{Q}}$ and $P \in X(\mathbb{Q})$. Here, as above, $\overline{\Phi}_{\mathbb{Q}}$ denotes the "quotient" map

$$\overline{\Phi}_{\mathbb{Q}} : X \times X \to \bar{Z}_{N,\varepsilon}.$$

Recall: The theory of correspondences shows that there is a close connection between the Neron-Severi group $NS(X \times X)$ of the product surface and the endomorphism ring $\mathrm{End}_{\mathbb{Q}}(J_X)$ of its Jacobian J_X. Thus:

Proposition 2. We have a canonical identification

$$\overline{NS}^0(Z_{N,\varepsilon}) \simeq \mathrm{End}^0_{G_N,\varepsilon}(J(N)),$$

where $J(N) = J_{X(N)}$ and

$$\mathrm{End}^0_{G_N,\varepsilon}(J(N)) = \{f \in \mathrm{End}^0_{\mathbb{Q}}(J(N)) : gf = f\alpha_\varepsilon(g), \forall g \in G_N\}.$$

Notation: Let $\mathbb{T}_\varepsilon \subset \mathrm{End}^0_{\mathbb{Q}}(J(N))$ denote the $\mathbb{Q}$-vector space generated by the $T_{n,k}$'s with $nk^2 \equiv \varepsilon \pmod N$. Thus

$$\mathbb{T}_\varepsilon \subset \mathrm{End}^0_{G_N,\varepsilon}(J(N)),$$

and

$$\sum_\varepsilon \mathbb{T}_\varepsilon = \langle T_n : (n, N) = 1\rangle =: \mathbb{T}'.$$

Theorem 5. If $N = p$ is prime, then

$$\overline{NS}^0(\bar{Z}_{p,\varepsilon}) \simeq \mathbb{T}_\varepsilon.$$

Conclusion: Thus, up to algebraic equivalence, all divisors in $Z_{p,\varepsilon}$ are modular, i.e. they are $\mathbb{Q}$-linear combinations of the divisors $\bar{T}_{n,k}$ with $nk^2 \equiv \varepsilon\,(N)$, together with the two curves X_P and X'_P.

Theorem 6. The rank of the Néron-Severi group (Picard number) is given by

$$\text{rk } NS(\bar{Z}_{p,\varepsilon}) = 2 + \frac{1}{24}(p-1)(p-5) + \frac{1}{2}\left(\frac{\varepsilon}{p}\right) h(p),$$

where

$$h(p) = \begin{cases} h(\mathbb{Q}(\sqrt{-p})) & \text{if } p \equiv 3(4) \\ 0 & \text{if } p \equiv 1(4) \end{cases}$$

Remarks. 1) The proofs of Theorems 5 and 6 depend on certain structure theorems for $\text{End}^0(J(N))$.

2) The analogous assertion of Theorem 5 for the Néron-Severi group of $\bar{Z}_{N,\varepsilon} \otimes \mathbb{C}$ is false for two reasons:

1) There exist modular curves on $\bar{Z}_{N,\varepsilon} \otimes \mathbb{C}$ which are not algebraically equivalent to linear combinations of the $T_{n,k}$'s;

2) The presence of CM-elliptic curves in $J_{X(N)} \otimes \mathbb{C}$ gives rise to cycles which are not modular.

6. The Structure of $\mathbf{End}^0_{\mathbb{Q}}(J(N))$

Let Γ be a congruence group with $\Gamma_1(N) \leq \Gamma \leq \Gamma_0(N)$,

$X_\Gamma = \Gamma\backslash\mathfrak{H}^*$ be the associated modular curve,

$X = X_{\Gamma,\mathbb{Q}}$ its canonical model over $\mathbb{Q}$,

$J = J_X$, its Jacobian variety of dimension g_X,

$\mathbb{E} = \mathrm{End}^0_{\mathbb{Q}}(J) = \mathrm{End}_{\mathbb{Q}}(J) \otimes \mathbb{Q}$.

Recall: $\mathbb{E}$ contains the (semi-simple) subalgebra

$$\mathbb{T}' \;=\; \langle T_n : n \geq 1, (n, N) = 1\rangle \;\subset\; \mathbb{E}.$$

Note: If $N = p$ is prime, then $\mathbb{T}' = \mathbb{E}$ (Ribet), but in general these two algebras are different.

Reason: For each pair (M, d) with $Md|N$, there is a degeneracy morphism (Mazur)

$$B_{M,d} : X \;\to\; X_M,$$

where X_M is the corresponding curve of level M, and these give rise to new endomorphisms

$$D_{M,d} := B^*_{M,1} \circ (B_{M,d})_*,\; {}^tD_{M,d} := B^*_{M,d} \circ (B_{M,1})_*$$

Theorem 7. $\mathbb{E} = \langle \mathbb{T}', \{D_{M,d}, {}^tD_{M,d} : Md|N\}\rangle$.

Corollary. $Z(\mathbb{E}) = \mathbb{T}'$.

Remark. The above result also applies to other modular curves such as $X = X(N)$. From this one obtains:

Corollary. If $N = p$ is prime, then there exist $\tau, \tau' \in \mathbb{Q}[G_N]$ such that

$$\mathrm{End}^0_{\mathbb{Q}}(J(N)) = \langle \mathbb{T}', \tau, \tau' \rangle.$$

Theorem 8. If $X = X(N)$, then

$$\mathbb{T}' = \bigoplus_{\varepsilon} \mathbb{T}_{\varepsilon}.$$