The Number of Curves of Genus 2 with a Given Refined Humbert Invariant

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1 Introduction

If C/K is a curve of genus 2 over an algebraically closed field K, then it comes equipped with a canonical quadratic form q_C called its *refined Humbert invariant*; cf. [K1], [K4]. This invariant is useful because many geometric properties of C are reflected in arithmetic properties of the quadratic form q_C; cf. [K1], [K5], [K7].

For example, the property that the Jacobian J_C of C is isogeneous to the self-product $E \times E$ for some elliptic curve E/K with complex multiplication can be characterized by a property of the form q_C; cf. Theorem 4 below. For convenience, let us call a curve C/K with this property a *curve of CM product type.* In this case the invariant q_C can be viewed as an equivalence class of positive integral *ternary* quadratic forms.

For such a curve C/K, it turns out that there are only finitely many isomorphism classes of curves C'/K which have the "same" refined Humbert invariant. The purpose of this paper is to determine this number precisely.

To state the result, let $d_C = d(q_C)$ denote the discriminant of q_C (in the sense of Brandt[B1] or of Watson[Wa]), and let $\Delta_C = \Delta_{q_C} := d_C/16$. Furthermore, let $\kappa_C = \kappa_{q_C} \geq 1$ be defined as follows. If q_C is a primitive form, then $\kappa_{q_C} = -I_1(q_C)/16$, where $I_1(q)$ is the genus invariant of a ternary form q (as defined by Brandt[B1]), and if q_C is imprimitive, then $\kappa_{q_C} = -I_1(q_C/4)$.

Theorem 1 *If C/K is a curve of genus 2 of CM product type, then the number N_C of isomorphism classes of genus 2 curves C'/K whose refined Humbert invariant $q_{C'}$ is equivalent to that of C is given by the formula*

$$N_C \;=\; 2^{\omega(\kappa_C)} h(\Delta_C) \frac{|\operatorname{Aut}(C)|}{|\operatorname{Aut}(q_C)|}, \tag{1}$$

where $\omega(\kappa_C)$ denotes the number of distinct prime divisors of κ_C, and $h(\Delta_C)$ denotes the number of proper equivalence classes of positive primitive binary forms of discriminant Δ_C.

By using standard finiteness results, the above theorem implies the following interesting fact.

Corollary 2 *For any algebraically closed field K and any integer $n \geq 1$, there are only finitely many isomorphism classes of genus 2 curves C/K of CM product type such that $N_C \leq n$. Moreover, there exist infinitely many curves C/K of CM product type such that $N_C > n$.*

The result of Theorem 1 is a special case of a more general formula which is valid for any principally polarized abelian surfaces (A, θ) of CM product type. For this, note that the refined Humbert invariant $q_{(A,\theta)}$ is defined for any principally polarized abelian surface (A, θ) (cf. §2), and that we have by definition that $q_C := q_{(J_C,\theta_C)}$, if C/K is a curve of genus 2.

In order to generalize formula (1) to a principally polarized abelian surface (A, θ), we first observe that all the quantities on the right hand of (1) naturally generalize to more general ternary forms, with the exception of the factor $|\operatorname{Aut}(C)|$ (cf. §2 below). But in [K7] it was shown that $|\operatorname{Aut}(C)|$ is given by an expression involving the number $r_n^*(q_C)$ of primitive representations of an integer n by q_C. More precisely, we have by Theorem 25 of [K7] that $|\operatorname{Aut}(C)| = 2a(q_C)$, where for any form q we put

$$(2) \qquad a(q) \; := \; \max(1, r_1^*(q)) \max(1, r_4^*(q), 3r_4^*(q) - 12).$$

This leads to the following generalization of Theorem 1.

Theorem 3 *If $(A, \theta)/K$ is a principally polarized abelian surface of CM product type with refined Humbert invariant $q := q_{(A,\theta)}$, then there are only finitely many isomorphism classes of principally polarized abelian surfaces $(A', \theta')/K$ such that $q_{(A',\theta')}$ is equivalent to q. Moreover, if $N_{(A,\theta)}$ denotes the number of these isomorphism classes, then we have that*

$$(3) \qquad N_{(A,\theta)} \; = \; 2^{\omega(\kappa_q)+1} h(\Delta_q) \frac{a(q)}{|\operatorname{Aut}(q)|},$$

except when q is equivalent to $x^2 + 4\kappa(y^2 + \varepsilon yz + z^2)$, for some $\kappa > 1$ and for some $\varepsilon = 0$ or 1. In this exceptional case we have that

$$(4) \qquad N_{(A,\theta)} \; = \; (2^{\omega(\kappa)-1} + 1 + \varepsilon) \frac{h(-(4-\varepsilon)\kappa^2)}{2+\varepsilon}.$$

Note that the exceptional case of Theorem 3 cannot occur when $(A, \theta) = (J_C, \theta_C)$ is a Jacobian; cf. §3 below.

In §4 we present some explicit examples which show that in certain cases the curve C/K is uniquely determined by its refined Humbert invariant q_C; cf. Proposition 14 and Examples 16 and 19.

2 The quadratic forms q_A and $q_{(A,\theta)}$

Let A/K be an abelian surface, where K is an arbitrary algebraically closed field and let $\mathrm{NS}(A) = \mathrm{Div}(A)/\equiv$ denote its Néron-Severi group. (Here $\equiv$ denotes the numerical equivalence of divisors on A.) The intersection product $(D_1.D_2)$ of divisors D_1, D_2 on A defines an integral quadratic form q_A on $\mathrm{NS}(A)$ which is given by the formula $q_A(D) = \frac{1}{2}(D.D)$, for $D \in \mathrm{NS}(A)$. Since $\mathrm{NS}(A) \simeq \mathbb{Z}^\rho$, where $\rho = \rho(A)$ is the Picard number of A, the form q_A is equivalent to an integral quadratic form q in ρ variables, i.e. we have an isomorphism $(\mathrm{NS}(A), q_A) \simeq (\mathbb{Z}^\rho, q)$ of quadratic modules. We then write $q_A \sim q$.

Now suppose that A has a principal polarization $\theta \in \mathcal{P}(A)$. (Here and below, $\mathcal{P}(A) \subset \mathrm{NS}(A)$ denotes the set of principal polarization on A.) Then the quadratic form $\tilde{q}_{(A,\theta)}$ on $\mathrm{NS}(A)$ is defined by the formula

$$\tilde{q}_{(A,\theta)}(D) = (D.\theta)^2 - 2(D.D), \quad \text{for } D \in \mathrm{NS}(A). \tag{5}$$

It is easy to see that $\tilde{q}_{(A,\theta)}(D + n\theta) = q_{(A,\theta)}(D)$, for all $n \in \mathbb{Z}$, so $\tilde{q}_{(A,\theta)}$ induces a quadratic form $q_{(A,\theta)}$ on the quotient module

$$\mathrm{NS}(A,\theta) = \mathrm{NS}(A)/\mathbb{Z}\theta.$$

Moreover, the Hodge Index Theorem shows that $q_{(A,\theta)}$ is a positive-definite form on $\mathrm{NS}(A,\theta) \simeq \mathbb{Z}^{\rho-1}$; cf. [K1]. The quadratic form $q_{(A,\theta)}$ or, more correctly, the quadratic module $(\mathrm{NS}(A,\theta), q_{(A,\theta)})$ is called the *refined Humbert invariant* of the principally polarized abelian surface (A,θ); cf. [K3], [K4].

We observe that if $(A,\theta) \simeq (A',\theta')$, i.e., if we have an isomorphism $\varphi : A \xrightarrow{\sim} A'$ of abelian surfaces such that $\varphi^*(\theta') = \theta$, then we have an induced module isomorphism $\varphi^* : \mathrm{NS}(A',\theta') \xrightarrow{\sim} \mathrm{NS}(A,\theta)$ such that $q_{(A',\theta')} = q_{(A,\theta)} \circ \varphi^*$, and hence the associated quadratic modules are isomorphic. Equivalently, this means that $q_{(A,\theta)} \sim q_{(A',\theta')}$.

