Abelian Subvarieties and the Shimura Construction

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1 Introduction

In his fundamental book, Shimura [Sh1] showed that each (Hecke) $\mathbb{T}$-eigenfunction $f \in S_2(N)$ on $\Gamma_1(N)$ gives to an abelian subvariety $A_f \subset J_1(N)$ on the Jacobian variety of the modular curve $X_1(N)/\mathbb{Q}$, and in a subsequent paper [Sh2] he explained that such Hecke eigenfunctions give more naturally rise to *quotient varieties* A'_f of $J_1(N)$. The purpose of this note is to show that both these constructions (and more) follow from a general "dictionary" that translates statements about subvarieties of abelian varieties into statements about ideals of the associated endomorphism algebras.

To explain this more precisely, let A be an abelian variety over an arbitrary field K, and let $\mathbf{Sub}(A/K) = \{B \leq A\}$ denote the set of abelian subvarieties B of A (which are defined over K). Then the aforementioned dictionary translates this set into the set $\mathbf{Id}_{\mathbb{E}}$ of right ideals of the endomorphism algebra $\mathbb{E} = \mathrm{End}_K(A) \otimes \mathbb{Q}$ of A as follows:

Theorem 1.1 *The map $B \mapsto I(B) := \{f \in \mathbb{E} : \mathrm{Im} f \subset B\}$ defines an inclusion-preserving bijection*

$$I_{A/K} : \mathbf{Sub}(A/K) \xrightarrow{\sim} \mathbf{Id}_{\mathbb{E}}$$

between the set of abelian subvarieties of A/K and the set of right ideals of $\mathbb{E} = \mathrm{End}^0_K(A)$. Furthermore, if $B_1, B_2 \in \mathbf{Sub}(A/K)$ are any two abelian subvarieties, then there is a canonical (functorial) isomorphism

$$\mathrm{Hom}^0_K(B_1, B_2) := \mathrm{Hom}_K(B_1, B_2) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{E}}(I(B_1), I(B_2)).$$

From this theorem (which is a special case of Theorem 2.4 below) one obtains as a consequence that if V is any faithful (left) $\mathbb{E}$-module, then there is a natural bijection between $\mathbf{Sub}(A/K)$ and certain *algebraic subspaces* of V; cf. section 3. In the case that V is finitely generated, this yields (via the *Morita theorems*) the following result.

Corollary 1.2 *Let V be a faithful, finitely generated left $\mathbb{E}$-module. Then the map $B \mapsto W_{\mathbb{E}}(B) := I(B)V$ defines an inclusion-preserving bijection*

$$W_{A/K} : \mathbf{Sub}(A/K) \xrightarrow{\sim} {}_{\tilde{\mathbb{E}}}\mathbf{Sub}(V)$$

between the set of abelian subvarieties of A/K and the set of left $\tilde{\mathbb{E}}$-submodules of V, where $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(V)$.

Two (natural) examples for which this result can be applied directly are the following.

Example 1.3 *(a) If $K \subset \mathbb{C}$, then the assignment $B \mapsto H_1(B) \subset H_1(A) := H_1((A \otimes \mathbb{C})^{an}, \mathbb{Q})$ defines an inclusion-preserving bijection between the set of abelian subvarieties of A/K and the set of left $\mathrm{End}_{\mathbb{E}}(H_1(A))$-submodules of $H_1(A)$.*

(b) If $K = \mathbb{Q}$, then the assignment $B \mapsto T_0(B) \subset T_0(A)$, where $T_0(A)$ denotes the tangent space of A at the origin, defines an inclusion-preserving bijection between $\mathbf{Sub}(A/\mathbb{Q})$ and the set of left $\mathrm{End}_{\mathbb{E}}(T_0(A))$-modules of $T_0(A)$.

Now this last example is actually the dual version of the one that gives rise to the Shimura construction, for the space $S_2(N, \mathbb{Q}) \subset S_2(N)$ of cusp forms of weight 2 on $\Gamma_1(N)$ with rational Fourier expansions can naturally be identified with the space of homomorphic 1-forms $H^0(X_1(N), \Omega^1_{X_1(N)/\mathbb{Q}}) \simeq T_0(J_1(N))^*$, which is the dual space of the tangent space $T_0(J_1(N))$. We thus require either an identification of T_0 with its dual space or a dual version of the above corollary. As we shall see, both these methods are feasible and lead to Shimura's two constructions.

More precisely, Shimura's first construction (in [Sh1]) may be deduced from the following general statement.

Corollary 1.4 *Let $A/\mathbb{Q}$ be an abelian variety, and suppose that there exists a commutative subring $\mathbb{T} \subset \mathbb{E} = \mathrm{End}^0_{\mathbb{Q}}(A)$ such that both $T_0(A)$ and $T_0(A)^*$ are free $\mathbb{T}$-modules of rank 1. Fix a $\mathbb{T}$-module isomorphism $\varphi : T_0(A)^* \xrightarrow{\sim} T_0(A)$. Then for any $\mathbb{T}$-submodule $W \subset T_0(A)^*$, there is a unique abelian subvariety $B_W \in \mathbf{Sub}(A/\mathbb{Q})$ such that $T_0(B_W) = \varphi(W)$.*

As will be explained in section 5, this can be applied directly to the case $A = J_1(N)$ (cf. Theorem 5.1); in fact, in this case Shimura[Sh2] constructed an explicit isomorphism φ.

A more natural method is to work with the module $\Omega(A) = T_0(A)^*$ directly. Since this is a *right* $\mathbb{E}$-module, we require a "dual version" of Corollary 1.2 for such modules. In this case, however, one has replace the set $\mathbf{Sub}(A/K)$ by either the set $\mathbf{Sub}(\hat{A}/K)$ of abelian subvarieties of the dual abelian variety $\hat{A}$, or equivalently, by the set $\mathbf{Quot}(A/K)$ of all *abelian quotient varieties of* A/K, i.e. by the set consisting of all (equivalence classes of) pairs (C, p) where $p : A \to C$ is a surjective homomorphism of abelian varieties such that $\mathrm{Ker}(p)$ is also an abelian variety; cf. section 4 below.

The precise analogue of Corollary 1.2 is given in Theorem 4.4. As a special case of this, one obtains

Theorem 1.5 *If A is an abelian variety defined over $\mathbb{Q}$ and $p : A \to C$ is a quotient of A, then the assignment $(C, p) \mapsto p^*\Omega(C) \subset \Omega(A) := H^0(A, \Omega^1_{A/\mathbb{Q}})$ induces a bijection*

$$W_\Omega : \mathbf{Quot}(A/\mathbb{Q}) \xrightarrow{\sim} \mathbf{Sub}(\Omega(A))_{\tilde{\mathbb{E}}}$$

between the set of abelian quotients of $A/\mathbb{Q}$ and the set of right $\tilde{\mathbb{E}}$-submodules of $\Omega(A)$, where $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(\Omega(A))$.

In particular, if there exists a commutative subring $\mathbb{T} \subset \mathbb{E}$ such that $\Omega(A)$ is a free $\mathbb{T}$-module of rank 1, then for every $\mathbb{T}$-submodule $W \subset \Omega(A)$ there exists a unique abelian quotient $p_W : A \to C_W$ such that $p_W^*\Omega(C_W) = W$. In addition, $\dim C_W = \dim_{\mathbb{Q}} W$.

As will be explained in section 5, the above theorem applies in particular to the case that $A = J_1(N)$ is the Jacobian of the modular curve $X_1(N)/\mathbb{Q}$ and $\mathbb{T} = \mathbb{T}_{\mathbb{Q}}$ is the Hecke algebra of $J_1(N)$, and we thus obtain a different characterization (and proof) of the (second) Shimura construction [Sh2].

2 Subvarieties of an Abelian Variety

As in the introduction, fix an arbitrary ground field K and let $A, B, \ldots$ denote abelian varieties defined over K. Throughout, we shall freely use the basic facts about abelian varieties as presented in Milne[Mi] and Mumford[Mu]. Some other standard facts (not explicitly mentioned there) are the following.

Proposition 2.1 *(a) If $f : A \to B$ is an isogeny of abelian varieties, then there is a unique isogeny $f' : B \to A$ such that*

$$f' \circ f = [e(f)]_A, \tag{1}$$

where $e(f)$ denotes the exponent *of* Ker$f$*, i.e., the smallest integer $n > 0$ such that* nKer$f = 0$. *Moreover, we then also have that $f \circ f' = [e(f)]_B$.*

(b) If $\lambda_{\mathcal{L}} : A \to \hat{A}$ is the canonical homomorphism defined by $\mathcal{L} \in \mathrm{Pic}(A)$, then for any homomorphism $h : B \to A$ of abelian varieties we have

$$\lambda_{h^*\mathcal{L}} = h^{\#}\lambda_{\mathcal{L}} := \hat{h} \circ \lambda_{\mathcal{L}} \circ h. \tag{2}$$

(c) If $\lambda : A \to \hat{A}$ is a polarization of A and $h : B \to A$ is a finite homomorphism (i.e. Ker(h) *is finite), then $h^{\#}\lambda := \hat{h} \circ \lambda \circ h : B \to \hat{B}$ is a polarization of B. In particular, $\hat{h}$ is surjective.*

Proof. (a) Since $\mathrm{Ker}(f) \subset \mathrm{Ker}([e(f)]_A)$, such a unique factorization exists by the universal property of quotients (viewing (B, f) as the quotient of A by $\mathrm{Ker}(f)$). Furthermore, if we write $h = f \circ f' : B \to B$ and $e = e(f)$, then $h \circ f = f \circ f' \circ f = f \circ [e]_A = [e]_B \circ f$, and hence $h = [e]_B$ because f is surjective.

