The Number of

Covers of an Elliptic Curve

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1. Introduction

Problem (Hurwitz, 1891): Let X be a compact Riemann surface, $W \subset X$ a finite subset and $n \geq 1$. Calculate $\#H(X,W,n)$, where

$$H(X,W,n) := \{Y \xrightarrow{f} X : \deg(f) = n, f \text{ is simply branched over } W\}/\simeq.$$

Observation: It is easier to calculate the weighted number of such covers:

$$N_X(w,n) := \frac{1}{n!}\varphi_X(w|n) := \sum_{f\in H(X,W,n)} \frac{1}{|\operatorname{Aut}(f)|}.$$

This depends only on $w = \#W$.

Notation (Hurwitz, 1901). Put

$$\begin{aligned}
\Phi_X(q,\lambda) &= \sum_{n\geq 1}\sum_{w\geq 0} N_X(w,n)q^n\frac{\lambda^w}{w!} \\
F_X(q,\lambda) &= e^{\Phi_X(q,\lambda)} - 1 \\
&= \sum_{n\geq 1}\sum_{w\geq 0} f_X(w|n)\frac{q^n}{n!}\frac{\lambda^w}{w!}
\end{aligned}$$

Thus: $\Phi_X(q,\lambda) = \log(1 + F_X(q,\lambda))$, and hence the $N_X(w,n)$'s can be computed from the $f_X(w|n)$'s.

Theorem A (Hurwitz, 1891/1901): If $X = \mathbb{P}^1_{\mathbb{C}}$, then

(a) $f_X(w|n) = \#\{(\tau_1, \ldots, \tau_w) \in T(S_n)^w :$
$\tau_1 \cdots \tau_w = 1\}$
where $T(S_n)$ denotes the set of transpositions of S_n.

(b) $f_X(w|n) = \sum_{\chi \in \mathrm{Irr}(S_n)} A_\chi B_\chi^w$, where

$$A_\chi = \frac{1}{n!}\deg(\chi)^2, \quad B_\chi = \frac{\binom{n}{2}\chi(\tau)}{\deg(\chi)}, \tau \in T(S_n)$$

(c) Let $\chi = \chi_\varkappa$, where $\varkappa = (\varkappa_1, \ldots, \varkappa_n)$ satisfies

$$\varkappa_1 + \ldots + \varkappa_n = \binom{n}{2}, \quad 0 \le \varkappa_1 < \ldots < \varkappa_n.$$

Then we have

$$A_\chi = \tfrac{1}{n!}\left(\frac{\Delta(\varkappa_1, \ldots \varkappa_n)}{\varkappa_1! \ldots \varkappa_n!}\right)^2,$$

$$B_\chi = \tfrac{1}{2}\sum_{i=1}^{n} \varkappa_i(\varkappa_1 - 1) - \tfrac{1}{6}n(n-1)(n+4).$$

Note: The formulae of part (c) are essentially due to Frobenius (1900).

Theorem B (Dijkgraaf 1995): If $X = E$ is an elliptic curve, then

(a) $f_E(w|n) = \#\{(\tau_1, \ldots, \tau_w, g_1, g_2) \in S_n^{w+2} : \tau_i \in T(S_n), \tau_1 \cdots \tau_w g_1 = g_2^{-1} g_1 g_2\}$.

(b) $f_E(w|n) = n! \sum_{\chi \in \mathrm{Irr}(S_n)} B_\chi^w$.