Suppose now that A is isogeneous to a product abelian surface $E \times E$, where E/K is an elliptic curve with complex multiplication, i.e. $\mathrm{End}(E)$ is isomorphic to an order in an imaginary quadratic field. The set of such abelian surfaces may be classified as follows.

Theorem 4 *If A/K is an abelian surface, then the following conditions are equivalent:*

(i) *A is isogeneous to a product $E \times E$, for some CM elliptic curve E/K.*

(ii) *A is isomorphic to a product $E_1 \times E_2$, where E_1/K and E_2/K are two isogeneous CM elliptic curves over K.*

(iii) *$\mathcal{P}(A) \neq \emptyset$, and for one (and hence for any) $\theta \in \mathcal{P}(A)$, the refined Humbert invariant $q_{(A,\theta)}$ is a ternary form such that $q_{(A,\theta)}(D) = n^2$ is a positive square, for some $D \in \mathrm{NS}(A,\theta)$.*

Proof. (i) $\Rightarrow$ (ii): If $K = \mathbb{C}$, then this is due to Shioda and Mitani[SM]. For a general ground field K, this is a special case of Theorem 2 of [K2].

(ii) $\Rightarrow$ (iii): Let θ_{E_1,E_2} denote the product polarization on $A' := E_1 \times E_2$, i.e., θ_{E_1,E_2} is the image of the divisor $E_1 \times \{0\} + \{0\} \times E_2 \in \mathrm{Div}(A')$ in $\mathrm{NS}(A') = \mathrm{Div}(A')/\equiv$. Then $\theta_{E_1,E_2} \in \mathcal{P}(A')$, and so $\mathcal{P}(A) \neq \emptyset$ because if $\varphi : A \xrightarrow{\sim} A'$ is any isomorphism, then $\varphi^*(\theta_{E_1,E_2}) \in \mathcal{P}(A)$.

Now let $\theta \in \mathcal{P}(A)$. Then $\mathrm{rank}(\mathrm{NS}(A,\theta)) = \rho(E_1 \times E_2) - 1 = \mathrm{rank}(\mathrm{Hom}(E_1,E_2)) + 1$ by Corollary 24 of [K3]. Since $E_1 \sim E_2$ are isogeneous CM elliptic curves by (ii), it follows that $\mathrm{rank}(\mathrm{Hom}(E_1,E_2)) = 2$, so $\mathrm{rank}(\mathrm{NS}(A,\theta)) = 3$, and hence $q_{(A,\theta))}$ is a ternary form.

Moreover, by Theorem 1.5 of [K1] we know that if $E \leq A$ is an elliptic subgroup on A, then $q_{(A,\theta)}([E]) = (E.\theta)^2 > 0$, where $[E] \in \mathrm{NS}(A,\theta)$ denotes the image of E in $\mathrm{NS}(A,\theta)$. Since $E = \varphi^{-1}(E_1 \times \{0\}) \leq A$, we see that (iii) holds.

(iii) $\Rightarrow$ (i): Let $\theta \in \mathcal{P}(A)$, and let $D \in \mathrm{NS}(A,\theta)$ be such that $q_{(A,\theta)}(D) = n^2 > 0$. Then $D \neq 0$, so we can write $D = mD'$, where $m \geq 1$ and D' is primitive in $\mathrm{NS}(A,\theta)$, and hence $q_{(A,\theta)}(D') = \left(\frac{n}{m}\right)^2$. By Theorem 1.5 of [K1] there exists an elliptic subgroup $E \leq A$ such that $[E] = D'$. Thus, by Poincaré, A is isogeneous to $E \times A'$, for some abelian subvariety A' of A (cf. [Mu], p. 173). Since $\dim(A) = 2$, it follows that $\dim(A') = 1$, so $A' = E'$ is an elliptic curve, and hence $A \sim E \times E'$.

By hypothesis, $\mathrm{rank}(\mathrm{NS}(A,\theta)) = 3$, so $4 = \rho(A) = \rho(E \times E') = \mathrm{rank}(\mathrm{Hom}(E,E')) + 2$, i.e., $\mathrm{rank}(\mathrm{Hom}(E,E')) = 2$. This means that E and E' are isogeneous CM curves, so A satisfies (i).

In view of property (ii), we call any surface satisfying these properties a *CM abelian product surface*. (In the introduction these were called abelian surfaces of CM product type.) Note that property (iii) gives a characterization of such surfaces in terms of the refined Humbert invariant $q_{(A,\theta)}$.

We recall from [K3], [K4] and [Ki] some basic arithmetic facts about the forms q_A and $q_{(A,\theta)}$ for such a surface. To state these, we will use discriminant $\mathrm{disc}(f) = d(f)$ of an integral quadratic form f as defined in Watson[Wa], p. 2. Moreover, the *content* $\mathrm{cont}(f)$ of such a form is the gcd of its values. (It is easy to see that when f is a ternary form, then $\mathrm{cont}(f) = t$ is the "coefficient-divisor" of [B1].) If $\mathrm{cont}(f) = 1$, then f is said to be *primitive*; cf. [Wa], p. 4. In addition, the basic invariants $I_k(f)$ for $k = 1, 2$ of an integral primitive ternary form f are defined as in Brandt[B1]. (These invariants are closely related to the invariants Ω and Δ as defined in Dickson[Di].) More precisely, $|I_1(f)| = \mathrm{cont}(\mathrm{adj}(f))$, where $\mathrm{adj}(f)$ denotes the *adjoint* of f as defined in Watson[Wa], p. 25, and the sign is assigned so that the *reciprocal* $F_f := \mathrm{adj}(f)/I_1(f)$ of f is a positive definite form when f is positive definite. Furthermore, $I_2(f) := I_1(F_f)$.

Proposition 5 *Let $A \simeq E \times E'$ be a CM abelian product surface, and let $q_{E,E'}$ denote the degree form on* $\mathrm{Hom}(E,E')$. *Let $\Delta = d(q_{E,E})$ denote its discriminant and*

$\kappa = \mathrm{cont}(q_{E,E'})$ *its content. If* $\theta \in \mathrm{NS}(A)$ *is a principal polarization, then*

$$d(q_A) \;=\; \Delta \quad \textit{and} \quad d(q_{(A,\theta)}) \;=\; 16\Delta. \tag{6}$$

Moreover, if $c := \mathrm{cont}(q_{(A,\theta)})$*, then* $c = 1$ *or* 4*, and we have that*

$$I_1(q_{(A,\theta)}/c) \;=\; -\frac{16\kappa}{c^2} \quad \textit{and} \quad I_2(q_{(A,\theta)}/c) \;=\; \frac{c\Delta}{\kappa^2}. \tag{7}$$

Proof. We have that $\mathrm{rank}(\mathrm{Hom}(E, E')) = 2$ because E and E' are isogeneous CM elliptic curves, and so $q_{E,E'}$ is a binary quadratic form. It thus follows from Corollary 24 of [K3] that $\det(q_A) = -\det(q_{E,E'})$, where (as in [K3]) $\det(f)$ denotes the determinant of the associated Gram matrix $M(f)$. Thus, since $\mathrm{rank}(\mathrm{NS}(A)) = 4$, it follows from [Wa], p. 2, that $d(q_A) = \det(q_A) = -\det(q_{E,E'}) = d(q_{E,E'})$. This proves the first equality of (6).

The second equality of (6) follows from Proposition 9 of [K3]. Indeed, by that result we have that $\det(q_{(A,\theta)}) = -2^5 \det(q_A)$, so $d(q_{(A,\theta)}) = -\frac{1}{2}\det(q_{(A,\theta)}) = 2^4 \det(q_A) = 16d(q_A) = 16\Delta$. This proves (6).

If $c = 1$, i.e., if $q_{(A,\theta)}$ is primitive, then formula (7) follows from Propositions 25 and 18 of [K6]. If $c > 1$, then by Corollary 14 of [Ki] and the proof of Proposition 17 of [Ki], we see that $c = 4$, and so in this case (7) follows directly from that proposition.

In view of the above result, it is useful to introduce the following notation (which was already used in Theorem 3).