(b) This follows immediately from the definitions of $\lambda_{\mathcal{L}}$, $\lambda_{h^*\mathcal{L}}$ and of $\hat{h}$. (See also Lang[La], p. 130).

(c) It is enough to verify this over $\overline{K}$, the algebraic closure of K. Then $\lambda = \lambda_{\mathcal{L}}$ for some ample sheaf $\mathcal{L} \in \mathrm{Pic}(A)$, and hence $h^*\mathcal{L}$ is also ample (because h is finite). Thus $h^{\#}\lambda = \lambda_{h^*\mathcal{L}}$ is a polarization. In particular, $\hat{h} \circ \lambda \circ h$ is an isogeny and hence is surjective, and thus $\hat{h}$ is also surjective.

The above facts lead to the following result which is of fundamental importance for the proof of Theorem 1.1:

Corollary 2.2 *Suppose $h : B \to A$ is a finite homomorphism and that $\lambda : A \to \hat{A}$ is a polarization. Put:*

$$p_h = p_{h,\lambda} := \hat{h} \circ \lambda : A \to \hat{B} \quad \text{and} \quad N_h = N_{h,\lambda} := (\lambda_B)' \circ p_h : A \to B, \tag{3}$$

where $\lambda_B = h^{\#}\lambda : B \to \hat{B}$ is the induced polarization on B and $(\lambda_B)' \circ \lambda_B = [e_B]_B$ with $e_B = e(\lambda_B)$. Then we have

$$N_h \circ h = [e_B]_B, \tag{4}$$

and so $\varepsilon_h = \varepsilon_{h,\lambda} := \frac{1}{e_B} h \circ N_h \in \mathbb{E} = \mathrm{End}^0_K(A)$ is an idempotent of $\mathbb{E}$.

Proof. By definition, $N_h \circ h = (\lambda_B)' \circ p_h \circ h = (\lambda_B)' \circ \lambda_B = [e_B]_B$, which proves (4). Thus $\varepsilon_h \circ \varepsilon_h = \frac{1}{e_B^2} h \circ N_h \circ h \circ N_h = \frac{1}{e_B^2} h \circ [e_B]_B \circ N_h = \frac{1}{e_B} h \circ N_h = \varepsilon_h$, which means that ε_h is an idempotent.

Remark. The above proposition applies in particular to the case when $h = j_B : B \hookrightarrow A$ is the inclusion map of an abelian subvariety $B \leq A$. In this case we shall write p_B, N_B and ε_B in place of p_{j_B}, N_{j_B} and ε_{j_B}, respectively.

Notation. If A and B are abelian varieties, then the group $\mathrm{Hom}_K(A, B)$ of K-homomorphisms is a lattice (of finite rank) in the $\mathbb{Q}$-vector space $\mathrm{Hom}^0(A, B) = \mathrm{Hom}_K(A, B) \otimes \mathbb{Q}$. In particular, if $A = B$, then the K-endomorphism algebra $\mathrm{End}_K(A)$ is a free $\mathbb{Z}$-algebra and a lattice in the (finite dimensional) $\mathbb{Q}$-algebra $\mathbb{E} = \mathbb{E}_{A/K} := \mathrm{End}^0_K(A) = \mathrm{End}_K(A) \otimes \mathbb{Q}$.

For any $f \in \mathrm{Hom}_K(B, A)$, its image $\mathrm{Im}(f) \subset A$ is an abelian subvariety of A. Since $\mathrm{Im}(nf) = \mathrm{Im}(f \circ [n]_B) = \mathrm{Im}(f)$, for any non-zero $n \in \mathbb{Z}$, we can extend this definition to $f \in \mathrm{Hom}^0_K(B, A)$ by setting $\mathrm{Im}(f) := \mathrm{Im}(nf) \subset A$, where $n \in \mathbb{Z}$ is chosen such that $nf \in \mathrm{Hom}_K(B, A)$ (and $n \neq 0$).

If $B \leq A$ is an abelian subvariety, then put

$$I(B) = I_{A/K}(B) = \{f \in \mathbb{E}_{A/K} : \mathrm{Im}(f) \subset B\}.$$

Note that $I(B)$ is a right ideal of $\mathbb{E}$ because we have $\mathrm{Im}(f+g) \subset \mathrm{Im}(f)+\mathrm{Im}(g)$ and $\mathrm{Im}(f \circ g) \subset \mathrm{Im}(f)$, for all $f, g \in \mathbb{E}$.

Proposition 2.3 *If $B \leq A$ is an abelian subvariety of A, then*

$$I(B) = j_B \mathrm{Hom}^0_K(A, B) = \varepsilon_{B,\lambda}\mathbb{E}, \tag{5}$$

where $\lambda : A \to \hat{A}$ is any polarization of A.

Proof. Clearly, $j_B \mathrm{Hom}^0_K(A, B) \subset I(B)$. Conversely, if $f \in I(B)$, then $\mathrm{Im}(f) \subset B$, and so we can write $f = j_B \circ f'$ for some $f' \in \mathrm{Hom}^0_K(A, B)$. Thus $f \in j_B \mathrm{Hom}^0_K(A, B)$, which proves the first equality of (5).

To prove the second equality, it clearly enough to show that

$$\mathrm{Hom}^0_K(A, B) = N_{B,\lambda}\mathbb{E}. \tag{6}$$

Now if $f \in \mathrm{Hom}_K(A, B)$, then by (4) we have $f = \frac{1}{e_B} N_{B,\lambda} \circ (j_B \circ f) \in N_{B,\lambda}\mathbb{E}$, and so $\mathrm{Hom}^0_K(A, B) \subset N_{B,\lambda}\mathbb{E}$. This proves (6) since the other inclusion is trivial.

The above proposition is the main tool required for proving Theorem 1.1 of the introduction, as we shall now see. In fact, we shall prove the following slightly more precise version of this theorem:

Theorem 2.4 *The above map* $B \mapsto I(B) := j_B \mathrm{Hom}^0_K(A,B) = \{f \in \mathbb{E} : \mathrm{Im} f \subset B\}$ *defines a lattice-preserving bijection*

$$I_{A/K} : \mathbf{Sub}(A/K) \xrightarrow{\sim} \mathbf{Id}_{\mathbb{E}}$$

whose inverse is given by $B_{A/K}(\mathfrak{a}) = \mathfrak{a}A := \sum_{f\in\mathfrak{a}} \mathrm{Im}(f)$. *In particular,* $I_{A/K}(B)$ *is a two-sided* $\mathbb{E}$*-ideal if and only if* B *is* $\mathbb{E}$*-stable, i.e. if and only if* $f(B) \subset B$*, for all* $f \in \mathbb{E}$.

Furthermore, for any $B_1, B_2 \in \mathbf{Sub}(A/K)$ *there is a unique functorial isomorphism*

$$(I_{A/K})_{B_1,B_2} : \mathrm{Hom}^0_K(B_1,B_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{E}}(I(B_1), I(B_2)) \tag{7}$$

such that for any $h \in \mathbb{E}$ *with* $h(B_1) \subset B_2$ *we have*

$$(I_{A/K})_{B_1,B_2}(h_{|B_1})(x) = hx, \quad \textit{for all } x \in I(B_1). \tag{8}$$

Proof. We first show that $I_{A/K}$ and $B_{A/K}$ are inverses of each other. For this, fix a polarization $\lambda : A \to \hat{A}$. Then for any $B \in \mathbf{Sub}(A/K)$ we have by equation (5) that $B_{A/K}(I_{A/K}(B)) = B_{A/K}(\varepsilon_{B,\lambda}\mathbb{E}) = \mathrm{Im}(\varepsilon_{B,\lambda}) = \mathrm{Im}(N_{B,\lambda}) = B$, the latter because $N_{B,\lambda} : A \to B$ is surjective (cf. (4)).

Now we also have $I_{A/K}(B_{A/K}(\mathfrak{a})) = \mathfrak{a}$, for all $\mathfrak{a} \in \mathbf{Id}_{\mathbb{E}}$. To see this, write $\mathfrak{a} = \varepsilon\mathbb{E}$, for some idempotent $\varepsilon \in \mathbb{E}$; such an ε exists by [CR], p. 44, since $\mathbb{E}$ semi-simple. Then $B := B_{A/K}(\mathfrak{a}) = \mathrm{Im}(\varepsilon)$, and so $\varepsilon \in I_{A/K}(B) = \varepsilon_{B,\lambda}\mathbb{E}$ by (5), i.e. $\varepsilon\mathbb{E} \subset \varepsilon_{B,\lambda}\mathbb{E}$. To prove the opposite inclusion, we first show that $\varepsilon j_B = j_B$. To see this, choose an $n > 0$ such that $g := n\varepsilon \in \mathrm{End}_K(A)$. Then $g^2 = ng$ and so $g \circ j_B = [n]_A \circ j_B$ because if $b = g(a) \in \mathrm{Im}(g)(\overline{K})$ then $g(b) = g^2(a) = ng(a) = nb$. Thus, $\varepsilon j_B = j_B$, and hence $\varepsilon\varepsilon_{B,\lambda} = \varepsilon_{B,\lambda}$. (Composing both sides with $\frac{1}{e_B}N_{B,\lambda}$.) Thus $\varepsilon_{B,\lambda} \in \varepsilon\mathbb{E}$ and so $\mathfrak{a} = \varepsilon\mathbb{E} = \varepsilon_{B,\lambda}\mathbb{E} = I_{A/K}(B) = I_{A/K}(B_{A/K}(\mathfrak{a}))$. Therefore, the maps $I_{A/K}$ and $B_{A/K}$ are inverses of each other and hence both are bijections.