(c) Let $\tilde{\Theta}(q, \lambda, \zeta)$ be defined by the formal product

$$\prod_{\substack{n \geq 1 \\ n \text{ odd}}} (1 - e^{n^2\lambda/8} q^{n/2} \zeta)(1 - e^{-n^2\lambda/8} q^{n/2} \zeta^{-1})$$

and let $\tilde{\Theta}_0(q, \lambda) \in \mathbb{Q}[[q, \lambda]]$ be the the coefficient of ζ^0 in $\tilde{\Theta}(q, \lambda, \zeta) \in \mathbb{Q}[\zeta, \zeta^{-1}][[q, \lambda]]$. Then

$$F_E(q, \lambda) = \tilde{\Theta}_0(q, \lambda) - 1,$$

and hence

$$\Phi_E(q, \lambda) = \log(\tilde{\Theta}_0(q, \lambda)).$$

Remark: Dijkgraaf attributes part (c) to a M.R. Douglas (preprint). Its proof uses the above-mentioned formula of Frobenius (1900).

Theorem C (Kaneko/Zagier, 1995): Write

$$\tilde{\Theta}_0(q,\lambda)\prod_{n\geq 1}(1-q^n)=\sum_{w=0}^{\infty}A_w(q)\lambda^w.$$

Then each A_w is a quasi-modular form of weight $6w$, i.e. there exist $c^w_{i,j,k}\in\mathbb{Q}$ such that

$$A_w=\sum_{\substack{i,j,k\geq 0\\ 2i+4j+6k=6w}} c^w_{i,j,k}E_2^iE_4^jE_6^k,$$

where for $k\geq 2$, k even, E_k is the Eisenstein series

$$E_k(q)=1-\frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n.$$

Corollary: For each $w\geq 2$, the function

$$\Phi_{E,w}(q)=\sum_{n\geq 1}N_E(w,n)q^n$$

is quasi-modular of weight $3w$.

Example:

$$\Phi_{E,2}(q)=\frac{1}{51840}(10E_2^3-6E_2E_4-4E_6).$$

2. Proof Sketch of Theorem B

(a) Let $T_{w,n} := \{(\tau_1, \ldots, \tau_w, g_1, g_2) \in T(S_n)^w \times S_n^2 :$
$$\tau_1 \cdots \tau_w g_1 = g_2^{-1} g_1 g_2\}.$$

The structure of the fundamental group $\pi_1(E \setminus W)$ shows that

$$\varphi_E(w|n) = \#\{t \in T_{w,n} : t \text{ generates a transitive subgroup of } S_n\}.$$

By a combinatorial argument one then derives the desired identity

$$f_E(w|n) = \#T_{w,n}.$$

(b) (Mike Roth) For each $g \in S_n$, let us put

$$P_g(w,n) = \{(\tau) \in T(S_n)^w : \tau_1 \cdots \tau_w g \in \mathfrak{c}(g)\}.$$

where $\mathfrak{c}(g)$ denotes the conjugacy class containing g. Then, if $\mathfrak{c}_1 = \{1\}, \mathfrak{c}_2 = T(S_n), \mathfrak{c}_3 \ldots, \mathfrak{c}_h$ denote all the conjugacy classes of S_n, we have

$$\frac{f(w|n)}{n!} = \sum_{g \in S_n} \frac{|P_g(w,n)|}{|\mathfrak{c}(g)|} = \sum_{i=1}^{h} \pi_i(w,n),$$

where $\pi_i(w,n) = |P_g(w,n)|$ with $\mathfrak{c}(g) = \mathfrak{c}_i$.

Let $Z = Z(\mathbb{C}[S_n])$ denote the centre of the group algebra of S_n. Z has two natural bases:

$$z := (z_1, \ldots, z_h) \quad \text{and} \quad \varepsilon := (\varepsilon_{\chi_1}, \ldots, \varepsilon_{\chi_h}),$$

in which

$$z_i = z_{\mathfrak{c}_i} = \sum_{g \in \mathfrak{c}_i} g, \text{ and } \varepsilon_{\chi_i} = \frac{\deg \chi_i}{n!} \sum_{g \in S_n} \chi_i(g) g^{-1}.$$

Consider the linear map $\mu_2 : Z \to Z$ defined by $z \mapsto z \cdot z_2$, and let its matrix wrt. the z-basis be

$$M_n = [\mu_2]_z.$$

Then the i-th diagonal element of M_n^w is

$$(M_n^w)_{i,i} = \pi_i(w, n), \text{ and so } f_E(w|n) = n!\mathrm{tr}(M_n^w).$$