Notation. If q is a positive ternary quadratic form such that $c := \mathrm{cont}(q) = 1$ or 4, then put

$$\Delta_q \;:=\; d(q)/16 \quad \text{and} \quad \kappa_q \;:=\; -c^2 I_1(q/c)/16.$$

It is clear from the definition that the form $q_{(A,\theta)}$ is determined by the form q_A and an element θ with $q_A(\theta) = 1$; this is the so-called θ-construction in [K4]. We now show conversely that the ternary form $q_{(A,\theta)}$ determines the quaternary form q_A (up to equivalence). More precisely, we prove the following result.

Theorem 6 *Let* A_1 *and* A_2 *be two CM abelian product surfaces, and let* $\theta_i \in \mathrm{NS}(A_i)$ *be a principal polarization on* A_i*, for* $i = 1, 2$*. If* $q_{(A_1,\theta_1)}$ *is equivalent to* $q_{(A_2,\theta_2)}$*, then* q_{A_1} *is equivalent to* q_{A_2}*.*

To prove this result, we will use several facts which were proven elsewhere. The first concerns the p-adic equivalence class of the form $q_{(A,\theta)}$, where the p-adic equivalence of two integral forms f_1 and f_2 is defined as in Jones[Jo], p. 82, and is denoted by $f_1 \sim_p f_2$. Recall also that f_1 and f_2 are *genus-equivalent* if $f_1 \sim_p f_2$, for all primes p (including $p = \infty$). We let $\mathrm{gen}(f_1)$ denote the set of (equivalence classes of) integral forms f_2 which are genus-equivalent to f_1.

Proposition 7 *Let $A = E \times E'$ be a CM abelian product surface, and let θ be a principal polarization on A. If $f_q := x^2 \perp 4q$, where $q = q_{E,E'}$ is the degree form, then*

$$q_{(A,\theta)} \sim_p f_q, \quad \text{for all odd primes } p. \tag{8}$$

Furthermore, if $q_{(A,\theta)}$ is primitive, then (8) also holds for $p = 2$, and so in this case $q_{(A,\theta)}$ is genus-equivalent to f_q.

Proof. The first assertion (8) follows immediately from Corollary 19 of [K4] because by equation (29) of [K6] we know that $f_q \sim q_{(A,\theta_{E,E'})}$, where $\theta_{E,E'}$ is the product polarization on A. The second assertion follows from Theorem 20 of [K4], as is explained in the proof of Proposition 25 of [K6].

In order to state the next result, we recall from [K6] and [Ki] that if $f : M \to \mathbb{Z}$ is a quadratic form on a module M, then $R(f) := \{f(x) : x \in M\}$ denotes the set of values represented by f, and if a, m are integers, then

$$R_{a,m}(f) \ := \ \{n \in R(f) : n \equiv a \pmod{m}\}.$$

Proposition 8 *Let $A = E \times E'$ be a CM abelian product surface, and let θ be a principal polarization on A. If $q_{(A,\theta)}$ is imprimitive, then $d(q_{E,E'}) \equiv 0 \pmod 4$ and $R_{3,4}(q_{E,E'}) \neq \emptyset$. In particular, $\kappa := \mathrm{cont}(q_{E,E'})$ is odd. Furthermore, there exists $n \in R_{3,4}(q_{E,E'})$ such that $\frac{n}{\kappa} \in R(F_{q_{(A,\theta)}/4})$.*

Proof. The first assertions follow immediately from Proposition 15 of [Ki]. Moreover, the last assertion follows from Propositions 16 and 19 of [Ki]. To see this, put $n = q_{E,E'}(h)$, where $h \in \mathrm{Hom}(E, E')$ is as in Proposition 19 of [Ki]. Then by Propositions 16 and 19 of [Ki] we have that $n \in R_{3,4}(q_{E,E'})$ and that $\frac{n}{\kappa} \in R(F_f)$, where F_f denotes the reciprocal of the primitive form $f := q_{(A,\theta)}/4$.

We are now ready to prove Theorem 6.

Proof of Theorem 6. For $i = 1, 2$, let E_i/K and E_i'/K be two (isogeneous) CM elliptic curves such that $A_i \simeq E_i \times E_i'$. We first observe that it suffices to show that the given hypothesis implies that the binary forms $q_1 := q_{E_1,E_1'}$ and $q_2 := q_{E_2,E_2'}$ are genus-equivalent, i.e., that

$$\mathrm{gen}(q_1) \ = \ \mathrm{gen}(q_2). \tag{9}$$

Indeed, since we have that

$$q_{A_i} \ \sim \ xy \perp (-q_i), \quad \text{for } i = 1, 2, \tag{10}$$

by [K3], Proposition 23 (or by [K4], formula (6)), it follows from Remark 27 of [K4] that (9) implies that $q_{A_1} \sim q_{A_2}$, as desired.

To prove that (9) holds, we will distinguish two cases.

Case 1. $q := q_{(A_1,\theta_1)}$ is primitive.

Since q is primitive, so is $q_{(A_2,\theta_2)} \sim q$. Thus, by Proposition 7 we have that $q_{(A_i,\theta_i)} \in \mathrm{gen}(f_{q_i})$, for $i = 1, 2$. Thus, $\mathrm{gen}(f_{q_1}) = \mathrm{gen}(f_{q_2})$ because $q_{(A_1,\theta_1)} \sim q_{(A_2,\theta_2)}$, and so $x^2 + 4q_1 \sim_p x^2 + 4q_2$, for all primes $p \geq 2$. Since x^2 is a form of unit determinant in the sense of Jones[Jo], it follows from the Cancellation Theorem 37 of [Jo] that $2q_1 \sim_p 2q_2$, for all primes $p \geq 2$, and so $\mathrm{gen}(2q_1) = \mathrm{gen}(2q_2)$ because $2q_1$ and $2q_2$ are both positive binary forms. Then also $\mathrm{gen}(q_1) = \mathrm{gen}(q_2)$, which proves (9) in this case.

Case 2. $q = q_{(A_1,\theta_1)}$ is imprimitive.

Since q is imprimitive, so is $q_{(A_2,\theta_2)} \sim q$, and hence both have content $c = 4$ by Proposition 5. Since $I_k(q_{(A_1,\theta_1)}/4) = I_k(q_{(A_2,\theta_2)}/4)$, for $k = 1, 2$, it therefore follows from (7) that q_1 and q_2 have the same content κ and the same discriminant Δ. Thus, $q_i' := q_i/\kappa$ are two primitive binary forms of the same discriminant $\Delta' = \Delta/\kappa^2$. It is clear that (9) follows once we have shown that $\mathrm{gen}(q_1') = \mathrm{gen}(q_2')$.

To prove this, we will use Theorem 3.21 and Lemma 3.20 of Cox[Co] which state that $\mathrm{gen}(q_1') = \mathrm{gen}(q_2')$ if and only if q_1' and q_2' have the same "complete character".

This latter condition is defined as follows. For an odd prime p, let χ_p denote the Legendre character which is defined by $\chi_p(a) = \left(\frac{a}{p}\right)$, for $a \in \mathbb{Z}$, and put $\chi_{-4}(n) = \delta(n) = (-1)^{(n-1)/2}$ and $\chi_8(n) = \epsilon(n) = (-1)^{(n^2-1)/8}$, for $n \equiv 1 \pmod 2$. (Here and below we will use the χ_n notation of [B1] instead of the δ, ϵ notation of [Co].) Then the list of *assigned characters* of discriminant Δ' is $X(\Delta') := \{\chi_p : p|\Delta'\} \cup X_s(\Delta')$, where $X_s(\Delta') \subset X_s := \{\chi_{-4}, \chi_8, \chi_{-4}\chi_8\}$ is the set of *supplementary characters* which is given in the table on p. 55 of [Co]. For each $\chi \in X(\Delta')$, the value $\chi(q_i') := \chi(r_i)$ does not depend on the choice of $r_i \in R(q_i')$, provided that $\gcd(r_i, \Delta') = 1$. Thus, by Cox[Co], loc. cit., we have that $\mathrm{gen}(q_1') = \mathrm{gen}(q_2')$ if and only if

$$\chi(q_1') = \chi(q_2'), \quad \text{for all } \chi \in X(\Delta'). \tag{11}$$

It thus suffices to verify (11). For this, we recall that there is a similar theory of assigned characters for primitive ternary forms; cf. Smith[Sm] and Brandt[B1], [B2].