Note that since $B_{A/K}(f\mathbb{E}) = \mathrm{Im}(f)$, the above implies that for any $f \in \mathbb{E}$ we have $I(B) = f\mathbb{E} \Leftrightarrow \mathrm{Im}(f) = B$, and so

$$fI(B) = I(f(B)), \quad \text{for all } B \in \mathbf{Sub}(A/K) \tag{9}$$

because $fI(B) = f\varepsilon_{B,\lambda}\mathbb{E}$ and $\mathrm{Im}(f\varepsilon_{B,\lambda}) = f(\mathrm{Im}(\varepsilon_{B,\lambda}) = f(B)$. Thus, if B is $\mathbb{E}$-stable, then clearly $I(B)$ is a two-sided ideal. Conversely, if $I(B)$ is a two-sided ideal, then $f(\mathrm{Im}(g)) = \mathrm{Im}(fg) \subset B$, for all $f \in \mathbb{E}$ and $g \in I(B)$.

Thus, if we take $g = \varepsilon_{B,\lambda}$ and use the fact that $\mathrm{Im}(\varepsilon_{B,\lambda}) = B$, then we see that B is $\mathbb{E}$-stable, as claimed.

Now since both $I_{A/K}$ and $B_{A/K}$ are inclusion-preserving, we have

$$(10) \quad B_1 \subset B_2 \quad \Leftrightarrow \quad I(B_1) \subset I(B_2), \quad \text{for all } B_1, B_2 \in \mathbf{Sub}(A/K),$$

and so it follows that $I_{A/K}$ is lattice-preserving, i.e. that we have

$$(11) \quad I(B_1 + B_2) = I(B_1) + I(B_2) \quad \text{and} \quad I((B_1 \cap B_2)^0) = I(B_1) \cap I(B_2),$$

because $B_1 + B_2$ (resp. $(B \cap B_2)^0)$) is the smallest (resp. largest) abelian subvariety containing (resp. contained in) B_1 and B_2.

To construct the map $(I_{A/K})_{B_1,B_2}$, put

$$(12) \qquad I_{A/K}(h)(x) = j_{B_2} \circ h \circ y, \quad \text{for } x = j_{B_1} \circ y \in I(B_1);$$

note that the right hand side is uniquely determined by x (and h) because j_{B_1} is injective. Thus, this rule defines an element $I(h) \in \mathrm{Hom}_{\mathbb{E}}(I(B_1), I(B_2))$.

We observe that $I(h)$ satisfies (8) because if $h(B_1) \subset B_2$, then $h \circ j_{B_1} = j_{B_2} \circ h'$ for a unique $h' \in \mathrm{Hom}_K^0(B_1, B_2)$, and hence $I(h_{|B_1})x := I(h')x \overset{(12)}{=} j_{B_2} \circ h' \circ y = h \circ j_{B_1} \circ y = h \circ x$, as claimed.

To see that the (additive) map $h \mapsto I(h)$ is injective, suppose that $I(h) = 0$. Then, since j_{B_2} is injective, we have $h \circ y = 0$, for all $y \in \mathrm{Hom}_K^0(A, B_1)$. In particular, $h \circ N_{B_1,\lambda} = 0$ and so $h = 0$ since $N_{B_1,\lambda}$ is an epimorphism. Thus $(I_{A/K})_{B_1,B_2}$ is injective.

To show that $(I_{A/K})_{B_1,B_2}$ is surjective, we shall use the elementary result in ring theory (analogous to Lemma (3.19) of [CR], p. 45) that for any idempotents $\varepsilon_1, \varepsilon_2 \in \mathbb{E}$, the left multiplication map $f \mapsto l_f := (x \mapsto fx)$ defines (by restriction to $\varepsilon_1\mathbb{E}$) an isomorphism

$$\varepsilon_2\mathbb{E}\varepsilon_1 \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{E}}(\varepsilon_1\mathbb{E}, \varepsilon_2\mathbb{E}).$$

Applying this here to the idempotents $\varepsilon_i = \varepsilon_{B_i,\lambda} = j_{B_i} \circ N'_{B_i}$ (where $N'_{B_i} = \frac{1}{e_{B_i}} N_{B_i,\lambda}$), we see that each $\varphi \in \mathrm{Hom}_{\mathbb{E}}(I(B_1), I(B_2))$ has the form $\varphi = (l_{\varepsilon_2 f \varepsilon_1})_{|I(B_1)}$, for some $f \in \mathbb{E}$. From this the surjectivity of $(I_{A/K})_{B_1,B_2}$ follows because by (8) (with $h = \varepsilon_2 f \varepsilon_1$) we have the formula

$$(13) \qquad I(N'_{B_2} \circ f \circ j_{B_1}) = (l_{\varepsilon_2 f \varepsilon_1})_{|I(B_1)}, \quad \text{for all } f \in \mathbb{E}.$$

Finally, we note that these isomorphisms are functorial in the sense that if $h \in \mathrm{Hom}_K^0(B_1, B_2)$ and $g \in \mathrm{Hom}_K^0(B_2, B_3)$, then we have

$$I_{A/K}(g \circ h) = I_{A/K}(g) \circ I_{A/K}(h). \tag{14}$$

Indeed, if $x = j_{B_1} y \in I(B_1)$ then we have $(I(g) \circ I(h))(x) = I(g)(I(h)(x)) = I(g)(j_{B_2} \circ h \circ y) = j_{B_3} \circ g \circ h \circ y = I(g \circ h)(x)$.

Remark 2.5 (a) The above theorem has a convenient reformulation in terms of *categories*: it asserts that $I_{A/K}$ defines a canonical *isomorphism of categories*

$$I_{A/K} : \underline{\mathrm{Sub}}^0_{A/K} \xrightarrow{\sim} \underline{\mathrm{Id}}_{\mathbb{E}}. \tag{15}$$

Here $\underline{\mathrm{Sub}}^0_{A/K}$ denotes the full subcategory of the category $\underline{\mathrm{Ab}}^0_{/K}$ "of abelian varieties up to isogeny' (cf. [Mu], p. 172) defined by $\mathbf{Sub}(A/K)$ (i.e. $\underline{\mathrm{Sub}}^0_{A/K}$ is the category whose objects are the elements of $\mathbf{Sub}(A/K)$ and whose morphisms from B_1 to B_2 are given by $\mathrm{Hom}_K^0(B_1, B_2)$), and similarly, $\underline{\mathrm{Id}}_{\mathbb{E}}$ denotes the full subcategory of the category $\underline{\mathrm{Mod}}_{\mathbb{E}}$ defined by the right ideals of $\mathbb{E}$, where $\underline{\mathrm{Mod}}_{\mathbb{E}}$ denotes the category of (finitely generated) right $\mathbb{E}$-modules.

(b) The above theorem is partially connected with the result of Lange[Lan] (cf. [LB], p. 126) that the map $B \mapsto \varepsilon_{B,\lambda}$ defines a bijection

$$\mathbf{Sub}(A/K) \quad \xrightarrow{\sim} \quad \{\text{symmetric idempotents of } \mathbb{E}\}$$

between the set of abelian subvarieties of A/K and the set of idempotents $\varepsilon \in \mathbb{E}$ which are symmetric with respect to Rosati involution $a \mapsto a^* = \lambda^{-1} \circ \hat{a} \circ \lambda$ defined by λ. (Indeed, this result is easily deduced from Theorem 2.4.) However, this bijection is not so useful for module-theoretic analysis below (cf. section 3). In addition, this bijection depends on the choice of the polarization λ (and hence is less functorial).

(c) Note that it follows from equation (7) of the above theorem that if $B_1, B_2 \in \mathbf{Sub}(A/K)$ are two abelian subvarieties, then

$$B_1 \text{ is isogenous to } B_2 \quad \Leftrightarrow \quad I(B_1) \simeq I(B_2) \;\; (\text{as } \mathbb{E}\text{-modules}).$$

From this it follows easily that if $B, B_1, \ldots, B_r \in \mathbf{Sub}(A/K)$ then

$$B \sim B_1^{n_1} \times \ldots \times B_r^{n_r} \quad \Leftrightarrow \quad I(B) \simeq I(B_1)^{n_1} \oplus \ldots \oplus I(B_r)^{n_r}. \tag{16}$$

Note that this statement immediately implies Theorem 1 of [KR].

3 Algebraic Subspaces of $\mathbb{E}$-Modules

As before, let A/K be an abelian variety and $\mathbb{E} = \mathrm{End}_K^0(A)$. In this section we suppose that we have a *faithful* representation

$$\rho : \mathbb{E} = \mathrm{End}_K^0(A) \to \mathrm{End}_F(V)$$

where V is a finite-dimensional vector space over a field $F \supset \mathbb{Q}$. In other words, V is a left $\mathbb{E} \otimes F$-module which is faithful as an $\mathbb{E}$-module.