On the other hand, by the orthogonality relations the matrix of μ_2 wrt. the ε-basis is

$$[\mu_2]_\varepsilon = \mathrm{diag}(B_{\chi_1}, \ldots, B_{\chi_h}),$$

and hence we obtain the desired formula

$$f_E(w|n) = n!\mathrm{tr}(M_n^w) = n! \sum_{i=1}^{h} B_{\chi_i}^w.$$

3. Hurwitz Moduli Spaces ($X = E$ and $w = 2$)

Reference: E.K., Collectanea Mathematica 54 (2003).

Let E/K be an elliptic curve over a field K (char $\neq 2$). Fix $N \geq 2$ prime to $\text{char}(K)$.

Study: the set $H^{(2)}(E/K, N)$ of all genus 2 covers of degee N of E/K:

$$f : Y \to E, \quad \deg(f) = N \text{ and } g_Y = 2.$$

More generally, the study of families of such covers yields a functor

$$\mathcal{H}^{(2)}_{E/K,N} : \underline{Sch} \to \underline{Sets}.$$

Theorem 1. If N is odd, then $\mathcal{H}^{(2)}_{E/K,N}$ is representable by a smooth, quasi-projective surface $H^{(2)}_{E/K,N}$ over K which has (over $\overline{K}$)

$$\sum_{d|N} \sigma(d) - \sigma(N)$$

irreducible components. Thus $H^{(2)}_{E/K,N}$ is irreducible if and only if N is prime.

Remarks: 1) The above result does not extend to the case that N is even. In that case the functor $\mathcal{H}$ is only coarsely representable by such a variety.

2) H breaks up into components because:

Each $Y \xrightarrow{f} E$ factors as $Y \xrightarrow{f'} E' \xrightarrow{u_f} E$, where $u_f : E' \to E$ is the max. unramified subcover of f.

Thus: $H^{(2)}_{E/K,N}$ is a union of components which are indexed by subgroups $G \leq E$ with $\#G|N$ (and $\#G \neq N$); explictly, $G = \mathrm{Ker}(\hat{u}_f)$.

Definition: A cover $f : Y \to E$ is called minimal if $\deg(u_f) = 1$.

Theorem 2. For every $N \geq 3$ (prime to $\mathrm{char}(K)$), the functor $\mathcal{H}^{(min)}_{E/K,N}$ which classifies minimal genus 2 covers is representable by a smooth, irreducible quasi-projective surface $H^{(min)}_{E/K,N}$ over K.

More precisely, we have

$$H^{(min)}_{E/K,N} \otimes_K \overline{K} \;\simeq\; E \times H_{E/K,N}$$

where $H_{E/K,N} \subset X(N)$ is an open subvariety (curve) of the modular curve $X(N)$ of (full) level N.

Remarks: 1) If $K = \mathbb{C}$, then $X(N) = \Gamma(N)\backslash\mathfrak{H}^*$, which is a Galois cover of $X(1) \simeq \mathbb{P}^1$ of degree

$$\overline{sl}(N) := |\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}|.$$

2) The reason that E appears as a factor of $\mathcal{H}^{(min)}_{E/K,N}$ is due to the fact that the group $E(K)$ acts on E and hence on $H^{(2)}_{E/K,N}$ etc. via translation: $f \mapsto T_x \circ f$.

Thus: introduce and study normalized covers.

Definition: A cover $f : Y \to E$ with $g_Y = 2$ is called normalized if it is minimal and if

$$f(W_Y) \subset E[2] \text{ and } \#(f^{-1}(0_E) \cap W_Y) = \begin{cases} 3 & N \text{ odd} \\ 0 & \text{else} \end{cases}$$

where $W_Y = \mathrm{Fix}(\sigma_Y)$ denotes the set of 6 Weierstrass points of Y. (Here: σ_Y is the hyperelliptic involution of Y.)