We will first apply this theory to the ternary forms f_{q_i}, for $i = 1, 2$. Since f_{q_i} is primitive and $f_{q_i} \sim q_{(A_i,\theta_{E_i,E_i'})}$ (cf. the proof of Proposition 7), it follows from Proposition 5 that

$$d(f_{q_i}) = 16\Delta, \quad I_1(f_{q_i}) = -16\kappa \quad \text{and} \quad I_2(f_{q_i}) = \Delta', \quad \text{for } i = 1, 2. \tag{12}$$

Let p be an odd prime. Since $q_{(A_1,\theta_1)} \sim q_{(A_2,\theta_2)}$ by hypothesis, it follows from (8) that $f_{q_1} \sim_p q_{(A_1,\theta_1)} \sim_p q_{(A_2,\theta_2)} \sim_p f_{q_2}$. This (together with (12)) implies that also their reciprocals $F_i := F_{f_{q_i}}$ are p-adically equivalent (for p odd); cf. Proposition 8 of [Ki]. Furthermore, from that proposition we also obtain that

$$(13) \qquad \chi_p(n_1) = \chi_p(n_2),\ \forall \text{ odd } p|\Delta',\ \forall n_i \in R(F_i) \text{ with } \gcd(n_i, \Delta') = 1,\ i = 1, 2.$$

Now by formula (44) of [K4] we have that $F_i \sim (-\Delta'\kappa)x^2 \perp q_i'$, so $R(q_i') \subset R(F_i)$, for $i = 1, 2$. Thus, if p is an odd prime with $p|\Delta'$, then $\chi_p \in X(\Delta')$, so if $n_i \in R(q_i')$ with $\gcd(n_i, \Delta') = 1$, then by (13) we obtain that $\chi_p(q_1') = \chi_p(n_1) = \chi_p(n_2) = \chi_p(q_2')$. This means (11) holds for all $\chi \in X(\Delta') \setminus X_s(\Delta')$.

It thus remains to show that (11) also holds for $\chi \in X_s(\Delta')$. For this, note first that by Proposition 8 we have that $\Delta' \equiv \Delta \equiv 0 \pmod 4$. As in [Co], p. 55, put $n := -\frac{\Delta'}{4}$.

Suppose first that $n \equiv 3 \pmod 4$. Then $X_s(\Delta') = \emptyset$ by [Co], so there is nothing to prove. Next, suppose that $n \equiv 1 \pmod 4$, so $X_s(\Delta') = \{\chi_{-4}\}$ by [Co]. Now by Proposition 8 there exists $n_i \in R_{3,4}(q_i)$, so $n_i' := \frac{n_i}{\kappa} \in R_{1,2}(q_i)$ because κ is odd. Thus, for $i = 1, 2$, we have that $\chi_{-4}(q_i') = \chi_{-4}(n_i') = \chi_{-4}(\kappa)^{-1}\chi_{-4}(n_i) = -\chi_{-4}(\kappa)$ because $n_i \equiv 3 \pmod 4$, and so $\chi_{-4}(q_1') = \chi_{-4}(q_2')$. Thus, (11) holds in this case as well.

We are thus left with the case that $n \equiv 0 \pmod 2$, i.e., that $\Delta' \equiv 0 \pmod 8$. In this case we will make use of the supplementary characters of the reciprocals F_{f_i} of the forms $f_i = q_{(A_i,\theta_i)}/4$, for $i = 1, 2$. By hypothesis, $f_1 \sim f_2$, and by Proposition 5 we see that f_i is a primitive ternary form with genus invariants $I_1(f_i) = -\kappa$ and $I_2(f_i) = 4\Delta'$. Since $\Delta' \equiv 0 \pmod 8$, we have that $I_1(F_{f_i}) = I_2(f_i) \equiv 0 \pmod{32}$. By Brandt[B1], p. 337, this implies that χ_{-4}, χ_8 and $\chi_{-8} := \chi_{-4}\chi_8$ are assigned characters of F_{f_i}. This means that if $\chi \in X_s = \{\chi_{-4}, \chi_8, \chi_{-8}\}$, and if $i = 1, 2$, then we have that

$$(14) \qquad \chi(r) \ = \ \chi(r'), \quad \text{for all } r, r' \in R_{1,2}(F_{f_i}).$$

Now let $\chi \in X_s(\Delta') \subset X_s$. Then $\chi(q_i') = \chi(n_i)$, for any $n_i \in R_{1,2}(q_i')$. By Proposition 8 there exists $n_i \in R_{1,2}(q_i') \cap R_{1,2}(F_{f_i})$. Now since $f_1 \sim f_2$, we also have that $F_{f_1} \sim F_{f_2}$ and so $R_{1,2}(F_{f_1}) = R_{1,2}(F_{f_2})$, and so $n_1, n_2 \in R_{1,2}(F_{f_1})$. Thus, by (14) we obtain that $\chi(q_1') = \chi(n_1) = \chi(n_2) = \chi(q_2')$, for all $\chi \in X_s(\Delta)$. This proves that (11) holds in all cases, and hence also (9), as desired.

3 The number $N_{(A,\theta)}$ of isomorphism classes

The purpose of this section is to prove the main results stated in the introduction. Let $\langle A, \theta\rangle$ denote the isomorphism class of a principally polarized abelian surface (A, θ), and let $\mathcal{A}_2(K)$ denote the set of isomorphism classes of principally polarized abelian surfaces $(A, \theta)/K$. When $(A, \theta)/K$ is a fixed principally polarized abelian surface, put

$$\mathcal{N}_{(A,\theta)} \ := \ \{\langle A', \theta'\rangle \in \mathcal{A}_2(K) \ : \ q_{(A',\theta')} \sim q_{(A,\theta)}\}.$$

By Theorem 4, it is clear that if A is a CM abelian product surface, then so is A', for any $\langle A', \theta' \rangle \in \mathcal{N}_{(A,\theta)}$. As in the introduction, let us put

$$N_{(A,\theta)} \;=\; |\mathcal{N}_{(A,\theta)}|.$$

The first step is to count the number of non-isomorphic abelian surfaces A'/K, which appear in the set $\mathcal{N}_{(A,\theta)}$. To formulate this, let $\langle A \rangle$ denote the isomorphism class of an abelian surface A/K. For a given quadratic form q, let us define

$$\mathcal{N}(q) \;:=\; \{\langle A \rangle \,:\, q_{(A,\theta)} \sim q, \text{ for some } \theta \in \mathcal{P}(A)\}$$

as the set of isomorphism classes of abelian surfaces A/K such that $q_{(A,\theta)}$ is equivalent to q, for some $\theta \in \mathcal{P}(A)$. Note that if $q_{(A,\theta)}$ is a ternary form which represents a square, then we see that $\mathcal{N}(q_{(A,\theta)})$ is a subset of the set of isomorphism classes of CM abelian product surfaces by Theorem 4.

In order to find $|\mathcal{N}(q_{(A,\theta)})|$ in this case, recall from the previous section that the form $q_{(A,\theta)}$ determines the form q_A (up to equivalence); cf. Theorem 6. This fact is the key tool to complete the first step because it provides the relation between the set $\mathcal{N}(q_{(A,\theta)})$ and the set $\mathcal{N}(A)$, where

$$\mathcal{N}(A) \;:=\; \{\langle A' \rangle \,:\, A' \sim A,\; q_{A'} \sim q_A\}$$

denotes the set of isomorphism classes of abelian surfaces A'/K which are isogeneous to a given A and whose intersection form $q_{A'}$ is equivalent to q_A. The advantage of studying this set is that [K2] gives an explicit formula for the cardinality $|\mathcal{N}(A)|$ when A is a CM abelian product surface. We now prove the following result, which will complete the first step.