Definition. An F-subspace $W \subset V$ is called $\mathbb{E}$-*algebraic* if it is of the form

$$W = \mathrm{Im}(\rho(a)), \quad \text{for some } a \in \mathbb{E}.$$

The set of all $\mathbb{E}$-algebraic subspaces of V is denoted by $\mathbf{Alg}_{\mathbb{E}}(V) = \mathbf{Alg}_{\mathbb{E},\rho}(V)$.

Remarks. (a) It is easy to see that the sum and intersection of algebraic subspaces is again algebraic, as will be clear from the proof of Theorem 3.1.

(b) Let $\tilde{\mathbb{E}}_F := \mathrm{End}_{\mathbb{E}\otimes F}(V) = \{f \in \mathrm{End}_F(V) : f\rho(a) = \rho(a)f, \text{for all } f \in \mathbb{E}\}$. Then every $W \in \mathbf{Alg}_{\mathbb{E}}(V)$ is a left $\tilde{\mathbb{E}}_F$-submodule of V, i.e.

$$\mathbf{Alg}_{\mathbb{E}}(V) \subset {}_{\tilde{\mathbb{E}}_F}\mathbf{Sub}(V). \tag{17}$$

(Indeed, if $w \in W = \mathrm{Im}(\rho(a))$, then $w = \rho(a)(v)$ for some $v \in V$ and then we have for $f \in \tilde{\mathbb{E}}_F$ that $fw = f(\rho(a)(v)) = \rho(a)(f(v)) \in \mathrm{Im}(\rho(a)) = W$.)

The set $\mathbf{Alg}_{\mathbb{E}}(V)$ of $\mathbb{E}$-algebraic subspaces of V is connected to the set $\mathbf{Sub}(A/K)$ of abelian subvarieties in the following way.

Theorem 3.1 *The map $B \mapsto W_{\mathbb{E}}(B) = W_\rho(B) := I(B) \otimes_{\mathbb{E}} V = I(B)V = \sum_{b\in I(B)} \mathrm{Im}\rho(b)$ induces a lattice-preserving bijection*

$$W_{\mathbb{E}} = W_{\mathbb{E},V} : \mathbf{Sub}(A/K) \xrightarrow{\sim} \mathbf{Alg}_{\mathbb{E}}(V);$$

in particular, the sum and intersection of algebraic subspaces is again algebraic. Furthermore, $W_{\mathbb{E}}(B)$ is a (left) $\mathbb{E}$-submodule of V if and only if B is an $\mathbb{E}$-stable subvariety.

Proof. By hypothesis, the ring homomorphism $\rho : \mathbb{E} \to \mathbb{A} := \mathrm{End}_F(V)$ is injective and hence faithfully flat because $\mathbb{E}$ is semi-simple. (Use the criterion of [Bou], Prop. I.3.9(b) (p. 33) and the fact that every $\mathbb{E}$-module is projective and hence flat). Thus, the map $\mathfrak{a} \mapsto \rho^*\mathfrak{a} = \mathfrak{a} \otimes_{\mathbb{E}} \mathbb{A} = \mathfrak{a}\mathbb{A}$ defines an injection

$$\rho^* : \mathbf{Id}_{\mathbb{E}} \hookrightarrow \mathbf{Id}_{\mathbb{A}}$$

because we have $\rho^{-1}(\mathfrak{a}\mathbb{A}) = \mathfrak{a}$, for all $\mathfrak{a} \in \mathbf{Id}_{\mathbb{E}}$ by [Bou], loc. cit.

Next we use the well-known fact that the map $\mathfrak{a} \mapsto W_V(\mathfrak{a}) := \mathfrak{a}V = \mathfrak{a} \otimes_{\mathbb{A}} V$ defines a bijection

$$W_V : \mathbf{Id}_{\mathbb{A}} \xrightarrow{\sim} \mathbf{Sub}_F(V)$$

between the set of right ideals of $\mathbb{A}$ and the set of F-subspaces of V. (This is in fact a special case of the Morita theorem(s) which are used below in the proof of Theorem 3.2; note that here $\mathrm{End}_{\mathbb{A}}(V) = F$.)

Now by definition we have $W_{\mathbb{E}}(B) = I(B)V = (I(B)\mathbb{A})V = W_V(\rho^*I(B))$, so $W_{\mathbb{E}} = W_V \circ \rho^*$ is the composition of two injections and hence is injective. Furthermore, since $I(B) = \varepsilon_{B,\lambda}\mathbb{E}$ by Proposition 2.3, we see that

$$W_{\mathbb{E}}(B) = W_V(\rho(\varepsilon_{B,\lambda})\mathbb{A}) = \mathrm{Im}(\rho(\varepsilon_{B,\lambda})), \tag{18}$$

for any polarization λ of A. This shows that $W_{\mathbb{E}}(B)$ is $\mathbb{E}$-algebraic, so $W_{\mathbb{E}}$ maps into the subset $\mathbf{Alg}_{\mathbb{E}}(V)$. Conversely, if $W = \mathrm{Im}(\rho(a))$ is $\mathbb{E}$-algebraic (with $a \in \mathbb{E}$), then by Theorem 2.4 there exists $B \in \mathbf{Sub}(A/K)$ such that $I(B) = a\mathbb{E}$, and then $W_{\mathbb{E}}(B) = \mathrm{Im}(\rho(a)) = W$. Thus $W_{\mathbb{E}}$ has image $\mathbf{Alg}_{\mathbb{E}}(V)$.

Note that $W_{\mathbb{E}} = W_V \circ \rho^* \circ I_{A/K}$ is lattice-preserving because each of the maps since $I_{A/K}$, ρ^* and W_V has this property.

To prove the last statement, suppose first that B is $\mathbb{E}$-stable. Then $I(B)$ is a two-sided $\mathbb{E}$-ideal (cf. Th. 2.4) and so for $f \in \mathbb{E}$ we have $fW_{\mathbb{E}}(B) = fI(B)V \subset I(B)V = W_{\mathbb{E}}(B)$, and hence $W_{\mathbb{E}}(B)$ is an $\mathbb{E}$-submodule of V.

Conversely, if $W_{\mathbb{E}}(B)$ is an $\mathbb{E}$-submodule, then for any $f \in \mathbb{E}$ we have $fI(B)\mathbb{A}V = fW_{\mathbb{E}}(B) \subset W_{\mathbb{E}}(B) = I(B)\mathbb{A}V$, so $fI(B)\mathbb{A} \subset I(B)\mathbb{A}$ (by Morita) and hence $fI(B) \subset I(B)$ (by faithful flatness). Thus, $I(B)$ is a two-sided $\mathbb{E}$-ideal and so B is $\mathbb{E}$-stable by Theorem 2.4.

In general, it is difficult to give a good characterization of the set $\mathbf{Alg}_{\mathbb{E}}(V)$ of algebraic subspaces of V. However, if the coefficient field can be chosen to be $F = \mathbb{Q}$, then such a characterization is indeed possible:

Theorem 3.2 *If V is a finitely generated, faithful left $\mathbb{E}$-module, then a subspace $W \subset V$ is algebraic if and only if W is a left $\tilde{\mathbb{E}}$-submodule of*

V, where $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(V)$. Thus the map $W_{\mathbb{E}}$ induces a (lattice-preserving) bijection

$$W_{\mathbb{E}} = W_{\mathbb{E},V} : \mathbf{Sub}(A/K) \xrightarrow{\sim} {}_{\tilde{\mathbb{E}}}\mathbf{Sub}(V).$$

Furthermore, for any two abelian subvarieties $B_1, B_2 \in \mathbf{Sub}(A/K)$ there is a functorial isomorphism

$$(19) \qquad (W_{\mathbb{E}})_{B_1,B_2} : \mathrm{Hom}^0_K(B_1, B_2) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{\mathbb{E}}}(W_{\mathbb{E}}(B_1), W_{\mathbb{E}}(B_2))$$

such that for any $h \in \mathbb{E}$ with $h(B_1) \subset B_2$ we have

$$(20) \qquad (W_{\mathbb{E}})_{B_1,B_2}(h_{|B_1})(w) = hw, \quad \textit{for all } w \in W_{\mathbb{E}}(B_1).$$

Proof. First note that since V is a finitely generated faithful $\mathbb{E}$-module and $\mathbb{E}$ is semi-simple, $\mathbb{E}$ is a direct factor of V^n (for some n). Thus by [CR], Lemma (3.39), V is a (pro)generator of ${}_{\mathbb{E}}\underline{\mathrm{Mod}}$ (in the sense of [CR], p. 56, 60). Next, view V as an $(\mathbb{E}, \tilde{\mathbb{E}}^{op})$-bimodule (where $\tilde{\mathbb{E}}^{op}$ denotes the opposite ring of $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(V)$), and put $Q = \mathrm{Hom}_{\mathbb{E}}(V, \mathbb{E})$. Then $P := V$ and Q (together with maps described on [CR], p. 59) satisfy the hypotheses of Morita's Theorem ([CR], p. 60), and so in particular (by [CR], Theorem (3.54)(vi)) the map $\mathfrak{a} \mapsto \mathfrak{a} \otimes_{\mathbb{E}} V$ defines a (lattice-preserving) bijection

$$T_V : \mathbf{Id}_{\mathbb{E}} \xrightarrow{\sim} \mathbf{Sub}(V)_{\tilde{\mathbb{E}}^{op}} = {}_{\tilde{\mathbb{E}}}\mathbf{Sub}(V).$$

But by definition $T_V(\mathfrak{a}) = \mathfrak{a}V = W_V(\rho^*(\mathfrak{a}))$ (in the notation of the proof of Theorem 3.1), and so it follows that $W_{\mathbb{E}} = T_V \circ I_{A/K}$ yields the desired bijection.