Theorem 3. For every $N \geq 3$ (as above), the functor $\mathcal{H}_{E/K,N}$ which classifies normalized genus 2 covers is representable by a smooth, irreducible affine curve $H_{E/K,N}/K$ such that $H_{E/K,N} \otimes \overline{K} \subset X(N)$.

Theorem 4: Let

$$D_{E/K,N} = X(N)_{/\overline{K}} \setminus (H_{E/K,N} \otimes \overline{K})$$

denote the degeneracy locus. Then

$$\# D_{E/K,N} \leq \tfrac{1}{12N}(5N+6)\overline{sl}(N),$$

and equality holds if and only if $\mathrm{char}(K) \nmid N!$.

– reinterpretation of results of Crelle J. 485 (1997) + J. No. Th. 64 (1997).

Theorem 5: The assignment $(Y \xrightarrow{f} E) \mapsto \mathrm{Disc}(f)$ is represented by a quasi-finite morphism

$$\delta = \delta_{E/K,N} : H_{E/K,N} \to \mathbb{P}^1_K \simeq (E^{(2)})^{sym}.$$

Furthermore, if $\mathrm{char}(K) \nmid N!$, then δ is finite and unramified outside of $\pi_E(E[2]) \subset \mathbb{P}^1$.

Theorem 6: If $\mathrm{char}(K) \nmid N!$, then

$$\deg(\delta_{E/K,N}) = \tfrac{1}{6}(N-1)\overline{sl}(N).$$

4. Application to Counting Covers

Corollary 7: If $W \subset E$ and $\#W = 2$ then

$$\#H(E,W,n) = \tfrac{2}{3}\sum_{d|n}\sigma_1(d)(n-d)\overline{sl}(n/d)$$
$$= \tfrac{n}{3}\left(\sigma_3(n) - n\sigma_1(n) + 3\sigma_1\left(\tfrac{n}{2}\right)\right),$$

where $\sigma_1(x) = 0$, if $x \notin \mathbb{Z}$.

Remark: Theorem 6 $\Rightarrow$ Corollary 7 by using

$$\sum_{d|n}\sigma_1(d)\mathrm{sl}(n/d) = \sigma_3(n),$$
$$\sum_{d|n}d\sigma_1(d)\mathrm{sl}(n/d) = n^2\sigma_1(n).$$

Corollary 8: The weighted number of covers is

$$N_E(2,n) = \tfrac{n}{3}\left(\sigma_3(n) - n\sigma_1(n)\right)$$

and hence

$$\Phi_{E,2}(q) = \tfrac{1}{51840}(10E_2^3 - 6E_2E_4 - 4E_6).$$

Lemma: If $f: Y \to E \in H(E,W,n)$, then

$$|\operatorname{Aut}(f)| = \begin{cases} 2 & \text{if } 2|n, \deg(u_f) = \frac{n}{2} \\ 1 & \text{otherwise} \end{cases}$$

5. Proof of Theorem 6 (Overview)

Remark: The proof of Theorem 6 uses the methods of Arithmetic Algebraic Geometry.

More precisely, it uses:

(I) a study of degenerations of the universal cover

$$f_{univ} : Y_H \to E \times H$$

1) study the degeneration of the minimal model $M(Y_H)/$ of Y_H; this uses the modular height of relative curves.

2) study whether or not f_{univ} extends to a cover

$$f : M(Y_H) \to E \times X(N).$$

(II) the intersection theory on $M(Y_H)$:

1) Calculate the modular height $h_{M(Y_H)/X(N)}$.

2) Calculate the self-intersection number of the relative dualizing sheaf $\omega^0_{M(Y_H)/X(N)}$.

3) Relate $\deg(\delta_{E/K,N})$ to intersection numbers involving the relative dualizing sheaf. (Relate the different divisor of f to $\omega^0_{M(Y_H)/X(N)}$.)