Proposition 9 *Let $A = E_1 \times E_2$ be a CM abelian product surface, and let $\theta \in \mathcal{P}(A)$. Put $q := q_{(A,\theta)}$, and let $q_1 := q_{E_1,E_2}$. Then we have that*

(15) $$\mathcal{N}(q) \;=\; \mathcal{N}(A).$$

Moreover, if we put $\Delta' = d(q_1)/\operatorname{cont}(q_1)^2$, then

(16) $$|\mathcal{N}(q)| \;=\; \frac{h(\Delta')}{g(\Delta')},$$

where $g(\Delta')$ denotes the number of genera of discriminant Δ'.

Proof. We first observe that

(17) $$\operatorname{End}^0(E_i) \;\simeq\; \mathbb{Q}(\sqrt{d(q_1)}), \quad \text{for } i = 1, 2.$$

Indeed, since E_1 and E_2 are two isogeneous CM elliptic curves, we have that $F := \mathrm{End}^0(E_1) \simeq \mathrm{End}^0(E_2)$ is an imaginary quadratic field, and by Corollary 42 of [K2] we have that $d(q_1) = \mathrm{lcm}(f_{E_1}, f_{E_2})^2\Delta_F$, where Δ_F is the discriminant of F and f_{E_i} is the *endomorphism conductor* of E_i as defined in [K2]. From this, (17) follows because $F \simeq \mathbb{Q}(\sqrt{\Delta_F})$.

To prove (15), suppose first that $\langle A'\rangle \in \mathcal{N}(q)$, so by definition there is a $\theta' \in \mathcal{P}(A')$ such that $q_{(A',\theta')} \sim q$. Since A is a CM abelian product surface, so is A' by Theorem 4, and hence we have by Theorem 6 that $q_A \sim q_{A'}$. Thus, in order to show that $\langle A'\rangle \in \mathcal{N}(A)$, it suffices to prove that $A \sim A'$.

Now since A' is a CM product surface, we have that $A' \simeq E_1' \times E_2'$, for some isogeneous CM elliptic curves E_1'/K and E_2'/K. If we put $q_1' := q_{E_1',E_2'}$, then by (6) we see that $d(q_1) = d(q)/16 = d(q_{(A',\theta')})/16 = d(q_1')$. Thus, equation (17) (applied to E_i') shows that $\mathrm{End}^0(E_i') \simeq \mathbb{Q}(\sqrt{d(q_1')}) = \mathbb{Q}(\sqrt{d(q_1)}) \simeq \mathrm{End}^0(E_i)$, for $i = 1, 2$, and so it follows that $E_i \sim E_i'$; cf. Proposition 36 of [K2]. It thus follows that $A' \simeq E_1' \times E_2' \sim E_1 \times E_2 = A$, and hence $\langle A'\rangle \in \mathcal{N}(A)$.

To prove the opposite inclusion, let $\langle A'\rangle \in \mathcal{N}(A)$, so we have that $q_{A'} \sim q_A$. Hence, since $\theta \in \mathcal{P}(A)$, it follows from Proposition 29 of [K6] that $q \sim q_{(A',\theta')}$, for some $\theta' \in \mathcal{P}(A')$, which proves that $\langle A'\rangle \in \mathcal{N}(q)$, and so (15) follows.

To prove (16), we will use Corollary 70 of [K2]. To apply it, note first that there exists an elliptic curve E/K with $E \sim E_i$ such that the discriminant Δ_E of the order $\mathrm{End}(E)$ equals $d(q_1)$. Indeed, by Proposition 29 of [K2] there exists an $E \sim E_i$ such that $f_E = \mathrm{lcm}(f_{E_1}, f_{E_2})$, and so by Corollary 42 of [K2] we have that $d(q_1) = \mathrm{lcm}(f_{E_1}, f_{E_2})^2\Delta_F$, where $F = \mathrm{End}^0(E) \simeq \mathrm{End}^0(E_i)$, and so $d(q_1) = f_E^2\Delta_F = \Delta_E$.

We thus have that $A \simeq E_1 \times E_2 \sim E \times E$. Moreover, by (10) we know that $q_A \sim xy \perp (-q_1)$, so it follows that $\mathcal{N}(A) = \mathcal{N}_{q_1} := \{A' \sim E \times E : q_{A'} \sim xy \perp (-q_1)\}/\simeq$. We thus see that formula (16) follows directly from that given in Corollary 70 of [K2].

The second step in finding a formula for $N_{(A,\theta)}$ is to determine the number of isomorphism classes of principal polarizations θ' on a given abelian surface A' with $\langle A'\rangle \in \mathcal{N}(q_{(A,\theta)})$. For this, we will use the results of [K7] and [K8]. As in those papers, let

$$\mathcal{P}(A, q) := \{\theta \in \mathcal{P}(A) \,:\, q_{(A,\theta)} \sim q\}$$

denote the set of principal polarizations θ on A such that $q_{(A,\theta)}$ is equivalent to a given quadratic form q. By Theorem 9 of [K7] we know that the automorphism group $\mathrm{Aut}(A)$ of A acts on $\mathcal{P}(A, q)$, so we can consider the set $\overline{\mathcal{P}}(A, q) := \mathrm{Aut}(A)\backslash\mathcal{P}(A, q)$ of orbits of $\mathcal{P}(A, q)$ under this action. We now show that the number $N_{(A,\theta)}$ can be expressed in terms of a sum of the cardinalities of suitable sets $\overline{\mathcal{P}}(A_i, q)$.

Proposition 10 *Let (A, θ) be a principally polarized abelian surface, where A is a CM abelian product surface, and let $q = q_{(A,\theta)}$. If $A_1, \ldots, A_n$ is a system of represen-*

tatives of the finite set $\mathcal{N}(q)$, then

$$\mathcal{N}_{(A,\theta)} \;=\; \{\langle A_i, \theta_i\rangle : \theta_i \in \mathcal{P}(A_i, q),\ \textit{for } 1 \le i \le n\}, \tag{18}$$

and hence

$$N_{(A,\theta)} \;=\; \sum_{i=1}^{n} |\overline{\mathcal{P}}(A_i, q)|. \tag{19}$$

In particular, $N_{(A,\theta)} < \infty$.

Proof. If $\langle A', \theta'\rangle \in \mathcal{N}_{(A,\theta)}$, then by definition $\langle A'\rangle \in \mathcal{N}(q)$ and $\theta' \in \mathcal{P}(A', q)$ because $q_{(A',\theta')} \sim q_{(A,\theta)} = q$. Thus, $A' \simeq A_i$, for some $1 \le i \le n$, and hence the left hand side of (18) is contained in the right hand side.

To prove the opposite inclusion, let $\langle A_i, \theta_i\rangle$ be an element contained in the right hand side of (18), so $\langle A_i\rangle \in \mathcal{N}(q)$ and $\theta_i \in \mathcal{P}(A_i, q)$. This implies that $q_{(A_i,\theta_i)} \sim q = q_{(A,\theta)}$, and hence $\langle A_i, \theta_i\rangle \in \mathcal{N}_{(A,\theta)}$, which verifies (18).

To verify (19), recall first from Proposition 9 that $\mathcal{N}(q)$ is a finite set, so the sum on the right hand side of (19) is a finite sum.

Next we observe that if $\langle A_i, \theta_i\rangle = \langle A_j, \theta_j\rangle$, then in particular $A_i \simeq A_j$ and so $i = j$. Thus, $\langle A_i, \theta_i\rangle = \langle A_j, \theta_j\rangle$ if and only if $i = j$ and θ_i and θ_j lie in the same $\mathrm{Aut}(A_i)$-orbit of $\mathcal{P}(A_i, q)$. This means that for each i, the number of $\langle A', \theta'\rangle \in \mathcal{N}_{(A,\theta)}$ with $A' \simeq A_i$ is equal to $|\overline{\mathcal{P}}(A_i, q)|$, and so (19) follows from (18).

Since $\overline{\mathcal{P}}(A_i, q)$ is always a finite set (cf. Theorem 1 of [K7], together with formula (7) of [K7]), it follows from (19) that $N_{(A,\theta)} = |\mathcal{N}_{(A,\theta)}|$ is also finite.