To prove the second assertion, we first recall that Morita's theorem (in the above context) also yields that the functor $T_V = * \otimes V : \underline{\mathrm{Mod}}_{\mathbb{E}} \to \underline{\mathrm{Mod}}_{\tilde{\mathbb{E}}^{op}} = {}_{\tilde{\mathbb{E}}}\underline{\mathrm{Mod}}$ is an equivalence of categories (cf. [CR], Th. (3.54)(v)) and that hence T_V induces a (functorial) isomorphism

$$\mathrm{Hom}_{\mathbb{E}}(\mathfrak{a}, \mathfrak{b}) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{\mathbb{E}}^{op}}(\mathfrak{a}V, \mathfrak{b}V), \quad \text{for all } \mathfrak{a}, \mathfrak{b} \in \mathbf{Id}_{\mathbb{E}}.$$

Note that by viewing the right $\tilde{\mathbb{E}}^{op}$-modules $\mathfrak{a}V$, $\mathfrak{b}V$ as left $\tilde{\mathbb{E}}$-modules, we have a canonical identification $\mathrm{Hom}_{\tilde{\mathbb{E}}^{op}}(\mathfrak{a}V, \mathfrak{b}V) = \mathrm{Hom}_{\tilde{\mathbb{E}}}(\mathfrak{a}V, \mathfrak{b}V)$, and so (19) follows by combining the above identification with that of (7). Explicitly, this isomorphism is given by the formula

$$(21) \qquad (W_{\mathbb{E}})_{B_1,B_2}(h)(w) = \rho(j_{B_2} \circ h \circ f)v = \rho(j_{B_2} \circ h \circ N'_{B_1})w,$$

in which $h \in \mathrm{Hom}_K^0(B_1, B_2)$, $w = \rho(j_{B_1} \circ f)v \in W_{\mathbb{E}}(B_1) = I(B_1)V$ with $f \in \mathrm{Hom}_K^0(A, B_1)$ and $v \in V$, and $N'_{B_1} = \frac{1}{e_{B_1}} N_{B_1,\lambda}$. [Indeed, by construction we have $W_{\mathbb{E}}(h)(w) = T_V(I_{A/K}(h)\rho(j_{B_1} \circ f)v) = \rho(I_{A/K}(h)(j_{B_1} \circ f))v \overset{(12)}{=} \rho(j_{B_2} \circ h \circ f)v = \rho(j_{B_2} \circ h \circ N'_{B_1} \circ j_{B_1} \circ f))v = \rho(j_{B_2} \circ h \circ N'_{B_1})w$.] Note that this formula immediately implies (20).

Remark 3.3 Note that if we take $V = \mathbb{E}$ in Theorem 3.2, then the isomorphism $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(\mathbb{E}) \simeq \mathbb{E}^{op}$ induces a canonical identification ${}_{\tilde{\mathbb{E}}}\mathbf{Sub}(\mathbb{E}) = \mathbf{Id}_{\mathbb{E}}$, and so Theorem 2.4 is actually a special case of Theorem 3.2.

The following corollary is of fundamental importance for the Shimura construction.

Corollary 3.4 *In the situation of Theorem 3.2, suppose that there exists a commutative subring $\mathbb{T} \subset \mathbb{E}$ such that V is a free $\mathbb{T}$-module of rank 1 (or, more generally, such that the image of $\mathbb{T}$ in $\mathrm{End}_{\mathbb{Q}}(V)$ is a maximal commutative subalgebra). Then for every $\mathbb{T}$-submodule $W \subset V$ there is a unique abelian subvariety $B_W \subset A$ such that $W_{\mathbb{E}}(B_W) = W$ and we have a natural ring embedding*

$$\theta_W : \mathrm{End}_{\mathbb{T}}(W) \hookrightarrow \mathrm{End}_K^0(B_W)$$

such that $\theta_W(\rho(t)_{|W}) = t_{|B_W}$, for all $t \in \mathbb{T}$. In particular, we have an induced embedding $\theta'_W : \mathbb{T}/\mathrm{Ann}_{\mathbb{T}}(W) \subset \mathrm{End}_{\mathbb{T}}(W) \hookrightarrow \mathrm{End}_K^0(B_W)$.

Proof. First note that if V is a free $\mathbb{T}$-module of rank 1, then $V \simeq \mathbb{T}$, and so $\mathrm{End}_{\mathbb{T}}(V) \simeq \mathrm{End}_{\mathbb{T}}(\mathbb{T}) = \mathbb{T}$. Thus $\mathrm{End}_{\mathbb{T}}(V) = \rho(\mathbb{T})$, the image of $\mathbb{T}$ in $\mathrm{End}_{\mathbb{Q}}(V)$, which means that $\rho(\mathbb{T})$ is a maximal commutative subalgebra. But then $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(V) \subset \mathrm{End}_{\mathbb{T}}(V) = \rho(\mathbb{T})$, and so every $\mathbb{T}$-submodule is *a fortiori* an $\tilde{\mathbb{E}}$-submodule, and every $\mathbb{T}$-homomorphism is also an $\tilde{\mathbb{E}}$-homomorphism. Thus, the first assertion follows directly from Theorem 3.2, as does the second by taking $\theta_W = (W_{\mathbb{E}}^{-1})_{|\mathrm{End}_{\mathbb{T}}(W)}$.

As we shall see presently, most of the interesting examples of faithful $\mathbb{E}$-modules V arise from faithful (covariant) functors

$$\mathcal{F} : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}_{/F}$$

from the category of abelian varieties to the category $\underline{\mathrm{Vec}}_{/F}$ of (finite dimensional) F-vector spaces. Some examples of such functors are the following.

Example 3.5 (a) (Homology functor) Suppose that $K \subset \mathbb{C}$. Then we can view $A_{\mathbb{C}} := A \otimes_K \mathbb{C}$ as a complex analytic space, and so homology theory yields a faithful functor $H_1 : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}_{/\mathbb{Q}}$ which is defined by $H_1(A) = H_1(A^{an}_{\mathbb{C}}, \mathbb{Q})$; cf. [Mu], p. 176.

(b) (Tangent space functor) Suppose that $\mathrm{char}(K) = 0$. Then the tangent space $T_0(A)$ of A at the origin is a K-vector space of dimension $d = \dim(A)$, and we obtain a faithful functor $T_0 : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}_{/K}$. (To see that T_0 is faithful, reduce to the case $K = \mathbb{C}$ and use [Mu], p. 176(top).)

(c) (Tate module functor) Let K be any field and fix a prime $\ell \neq \mathrm{char}(K)$. For any abelian variety A/K, its Tate module $T_\ell(A) := T_\ell(A \otimes \overline{K})$ a free $\mathbb{Z}_\ell$-module of rank $2d$ and so $T^0_\ell(A) = T_\ell \otimes \mathbb{Q}_\ell$ is a $\mathbb{Q}_\ell$-vector space of dimension $2d$. Moreover, the induced functor $T^0_\ell : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}_{\mathbb{Q}_\ell}$ is faithful by [Mu], p. 176ff.

Now if we have a faithful functor $\mathcal{F} : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}_{/F}$, then the above theorems can be restated in the following manner.

Theorem 3.6 *If $\mathcal{F} : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}_{/F}$ is a faithful (covariant) functor from the category of abelian varieties to the category $\underline{\mathrm{Vec}}_{/F}$ of (finite dimensional) F-vector spaces, then the map $B \mapsto W_{\mathcal{F}}(B) = \mathrm{Im}(\mathcal{F}(j_B)) = I(B)\mathcal{F}(A) \subset \mathcal{F}(A)$ induces for each A/K a lattice-preserving bijection*

$$W_{\mathcal{F}} = W_{\mathcal{F},A/K} : \mathbf{Sub}(A/K) \xrightarrow{\sim} \mathbf{Alg}_{\mathcal{F}}(\mathcal{F}(A))$$

between the set of abelian subvarieties of A/K and the set $\mathbf{Alg}_{\mathcal{F}}(\mathcal{F}(A)) = \{\mathrm{Im}(\mathcal{F}(a)) : a \in \mathbb{E}\}$ of $\mathcal{F}$-algebraic subspaces of $\mathcal{F}(A)$.

Moreover, if $F = \mathbb{Q}$, then for every abelian variety A/K we have the identification

$$\mathbf{Alg}_{\mathcal{F}}(\mathcal{F}(A)) = {}_{\tilde{\mathbb{E}}}\mathbf{Sub}(\mathcal{F}(A))$$

where $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(\mathcal{F}(A))$ and $\mathbb{E} = \mathrm{End}^0_K(A)$, and the map $h \mapsto \mathcal{F}(h)$ induces for any $B_1, B_2 \in \mathbf{Sub}(A/K)$ a (functorial) isomorphism

$$(W_{\mathcal{F}})_{B_1,B_2} : \mathrm{Hom}^0_K(B_1, B_2) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{\mathbb{E}}}(\mathcal{F}(B_1), \mathcal{F}(B_2)), \tag{22}$$

where we view $\mathcal{F}(B_i)$ as an $\tilde{\mathbb{E}}$-module via the identification $\mathcal{F}(j_{B_i}) : \mathcal{F}(B_i) \xrightarrow{\sim} W_{\mathcal{F}}(B_i)$.

Proof. By functoriality, the map $a \mapsto \mathcal{F}(a)$ defines a ring homomorphism

$$\rho := \rho_{\mathcal{F},A/K} : \mathrm{End}_K(A) \to \mathrm{End}_F(\mathcal{F}(A)).$$

Since $\mathcal{F}$ is faithful, the map ρ is injective, and hence ρ extends to an injective ring homomorphism $\rho : \mathbb{E} = \mathrm{End}^0_K(A) \to \mathrm{End}_F(\mathcal{F}(A))$.

Since clearly $\mathbf{Alg}_{\mathbb{E},\rho}(\mathcal{F}(A)) = \mathbf{Alg}_{\mathcal{F}}(\mathcal{F}(A))$, the first assertion follows from Theorem 3.1 once we have shown that $W_\rho(B) = W_{\mathcal{F},A/K}(B)$. Now since $\mathcal{F}(N_{B,\lambda}) \circ \mathcal{F}(j_B) = e_B id_{\mathcal{F}(B)}$, the map $\mathcal{F}(N_{B,\lambda})$ is surjective (and $\mathcal{F}(j_B)$ is injective) and hence by (18) we obtain $W_\rho(B) = \mathrm{Im}(\rho(\varepsilon_{B,\lambda})) = \mathrm{Im}(\mathcal{F}(j_B) \circ \mathcal{F}(N_{B,\lambda})) = \mathrm{Im}(\mathcal{F}(j_B)) = W_{\mathcal{F},A/K}(B)$, as desired. Note also that $I(B)\mathcal{F}(A) = W_{\mathcal{F}}(B)$ by Theorem 3.1.

Now suppose that $F = \mathbb{Q}$. Then $\mathcal{F}(A)$ is a finitely generated faithful $\mathbb{E}$-module, and so the second assertion follows directly from Theorem 3.2.

To verify the last assertion, first note that the map $g \mapsto \Phi(g) := \mathcal{F}(j_{B_2})^{-1} \circ g \circ \mathcal{F}(j_{B_1})$ defines an isomorphism

$$\Phi : \mathrm{Hom}_{\tilde{\mathbb{E}}}(W_{\mathcal{F}}(B_1), W_{\mathcal{F}}(B_2)) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{\mathbb{E}}}(\mathcal{F}(B_1), \mathcal{F}(B_2)).$$

Thus, if we put $(W_{\mathcal{F}})_{B_1,B_2} = \Phi \circ W_{\mathbb{E}}$, then by (19) we obtain an isomorphism (22). Moreover, since $(W_{\mathcal{F}})_{B_1,B_2}(h) = \mathcal{F}(h)$ if $h \in \mathrm{Hom}^0_K(B_1, B_2)$ (because by (21) we have $W_{\mathbb{E}}(h) = \mathcal{F}(j_{B_2} \circ h \circ N'_{B_1})_{|W_{\mathcal{F}}(B_1)} = \mathcal{F}(j_{B_2}) \circ \mathcal{F}(h) \circ \mathcal{F}(N'_{B_1})_{|W_{\mathcal{F}}(B_1)} = \Phi^{-1}(\mathcal{F}(h)))$, the last assertion follows.

Remark. Note that Example 1.3 of the introduction is an immediate consequence of Example 3.5 and Theorem 3.6. Similarly, Corollary 1.4 follows immediately from Example 1.3(b) and Corollary 3.4.

4 Quotient Varieties

In the previous sections we saw that the set $\mathbf{Id}_{\mathbb{E}}$ of *right* ideals of $\mathbb{E} = \mathrm{End}^0_K(A)$ naturally corresponds to the set $\mathbf{Sub}(A/K)$ of abelian subvarieties of A/K, and that there is a similar assertion for finitely generated faithful *left* $\mathbb{E}$-modules V (in place of $\mathbb{E}$); cf. Theorem 3.2 and Remark 3.3. Here we shall now show that there is an analogous *dual* statement involving the set ${}_{\mathbb{E}}\mathbf{Id}$ of *left* ideals which then generalizes to an assertion involving *right* $\mathbb{E}$-modules. In this case, however, the set $\mathbf{Sub}(A/K)$ has to be replaced by either the set $\mathbf{Sub}(\hat{A}/K)$ of subvarieties of the dual abelian variety $\hat{A}$ or by the set $\mathbf{Quot}(A/K)$ of all *abelian quotients of* A/K which is defined as follows.

Definition. If A/K is an abelian variety, then we call a homomorphism $p : A \to C$ an *abelian quotient* if p is surjective and if $\mathrm{Ker}(p)$ is an abelian subvariety of A. Furthermore, if $p' : A \to C'$ is another abelian quotient, then p' is called *equivalent to* p if there exists an isomorphism $h : C \xrightarrow{\sim} C'$ such that $p' = h \circ p$. The set of equivalence classes of abelian quotients of A is denoted by $\mathbf{Quot}(A/K)$.

Remark 4.1 (a) Note that if $p : A \to C$ is any surjective homomorphism of abelian varieties, then p is faithfully flat and hence (C, p) is the quotient of A by $\mathrm{Ker}(p)$.

(b) If $p : A \to C$ is any homomorphism of abelian varieties, then it is well-known (cf. [La], p. 216) that

(23) $\qquad p$ is an abelian quotient $\quad\Leftrightarrow\quad \hat{p} : \hat{C} \to \hat{A}$ is injective.

It thus follows that for any such p there is a finite homomorphism $j_p : C \to A$ such that

(24) $$p \circ j_p = [e_C]_C,$$

for some integer $e_C > 0$. [Indeed, apply Corollary 2.2 to $h = \hat{p} : \hat{C} \hookrightarrow \hat{A}$ and take $j_p := \kappa_A^{-1} \circ \hat{N}_{\hat{p}} \circ \kappa_C$.]

Proposition 4.2 *The map* $(C, p) \mapsto \mathrm{Im}(\hat{p})$ *defines a bijection*

$$D = D_{A/K} : \mathbf{Quot}(A/K) \xrightarrow{\sim} \mathbf{Sub}(\hat{A}/K).$$

Proof. For $B \in \mathbf{Sub}(\hat{A}/K)$ put $D'(B) = (\hat{B}, \hat{j}_B \circ \kappa_A)$. Then by the above equivalence (23) we see that $D'(B) \in \mathbf{Quot}(A/K)$ and that D and D' are inverses of each other.

We can use the above bijection to prove the following dual version of Theorem 2.4.

Theorem 4.3 *The map* $(C, p) \mapsto \hat{I}_{A/K}(C, p) := \mathrm{Hom}^0(C, A)p$ *induces a bijection*

$$\hat{I}_{A/K} : \mathbf{Quot}(A/K) \xrightarrow{\sim} {}_{\mathbb{E}}\mathbf{Id}$$

whose inverse is given by the map $C_{A/K}(\mathfrak{a}) = (C_{\mathfrak{a}}, p_{\mathfrak{a}})$*, where* $C_{\mathfrak{a}} = A/r_{\mathbb{E}}(\mathfrak{a})A$*,* $r_{\mathbb{E}}(\mathfrak{a}) = \{f \in \mathbb{E} : \mathfrak{a}f = 0\}$ *is the right annihilator of* $\mathfrak{a}$ *and* $p_a : A \to C_a$ *is* *the quotient map. Moreover, for any pair* $\underline{C}_i = (C_i, p_i) \in \mathbf{Quot}(A/K)$ *there is a functorial isomorphism*

$$(\hat{I}_{A/K})_{\underline{C}_1,\underline{C}_2} : \mathrm{Hom}^0_K(C_1, C_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{E}}(\hat{I}(C_2, p_2), \hat{I}(C_1, p_1)) \tag{25}$$

such that for any $h \in \mathbb{E}$ *with the property that* $p_2 \circ h = h' \circ p_1$*, for some* $h' \in \mathrm{Hom}^0_K(C_1, C_2)$*, we have*

$$(\hat{I}_{A/K})_{\underline{C}_1,\underline{C}_2}(h')(x) = xh, \quad \textit{for all } x \in \hat{I}_{A/K}(C_2, p_2). \tag{26}$$

Proof. Since $f \mapsto \hat{f}$ induces an isomorphism $\mathbb{E} = \mathbb{E}_{A/K} \xrightarrow{\sim} \hat{\mathbb{E}}^{op} := \mathbb{E}^{op}_{\hat{A}/K}$, we have an induced bijection $D_{\mathbb{E}} : {}_{\mathbb{E}}\mathbf{Id} \xrightarrow{\sim} {}_{\hat{\mathbb{E}}^{op}}\mathbf{Id} = \mathbf{Id}_{\hat{\mathbb{E}}}$. Now

$$D_{\mathbb{E}} \circ \hat{I}_{A/K} = I_{\hat{A}/K} \circ D_{A/K}. \tag{27}$$

because $D_{\mathbb{E}}(\hat{I}_{A/K}(C, p)) = \{\hat{f} : f \in \mathrm{Hom}^0_K(C, A)p\} = \hat{p}\mathrm{Hom}^0_K(\hat{A}, \hat{C}) = I_{\hat{A}/K}(\hat{p}(\hat{C})) = I_{\hat{A}/K}(D_{A/K}(C, p))$, and so we see that $\hat{I}_{A/K}$ is a bijection since $D_{A/K}$, $D_{\mathbb{E}}$ and $I_{\hat{A}/K}$ are all bijections.