To analyze $N_{(A,\theta)}$ further, we will make use of a formula for the cardinality of the set $\overline{\mathcal{P}}(A, q)$ found in [K7] and [K8]. One of the ingredients of this formula is the quantity $a(q)$ which was mentioned in (2), and which used certain "representation numbers" $r_n^*(q)$. These numbers are defined as follows. If $n > 0$ is an integer, and if q is a positive ternary form, then the *number of primitive representations* of n by q is defined by

$$r_n^*(q) := |\{(x_1, x_2, x_3) \in R_n(q) \;:\; \gcd(x_1, x_2, x_3) = 1\}|,$$

where $R_n(q) := \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \;:\; q(x_1, x_2, x_3) = n\}$ denotes the set of all representations of an integer n by q.

Proposition 11 *Let $A = E \times E'$ be a CM abelian product surface, and suppose that $\mathcal{P}(A, q)$ is nonempty. Let $\Delta = d(q_{E,E'})$ denote the discriminant of the degree map $q_{E,E'}$, and let $\kappa = \mathrm{cont}(q_{E,E'})$ be its content, and put $\Delta' = \Delta/\kappa^2$. If q is not equivalent to $x^2 + 4\kappa(y^2 + \varepsilon yz + z^2)$, for any $\kappa > 1$ and $\varepsilon = 0$ or 1, then we have that*

$$|\overline{\mathcal{P}}(A, q)| \;=\; \frac{2^{\omega(\kappa)+1} g(\Delta') h(\Delta) a(q)}{|\,\mathrm{Aut}(q)| h(\Delta')}. \tag{20}$$

On the other hand, if q is equivalent to $x^2+4\kappa(y^2+\varepsilon yz+z^2)$, for some $\kappa>1$ and for some $\varepsilon=0$ or 1, then we have that

$$(21)\qquad |\overline{\mathcal{P}}(A,q)| \;=\; (2^{\omega(\kappa)-1}+1+\varepsilon)\frac{h((\varepsilon-4)\kappa^2)}{2+\varepsilon}.$$

Proof. Since $\mathcal{P}(A,q)$ is non-empty, there is a $\theta\in\mathcal{P}(A)$ with $q_{(A,\theta)}\sim q$. Note that q is a ternary form by Theorem 4.

To prove (20), suppose first that q does not represent 1. Then (20) follows directly from Theorem 3 and formula (7) of [K7]. (Note that in this case (2) reduces to the formula (3) of [K7]; cf. Corollary 18 of [K8].)

Next, if q represents 1 but is not equivalent to $x^2+4\kappa(y^2+\varepsilon yz+z^2)$, for any $\kappa>1$ and $\varepsilon\in\{0,1\}$, then from the proof of Corollary 18 of [K8] we see that $a(q)=2\max(1,r_4^*(q))$, and hence (20) follows from Theorem 1 of [K8] together with formula (7) of [K7].

We are thus left with the case that $q\sim x^2+4\kappa(y^2+\varepsilon yz+z^2)$, for some $\kappa>1$ and $\varepsilon\in\{0,1\}$. Put $q_{\varepsilon,\kappa}:=\kappa(y^2+\varepsilon yz+z^2)$, so $\mathrm{cont}(q_{\varepsilon,\kappa})=\kappa$ and $d(q_{\varepsilon,\kappa})/\kappa^2=\varepsilon-4$. Since $x^2+4\kappa(y^2+\varepsilon yz+z^2)=f_{q_{\varepsilon,\kappa}}$ in the notation of [K4], the formula given in the proof of Proposition 53 of [K4] shows that $I_1(q)=I_1(f_{q_{\varepsilon,\kappa}})=-16\kappa$ and $d(q)=d(f_{q_{\varepsilon,\kappa}})=16\kappa^2(\varepsilon-4)$. It thus follows from Proposition 5 that in this case $\Delta=d(q_{E,E'})=\kappa^2(\varepsilon-4)$ and $\Delta'=\varepsilon-4$.

Put $u(\Delta'):=|\operatorname{Aut}^+(q_{\varepsilon,\kappa}/\kappa)|$ (as in [K8]). Thus, $u(\Delta')=4+2\varepsilon$, as is well-known; cf. [Jo], Theorem 51a. Moreover, since $h(\Delta')=1$ (cf. Theorem 7.30 of [Co]), we also have that $g(\Delta')=1$, and so we obtain from Proposition 20 of [K8] that

$$|\overline{\mathcal{P}}(A,q)|=(2^{\omega(\kappa)}+u(\Delta')-2)\frac{2g(\Delta')h(\Delta)}{|\operatorname{Aut}(q_{\varepsilon,\kappa})|h(\Delta')}=(2^{\omega(\kappa)}+2+2\varepsilon)\frac{2h(\Delta)}{|\operatorname{Aut}(q_{\varepsilon,\kappa})|}.$$

This shows that (21) holds because $|\operatorname{Aut}(q_{\varepsilon,\kappa})|=2u(\Delta')=8+4\varepsilon$.

It follows from the above proposition together with Proposition 5 that $|\overline{\mathcal{P}}(A_i,q)|$ does not depend on the choice of $\langle A_i\rangle\in\mathcal{N}(q)$.

Corollary 12 *Let A be a CM abelian product surface. If $\mathcal{P}(A,q)$ is nonempty, then*

$$(22)\qquad |\overline{\mathcal{P}}(A,q)| \;=\; |\overline{\mathcal{P}}(A',q)|,\ \textit{for every}\ \langle A'\rangle\in\mathcal{N}(q).$$

Proof. By hypothesis, $A\simeq E_1\times E_2$, for some isogeneous CM elliptic curves E_i/K. Since $\mathcal{P}(A,q)\neq\emptyset$, there is a $\theta\in\mathcal{P}(A)$ such that $q_{(A,\theta)}\sim q$, and so $\langle A\rangle\in\mathcal{N}(q)$.

Let $\langle A'\rangle\in\mathcal{N}(q)$. Then by definition there exists a $\theta'\in\mathcal{P}(A')$ such that $q_{(A',\theta')}\sim q\sim q_{(A,\theta)}$, and hence A' is again a CM abelian product surface by Theorem 4. Thus, $A'\simeq E_1'\times E_2'$, for some isogeneous CM elliptic curves E_i'/K. Since $q_{(A',\theta')}\sim q_{(A,\theta)}$, we have that $d(q_{(A',\theta')}) = d(q_{(A,\theta)})$, $\mathrm{cont}(q_{(A',\theta')}) = \mathrm{cont}(q_{(A,\theta)})$ and that $I_1(q_{(A',\theta')}) = I_1(q_{(A,\theta)})$. It thus follows from Proposition 5 that

$$d(q_{E_1',E_2'}) = d(q_{E_1,E_2}) \quad \text{and} \quad \mathrm{cont}(q_{E_1',E_2'}) = \mathrm{cont}(q_{E_1,E_2}). \tag{23}$$

Now if $q_{(A',\theta')}$ is not equivalent to $x^2+4\kappa(y^2+\varepsilon yz+z^2)$, for any $\kappa > 1$ and $\varepsilon \in \{0,1\}$, then the same is true for $q_{(A,\theta)} \sim q_{(A',\theta')}$, and then formula (20), together with (23), shows that $|\overline{\mathcal{P}}(A,q)| = |\overline{\mathcal{P}}(A',q)|$. On the other hand, if $q_{(A',\theta')} \sim x^2+4\kappa(y^2+\varepsilon yz+z^2)$, for some $\kappa > 1$ and $\varepsilon \in \{0,1\}$, then the assertion follows from (21).

We are now ready to prove Theorem 3.

Proof of Theorem 3. By hypothesis, A satisfies condition (i) of Theorem 4, so by that theorem there exist two isogeneous CM elliptic curves E_i/K such that $A \simeq E_1 \times E_2$. In addition, $q := q_{(A,\theta)}$ is a ternary form. Thus $N_{(A,\theta)} < \infty$ by Proposition 10.