Next we show that $C_{A/K} : {}_{\mathbb{E}}\mathbf{Id} \to \mathbf{Quot}(A/K)$ is the inverse of $\hat{I}_{A/K}$. Since $\hat{I}_{A/K}$ is a bijection, it is enough to verify that $C_{A/K}(\hat{I}_{A/K}(C, p)) \simeq (C, p)$, for all $(C, p) \in \mathbf{Quot}(A/K)$, and for this it is enough to show that $\mathrm{Ker}(p) = r_{\mathbb{E}}(\hat{I}_{A/K}(C, p))A$ or, equivalently (by Theorem 2.4) that

$$I_{A/K}(\mathrm{Ker}(p)) = r_{\mathbb{E}}(\hat{I}_{A/K}(C, p)). \tag{28}$$

Now if $f \in \mathbb{E}$, then $f \in I_{A/K}(\mathrm{Ker}(p)) \Leftrightarrow \mathrm{Im}(f) \subset \mathrm{Ker}(p) \Leftrightarrow p \circ f = 0$. Thus, if $f \in I_{A/K}(\mathrm{Ker}(p))$, then $\hat{I}_{A/K}(C, p)f = \mathrm{Hom}^0_K(C, A)pf = 0$, and so $f \in r_{\mathbb{E}}(\hat{I}_{A/K}(C, p))$. Conversely, if $\hat{I}_{A/K}(C, p)f = 0$ then in particular $j_p \circ p \circ f = 0$, where $j_p \in \mathrm{Hom}_K(C, A)$ is as in (24), and hence $e_C(p \circ f) = p \circ j_p \circ p \circ f = 0$, which means that $f \in I_{A/K}(\mathrm{Ker}(p))$. This proves (28), and hence that $C_{A/K}$ is the inverse of $\hat{I}_{A/K}$.

We now construct the bijection $(\hat{I}_{A/K})_{\underline{C}_1,\underline{C}_2}$. For this, we first observe that for any two left $\mathbb{E}$-ideals $\mathfrak{a}, \mathfrak{b} \in {}_{\mathbb{E}}\mathbf{Id}$ there is a canonical isomorphism

$$D_{\mathbb{E}} = (D_{\mathbb{E}})_{\mathfrak{a},\mathfrak{b}} : \mathrm{Hom}_{\hat{E}}(D_{\mathbb{E}}(\mathfrak{a}), D_{\mathbb{E}}(\mathfrak{b})) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{E}}(\mathfrak{a}, \mathfrak{b})$$

such that for all $f \in \mathrm{Hom}_{\hat{E}}(D_{\mathbb{E}}(\mathfrak{a}), D_{\mathbb{E}}(\mathfrak{b}))$ we have

$$(D_{\mathbb{E}}(f)(x))\hat{} = f(\hat{x}), \quad \text{for all } x \in \mathfrak{a}. \tag{29}$$

To define $(\hat{I}_{A/K})_{\underline{C}_1,\underline{C}_2}$, let $\alpha_i : \hat{C}_i \xrightarrow{\sim} \hat{C}'_i := \mathrm{Im}(\hat{p}_i) \in \mathbf{Sub}(\hat{A})$ be the unique isomorphism such that $\hat{p}_i = j_{\hat{C}'_i} \circ \alpha_i$, and put for $h \in \mathrm{Hom}^0_K(C_1, C_2)$

$$(\hat{I}_{A/K})_{\underline{C}_1,\underline{C}_2}(h) = D_{\mathbb{E}}((I_{\hat{A}/K})_{\hat{C}'_2,\hat{C}'_1}(\alpha_1 \circ \hat{h} \circ \alpha_2^{-1})). \tag{30}$$

Clearly, $(\hat{I}_{A/K})_{\underline{C}_1,\underline{C}_2}$ defines an isomorphism (25) because $D_{\mathbb{E}}$ and $(I_{\hat{A}/K})_{\hat{C}'_2,\hat{C}'_1}$ are both isomorphisms (the latter by Theorem 2.4), and one verifies without difficulty (by using (8) and (29)) that (26) holds.

Remark. Note that the above proof shows that even though the ideal $\hat{I}(C_i, p_i)$ only depends on the equivalence class of (C_i, p_i), the above isomorphism $(\hat{I}_{A/K})_{\underline{C}_1,\underline{C}_2} : \mathrm{Hom}^0_K(C_1, C_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{E}}(\hat{I}(C_2, p_2), \hat{I}(C_1, p_1))$ actually depends on the choice of the p_i's.

Similarly, we have the following dual version of Theorem 3.2.

Theorem 4.4 *If V is a faithful, finitely generated right $\mathbb{E}$-module, then the map $(C, p) \mapsto \hat{W}_{\mathbb{E}}(C, p) := V \otimes_{\mathbb{E}} \hat{I}(C, p) = V\hat{I}(C, p) = \sum_{g \in \mathrm{Hom}^0_K(C,A)} V(g \circ p)$ defines a bijection*

$$\hat{W}_{\mathbb{E}} = \hat{W}_{\mathbb{E},V} : \mathbf{Quot}(A/K) \xrightarrow{\sim} \mathbf{Sub}(V)_{\tilde{\mathbb{E}}}.$$

whose inverse is given by $C_{\mathbb{E},V}(W) = (C_W, p_W)$, where $C_W = A/r_{\mathbb{E}}(W)A$, $r_{\mathbb{E}}(W) = \{f \in \mathbb{E} : Wf = 0\}$ is the right annihilator and $p_W : A \to C_W$ is the quotient map. Furthermore, for any two quotients $\underline{C}_i = (C_i, p_i) \in$ $\mathbf{Quot}(A/K)$ we have an induced functorial isomorphism

$$(\hat{W}_{\mathbb{E}})_{\underline{C}_1,\underline{C}_2} : \mathrm{Hom}^0_K(C_1, C_2) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{\mathbb{E}}}(\hat{W}_{\mathbb{E}}(\underline{C}_2), \hat{W}_{\mathbb{E}}(\underline{C}_1)). \tag{31}$$

such that the analogue of (26) holds.

Proof. We can deduce this from Theorem 3.2 as follows. Let $\hat{\mathbb{E}} = \mathrm{End}^0_K(\hat{A})$ and view V as a *left* $\hat{\mathbb{E}}$-module via the rule $\hat{f} \cdot v = vf$, if $v \in V$ and $f \in \mathbb{E}$. Then we have the identification $(\hat{\mathbb{E}})\tilde{} = \mathrm{End}_{\hat{\mathbb{E}}}(V) = \mathrm{End}_{\mathbb{E}}(V) = \tilde{\mathbb{E}}$ as subrings of $\mathrm{End}_{\mathbb{Q}}(V)$, and this yields the identification ${}_{(\hat{\mathbb{E}})\tilde{}}\mathbf{Sub}(V) = \mathbf{Sub}(V)_{\tilde{\mathbb{E}}}$. Thus, since $W_{\hat{A}/K}$ and $D_{A/K}$ are both bijections (cf. Th. 3.2 and Prop. 4.2), so is

$$\hat{W}_{\mathbb{E},V} = W_{\hat{\mathbb{E}},V} \circ D_{A/K}. \tag{32}$$

Moreover, since $r_{\mathbb{E}}(V\hat{I}_{A/K}(C, p)) = r_{\mathbb{E}}(\hat{I}_{A/K}(C, p))$ (because V is faithful), we see by using Theorem 4.3 that $C_{\mathbb{E},V}$ is the inverse map of $\hat{W}_{\mathbb{E},V}$.

Finally, the last assertion can be deduced from Theorem 3.2 by an argument similar to that of the proof of Theorem 4.3.

Just as *covariant* functors $\mathcal{F} : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}_{/F}$ give rise to *left* $\mathbb{E}$-modules, *contravariant* functors

$$\mathcal{G} : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}^0_{/F},$$

give rise to *right* $\mathbb{E}$-modules. The following are examples of such functors.

Example 4.5 (a) (Duals of covariant functors) Let $D_F : \underline{\mathrm{Vec}}_{/F} \xrightarrow{\sim} \underline{\mathrm{Vec}}^0_{/F}$ denote the (contravariant) duality functor defined by $D_F(V) = V^* = \mathrm{Hom}(V, F)$. Clearly, if $\mathcal{F} : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}_{/F}$ is any faithful *covariant* functor, then its "dual" $\mathcal{F}^* = D_F \circ \mathcal{F} : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}^0_{/F}$ is a faithful *contravariant* functor. In particular, the duals of the functors H_1, T_0 and T^0_ℓ considered in Example 3.5 are called the *cohomology functor* $H^1 = (H_1)^*$, the *cotangent functor* $T^*_0 = (T_0)^*$ and the *étale cohomology functor* $H^1_{et}(\cdot, \mathbb{Q}_\ell) = (T^0_\ell(\cdot))^*$, respectively.

(b) (The functor of holomorphic differentials) Let $\mathrm{char}(K) = 0$ and let $\Omega : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}^0_{/K}$ denote the *functor of holomorphic differentials* defined by $\Omega(A) = H^0(A, \Omega^1_{A/K})$. Since the map $\omega \mapsto \omega_0 \in T^*_0(A)$ defines an isomorphism of functors $\Omega \simeq T^*_0$, this functor is again faithful.