Let $\Delta = d(q_{E_1,E_2})$ and $\kappa = \mathrm{cont}(q_{E_1,E_2})$. Then from (6), (7) and the definitions we see that $\Delta_q = \Delta$ and that $\kappa_q = \kappa$. Put $\Delta' = \Delta/\kappa^2$.

Now if q is not equivalent to $f_{q_{\varepsilon,\kappa}} := x^2 + 4\kappa(y^2 + \varepsilon yz + z^2)$, for any $\kappa > 1$ and $\varepsilon \in \{0,1\}$, then (3) follows from (19), (22),(16) and (20) because

$$\begin{aligned} N_{(A,\theta)} &= \sum_{\langle A_i\rangle \in \mathcal{N}(q)} |\overline{\mathcal{P}}(A_i,q)| \;=\; |\mathcal{N}(q)||\overline{\mathcal{P}}(A,q)| \\ &\overset{(16)}{=} \frac{h(\Delta')}{g(\Delta')}|\overline{\mathcal{P}}(A,q)| \overset{(20)}{=} \frac{h(\Delta')}{g(\Delta')}\frac{2^{\omega(\kappa)+1}g(\Delta')h(\Delta)a(q)}{|\operatorname{Aut}(q)|h(\Delta')} \\ &= 2^{\omega(\kappa)+1}h(\Delta)\frac{a(q)}{|\operatorname{Aut}(q)|} \;=\; 2^{\omega(\kappa_q)+1}h(\Delta_q)\frac{a(q)}{|\operatorname{Aut}(q)|}. \end{aligned}$$

On the other hand, if $q \sim f_{q_{\varepsilon,\kappa}}$, for some $\kappa > 1$ and some $\varepsilon \in \{0,1\}$, then $h(\Delta') = g(\Delta') = 1$, as we saw in the proof of Proposition 11. This implies by (16) that $|\mathcal{N}(q)| = 1$, so $N_{(A,\theta)} = |\overline{\mathcal{P}}(A,q)|$ by (19), and hence (4) follows from (21).

We will now deduce Theorem 1 from Theorem 3. For this, recall (from [Mi], for example) that if C/K is a curve of genus 2, then its Jacobian J_C is an abelian surface and the image $j(C)$ of C via the canonical embedding $j : C \hookrightarrow J_C$ is an ample divisor on J_C. Furthermore, the image $\theta_C \in \mathrm{NS}(J_C)$ of $j(C)$ in the Néron-Severi group is a principal polarization. We then write $q_C := q_{(J_C,\theta_C)}$ for its associated refined Humbert invariant.

It turns out that whether or not a given principally polarized abelian surface (A,θ) is a Jacobian can be determined from its refined Humbert invariant $q_{(A,\theta)}$. Indeed, by by Proposition 6 of [K3] we have that

$$(A,\theta) \simeq (J_C,\theta_C), \text{ for some curve } C/K \;\Leftrightarrow\; q_{(A,\theta)}(D) \neq 1, \forall D \in \mathrm{NS}(A,\theta). \tag{24}$$

This result, together with Torelli's Theorem, implies that that the number N_C defined in the introduction equals the number $N_{(J_C,\theta_C)}$, if C is of CM product type. More precisely:

Lemma 13 *If C/K is a curve of genus 2, then the map $C' \mapsto (J_{C'}, \theta_{C'})$ induces a bijection $\mathcal{N}_C \xrightarrow{\sim} \mathcal{N}_{(J_C,\theta_C)}$, where $\mathcal{N}_C$ denotes the set of isomorphism classes $\langle C' \rangle$ of curves C'/K of genus 2 such that $q_{C'} \sim q_C$. Thus, if C/K is of CM product type, then $N_C = N_{(J_C,\theta_C)}$.*

Proof. It is clear from the definitions that the given rule induces a map $\varphi_C : \mathcal{N}_C \to \mathcal{N}_{(J_C,\theta_C)}$. This map is injective by Torelli's Theorem; cf. Theorem 12.1 in [Mi]. Moreover, φ_C is surjective because if $\langle A, \theta \rangle \in \mathcal{N}_{(J_C,\theta_C)}$, then $q_{(A,\theta)} \sim q_{(J_C,\theta_C)}$. Since $q_{(J_C,\theta_C)}$ does not represent 1 by (24), it follows that also $q_{(A,\theta)}$ does not represent 1, and so by (24) again we have that $(A, \theta) \simeq (J_{C'}, \theta_{C'})$, for some curve C'/K of genus 2. Since $q_{C'} = q_{(J_{C'},\theta_{C'})} \sim q_{(A,\theta)} \sim q_{(J_C,\theta_C)} = q_C$, we see that $\langle C' \rangle \in \mathcal{N}_C$, so $\langle A, \theta \rangle = \varphi_C(\langle C' \rangle)$ lies in the image of φ_C. Thus, φ_C is surjective and hence bijective.

If C is of CM product type, then J_C is a CM abelian product surface by Theorem 4, and so $N_{(J_C,\theta_C)} < \infty$ by Proposition 10. Thus, the first assertion shows that $N_C := |\mathcal{N}_C| = |\mathcal{N}_{(J_C,\theta_C)}| = N_{(J_C,\theta_C)}$.

Proof of Theorem 1. Since C is of CM product type, we have by Lemma 13 that $N_C = N_{(J_C,\theta_C)}$. As was mentioned in its proof, J_C is a CM abelian product surface. By (24) we know that $q_C = q_{(J_C,\theta_C)}$ cannot represent 1, so q_C cannot be equivalent to one of the exceptional forms of Theorem 3, and hence it follows from that theorem that

$$N_C \;=\; N_{(J_C,\theta_C)} \;=\; 2^{\omega(\kappa_{q_C})+1} h(\Delta_{q_C}) \frac{a(q_C)}{|\operatorname{Aut}(q_C)|}.$$

Since q_C represents a square by Theorem 4, we have by Theorem 25 of [K7] that

$$2a(q_C) \;=\; |\operatorname{Aut}(C)|, \tag{25}$$

and so (1) follows.

Proof of Corollary 2. To prove the first assertion, fix an integer $n \geq 1$ and let

$$Q_n \;=\; \{q_C : C/K \text{ is a curve of genus 2 of product CM type with } N_C \leq n\}/\sim$$

denote the set of equivalence classes of those refined Humbert invariants which arise from curves C/K of genus 2 of product CM type with $N_C \leq n$. It clearly suffices to show that $|Q_n| < \infty$ because by definition the number of isomorphism classes of such curves with a given $q \in Q_n$ is equal to $N_C \leq n$.

Now if $q \in Q_n$, then by Theorem 4 we know that q is a positive ternary form and so $|\operatorname{Aut}(q)| \leq 48$ because $|\operatorname{Aut}^+(q)| \leq 24$ by Theorem 105 of [Di] and because

$|\operatorname{Aut}(q)| = 2|\operatorname{Aut}^+(q)|$ since $-1 \in \operatorname{Aut}(q) \setminus \operatorname{Aut}^+(q)$. We thus obtain from (1) that if q is in Q_n, then

$$\frac{h(\Delta_q)}{48} \;\le\; 2^{\omega(\kappa_q)} h(\Delta_q) \frac{|\operatorname{Aut}(C)|}{|\operatorname{Aut}(q)|} \;=\; N_C \;\le\; n.$$

Now if $q \in Q_n$, then $\Delta_q = d(q)/16$ is a negative quadratic discriminant by Proposition 5. But by the celebrated result of Heilbronn[He] we know that there are only finitely many negative discriminants Δ such that $h(\Delta) \le 48n$, so there exists a bound B_n such that $h(\Delta) \le 48n \Rightarrow |\Delta| \le B_n$. We therefore see that Q_n is a subset of the set

$$\tilde{Q}_n = \{q : q \text{ is a positive ternary form with } |d(q)| \le 16B_n\}/\!\sim .$$

Now by Theorem 11 of [Wa], there are up to equivalence only finitely many forms of a given rank and discriminant, so $\tilde{Q}_n$ is a finite set, and hence so is its subset Q_n. This proves the first assertion.