In particular, if $K = \mathbb{Q}$, then $V = \Omega(A)$ satisfies the hypothesis of Theorem 4.4 and so Theorem 1.5 of the introduction follows from Theorem 4.4 together with (the proof of) Corollary 3.4.

For contravariant functors, the dual analogue of Theorem 3.6 is the following result which is proven in a similar manner.

Corollary 4.6 *If $\mathcal{G} : \underline{\mathrm{Ab}}_{/K} \to \underline{\mathrm{Vec}}^0_{/\mathbb{Q}}$ is a faithful, contravariant functor then for each A/K the map $(C, p) \mapsto \hat{W}_\mathcal{G}(C, p) := \mathrm{Im}(\mathcal{G}(p)) \subset \mathcal{G}(A)$ defines a bijection*

$$\hat{W}_\mathcal{G} = \hat{W}_{A/K,\mathcal{G}} : \mathbf{Quot}(A/K) \xrightarrow{\sim} \mathbf{Sub}(\mathcal{G}(A))_{\tilde{\mathbb{E}}},$$

where $\tilde{\mathbb{E}} = \mathrm{End}_\mathbb{E}(\mathcal{G}(A)) = \{f \in \mathrm{End}(\mathcal{G}(A)) : (vf)a = (va)f, \forall v \in \mathcal{G}(A), a \in \mathbb{E}\}$. Furthermore, the rule $h \mapsto \mathcal{G}(h)$ defines for $\underline{C}_i = (C_i, p_i)$ an isomorphism

$$(\hat{W}_\mathcal{G})_{\underline{C}_1,\underline{C}_2} : \mathrm{Hom}^0_K(C_1, C_2) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{\mathbb{E}}}(\mathcal{G}(C_2), \mathcal{G}(C_1)),$$

where we view $\mathcal{G}(C_i)$ as a right $\tilde{\mathbb{E}}$-module via the isomorphism $\mathcal{G}(p) : \mathcal{G}(C_i) \xrightarrow{\sim} \hat{W}_\mathcal{G}(C_i, p_i)$.

5 Applications to Modular Curves

Let $X_{\Gamma,\mathbb{C}} = \Gamma\backslash\mathfrak{H}^*$ be the complex modular curve attached to a subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ *of level* $N \geq 1$, i.e. $\Gamma_0(N) \leq \Gamma \leq \Gamma_1(N)$. (We could also take $\Gamma = \Gamma(N)$.) By Shimura[Sh1], $X_{\Gamma,\mathbb{C}}$ has a "canonical" model $X = X_\Gamma/\mathbb{Q}$ over $\mathbb{Q}$ such that the map $f \mapsto f dz$ induces a natural identification $S_2(\Gamma,\mathbb{Q}) \xrightarrow{\sim} H^0(X,\Omega^1_{X/\mathbb{Q}})$ where $S_2(\Gamma,\mathbb{Q}) \subset S_2(\Gamma)$ denotes the subspace of all cusp forms of weight 2 on Γ whose Fourier expansion (at the cusp ∞) have rational coefficients; cf. [Sh1], p. 156 and p. 140 or [DDT], p. 35. Thus, if $J = J_\Gamma/\mathbb{Q}$ denotes the Jacobian variety of X_Γ, then we have a canonical identification

$$\Omega(J) := H^0(J,\Omega^1_{J/\mathbb{Q}}) = H^0(X,\Omega^1_{X/\mathbb{Q}}) = S_2(\Gamma,\mathbb{Q}).$$

By Hecke's theory, there is a commutative subring (called the *Hecke algebra*) $\mathbb{T} = \mathbb{Q}[\{T_n\}_{n\geq 1}] \subset \mathbb{E} = \mathrm{End}^0_{\mathbb{Q}}(J)$ such that $S_2(\Gamma,\mathbb{Q})^* \simeq T_0(J)$ is a free $\mathbb{T}$-module of rank 1; cf. [Sh1], Theorem 3.51 or [DDT], Lemma 1.34. Furthermore, $\Omega(J) = S_2(\Gamma,\mathbb{Q})$ is also a free $\mathbb{T}$-module of rank 1, as can be seen either from Atin-Lehner Theory (cf. [DDT], Lemma 1.35) or by constructing an explicit $\mathbb{T}$-module isomorphism

$$\varphi_\Gamma : S_2(\Gamma,\mathbb{Q}) \xrightarrow{\sim} T_0(J)$$

as in [Sh2], §2; cf. also [Sh1], Theorem 3.51. (Actually, Shimura constructs only a $\mathbb{T}\otimes\mathbb{C}$-module isomorphism $\tilde{\varphi} : S_2(\Gamma) = S_2(\Gamma,\mathbb{Q})\otimes\mathbb{C} \xrightarrow{\sim} T_0(J\otimes\mathbb{C}) = T_0(J)\otimes\mathbb{C}$, but it is easy to check that $\tilde{\varphi}$ is $G_{\mathbb{Q}}$-equivariant.) We are now ready to prove the following generalization of the Shimura construction:

Theorem 5.1 *Let $W \subset S_2(\Gamma,\mathbb{Q})$ be any $\mathbb{T}$-submodule. Then:*

*(a) There exists a unique abelian subvariety $B_W \in$ **Sub**(J_Γ) such that $\varphi_\Gamma(W) = T_0(B_W) \subset T_0(J)$. Furthermore,* $\dim B_W = \dim_{\mathbb{Q}} W$ *and there exists a ring injection $\theta'_W : \mathrm{End}_{\mathbb{T}}(W) \hookrightarrow \mathrm{End}^0(B_W)$ such that $\theta'_W(t_{|W}) = t_{|B_W}$, for all $t \in \mathbb{T}$.*

(b) There exists a unique abelian quotient $p_W : J_\Gamma \to C_W$ such that $p_W^*\Omega(C_W) = W$. Furthermore, $\dim C_W = \dim_{\mathbb{Q}} W$ *and we have an injective ring homomorphism $\theta_W : \mathrm{End}_{\mathbb{T}}(W) \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(C_W)$ such that $\theta_W(t_{|W}) \circ p_W = p_W \circ t$, for all $t \in \mathbb{T}$.*

Proof. (a) The first assertion follows directly from Corollary 1.4, and the second is clear because $\dim B_W = \dim_{\mathbb{Q}} T_0(B_W) = \dim_{\mathbb{Q}} W$. Finally, the last assertion follows from Corollary 3.4 (applied to $\varphi(W) \subset V = T_0(J)$).

(b) This follows directly from Theorem 1.5, together with (the dual analogue of) Corollary 3.4.

The classical Shimura construction is the following special case of the above Theorem.

Example 5.2 (Shimura) Let $f \in S_2(\Gamma)$ be a (normalized) $\mathbb{T}$-eigenfunction, and put $\tilde{W}_f = \sum \mathbb{C} f^\sigma$, where the sum is over all $\mathrm{Aut}(\mathbb{C})$-conjugates of f. Clearly, $\tilde{W}_f$ is $\mathrm{Aut}(\mathbb{C})$-invariant and hence is of the form $\tilde{W}_f = W_f \otimes \mathbb{C}$ for a unique subspace $W_f \subset S_2(\Gamma, \mathbb{Q})$. Moreover, $\dim_{\mathbb{Q}} W_f = \dim_{\mathbb{C}} \tilde{W}_f = [K_f : \mathbb{Q}]$, where K_f is the field generated by the Fourier coefficients $a_n(f)$ of f. Let $\lambda_f : \mathbb{T} \to K_f$ denote the canonical surjective homomorphism defined by $f|t = \lambda_f(t)f$, for $t \in \mathbb{T}$. (In particular, $\lambda_f(T_n) = a_n(f)$, where T_n is the n-th Hecke operator.) It is then immediate that $\mathrm{Ann}_{\mathbb{T}}(W_f) = \mathrm{Ker}(\lambda_f)$, and thus we have a natural injection $K_f = \mathbb{T}/\mathrm{Ann}_{\mathbb{T}}(W_f) \hookrightarrow \mathrm{End}_{\mathbb{T}}(W)$. Thus, by the above theorem there exists an abelian subvariety $B_f = B_{W_f} \leq J = J_\Gamma$ and an abelian quotient $p = p_f : J \to C_f := C_{W_f}$ together with maps $\theta'_f : K_f \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(B_f)$ and $\theta_f : K_f \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(C_f)$ such that (B_f, θ'_f) and (C_f, p_f, θ_f) satisfy the following conditions (which are in fact identical to those of Theorems 1 and 2 of [Sh2]):

(i) $B_f \in \mathbf{Sub}(J/\mathbb{Q})$ and $(C_f, p_f) \in \mathbf{Quot}(J/\mathbb{Q})$.

(ii) $\theta'_f : K_f \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(B_f)$ and $\theta_f : K_f \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(C_f)$ are injective ring homomorphisms such that $\theta'_f(a_n(f)) = (T_n)_{|B_f}$ and $\theta_f(a_n(f)) \circ p_f = p_f \circ T_n$, for all $n \geq 1$.

(iii) $\dim B_f = \dim C_f = [K_f : \mathbb{Q}]$.

(iv) $T_0(B_f) = \varphi_\Gamma(W_f)$ and $p_f^* \Omega(C_f) = W_f$.

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