In order to prove the second assertion, it suffices in view of the first assertion to show that there exist infinitely many non-isomorphic genus 2 curves C/K of CM product type.

To verify this, we first show that the set $\mathcal{A}_{CM}(K) = \{\langle A\rangle\}$ consisting of the set of isomorphism classes of CM abelian product surfaces A/K is infinite. If $\operatorname{char}(K) = 0$, then this follows immediately from Theorem 71 of [K2] (and its proof) because there are clearly infinitely many negative discriminants $\Delta \equiv 0, 1 \pmod 4$, and hence there are infinitely many equivalence classes of positive binary quadratic forms. If $\operatorname{char}(K) = p > 0$, then this follows from Remark 73 of [K2] because there exist infinitely many negative numbers Δ with $\Delta \equiv 1 \pmod{4p}$, so there exist infinitely many positive binary quadratic forms whose discriminant Δ satisfies $(\frac{\Delta}{p}) = 1$.

Next, let $\mathcal{A}^*_{CM}(K)$ denote the subset of $\mathcal{A}_{CM}(K)$ consisting the isomorphism classes of those CM abelian product surfaces A/K which contain a smooth genus 2 curve C/K. By Theorem 2 of [K4] we know that $|\mathcal{A}_{CM}(K) \setminus \mathcal{A}^*_{CM}(K)| \le 15$, so $\mathcal{A}^*_{CM}(K)$ is also an infinite set.

Now if C/K is a smooth genus 2 curve lying on an abelian surface A/K, then $A \simeq J_C$ (by the universal property of the Jacobian), and so if $C \subset A$, where $\langle A\rangle \in \mathcal{A}^*_{CM}(K)$, then C is a curve of CM product type. Furthermore, since $C \simeq C'$ implies that $J_C \simeq J_{C'}$, it therefore follows that the infinitely many non-isomorphic abelian surfaces in $\mathcal{A}^*_{CM}(K)$ give rise to infinitely many non-isomorphic curves C/K of CM product type, as claimed. This proves the second assertion.

4 Examples

In this section, we want to illustrate how Theorem 1 can be applied to some explicit cases. In this context, the following result discusses some cases satisfying $N_C = 1$.

Proposition 14 *Let A/K be a CM abelian product surface, and let $\theta \in \mathcal{P}(A)$ be a principal polarization on A. Suppose that $\kappa_{q_{(A,\theta)}} = 1$ and that $h(\Delta) = 1$, where $\Delta = \Delta_{q_{(A,\theta)}}$. Then $A \simeq E \times E$, where E/K is up to isomorphism the unique CM elliptic curve such that* $\mathrm{End}(E)$ *has discriminant Δ.*

Conversely, if E/K is a CM elliptic curve such that $h(\Delta) = 1$, where Δ is the discriminant of $\mathrm{End}(E)$*, then $A = E \times E$ is a CM abelian product surface, and $\Delta_{q_{(A,\theta)}} = \Delta$, $\kappa_{q_{(A,\theta)}} = 1$, for every principal polarization $\theta \in \mathcal{P}(A)$ on A. In addition, we have that $N_{(A,\theta)} = 1$ and $|\operatorname{Aut}(q_{(A,\theta)})| = 2a(q_{(A,\theta)})$.*

In particular, if C is a curve of CM product type with $\kappa_C = 1$ and $h(\Delta_C) = 1$, then $N_C = 1$, $|\operatorname{Aut}(C)| = |\operatorname{Aut}(q_C)|$ and $J_C \simeq E \times E$, where E/K is up to isomorphism the unique elliptic curve such that $\mathrm{End}(E)$ *has discriminant Δ_C.*

Proof. Since A/K is a CM abelian product surface, we have that $A \simeq E_1 \times E_2$, where E_i/K are two isogeneous CM elliptic curves E_i/K. By Proposition 5 we have that $d(q_{E_1,E_2}) = d(q_{(A,\theta)})/16 = \Delta_{q_{(A,\theta)}} = \Delta$ and $\mathrm{cont}(q_{E_1,E_2}) = \kappa_{q_{(A,\theta)}} = 1$. Thus, q_{E_1,E_2} is a primitive positive binary quadratic form of discriminant Δ. Since $h(\Delta) = 1$ by hypothesis, q_{E_1,E_2} is equivalent to the principal form 1_Δ of discriminant Δ, and hence q_{E_1,E_2} represents 1. Thus, there exists an $h \in \mathrm{Hom}(E_1, E_2)$ with $\deg(h) = 1$, so $E := E_1 \simeq E_2$, and hence $q_{E_1,E_2} \sim q_{E,E}$. This means that $\mathrm{End}(E)$ has discriminant Δ, and that $A \simeq E \times E$.

Since $h(\Delta) = 1$, then, as is well-known, the isomorphism class of the elliptic curve E/K is uniquely determined by the fact that the endomorphism ring of E/K has discriminant Δ; cf. [K2], equation (55).

Conversely, suppose that $A = E \times E$ where E/K is a CM elliptic curve, and let Δ be the discriminant of $\mathrm{End}(E)$. Then $\Delta = d(q_{E,E})$ (by definition), and clearly $\mathrm{cont}(q_{E,E}) = 1$ because $1_E \in \mathrm{End}(E)$. Thus, by Proposition 5 (and the definitions) we have that $\Delta_{q_{(A,\theta)}} = \Delta$ and $\kappa_{q_{(A,\theta)}} = 1$, for every $\theta \in \mathcal{P}(A)$.

It remains to show that $N_{(A,\theta)} = 1$ and that $2a(q) = |\operatorname{Aut}(q)|$, where $q = q_{(A,\theta)}$. Now since $\kappa_q = 1$, we see that q is not one of the exceptional forms of Theorem 3, and so it follows from (3) that $N_{(A,\theta)} = \frac{2a(q)}{|\operatorname{Aut}(q)|}$. Since $a(q) = a(\theta)$ by Corollary 17 of [K8] (where $a(\theta)$ is defined as in [K8]) and since $a(\theta) \mid |\operatorname{Aut}^+(q)| = \frac{1}{2}|\operatorname{Aut}(q)|$ by Proposition 16 of [K7], we see that $N_{(A,\theta)} \le 1$. But since $N_{(A,\theta)} \ge 1$ (because $\langle A, \theta\rangle \in \mathcal{N}_{(A,\theta)}$), it follows that $N_{(A,\theta)} = 1$ and $2a(q) = |\operatorname{Aut}(q)|$, as claimed.

The last assertion follows directly from the first part by using equation (25).

Remark 15 (a) If $\mathrm{char}(K) = 0$ and if E/K is a CM elliptic curve such that $h(\Delta_E) = 1$, where Δ_E is the discriminant $\mathrm{End}(E)$, then E/K comes from an elliptic curve defined over $\mathbb{Q}$. An equation of such a curve, together with its j-invariant, is given in the tables on p. 483 of Silverman[Si].

Similarly, if $\mathrm{char}(K) = p > 0$, then the CM elliptic curves with $h(\Delta_E) = 1$ are obtained by reduction mod p from the corresponding CM curve over $\mathbb{Q}$, provided that $(\frac{\Delta_E}{p}) = 1$. On the other hand, if $(\frac{\Delta_E}{p}) \neq 1$, then there is no such CM elliptic curve.

(b) In the situation of Proposition 14 there may not be a curve C/K satisfying the given conditions. Indeed, if $\Delta = -3, -4$ or -7, then there is no such curve; cf. Hayashida/Nishi[HN], Theorem 1, or [K4], Theorem 2, or the table in [K7].

We now want to show how Proposition 14 can be used to deduce the uniqueness of a curve C/K when we start with a fixed ternary quadratic form.

Example 16 *Consider the following ternary quadratic form*

$$q(x, y, z) = 4x^2 + 4y^2 + 4z^2 + 4yz + 4xz + 4xy.$$

If $\mathrm{char}(K) = 0$*, then up to isomorphism there is a unique genus* 2 *curve* C/K *such that* $q_C \sim q$*. Moreover,* $|\,\mathrm{Aut}(C)| = 48$ *and the Jacobian* J_C *of* C *is isomorphic to* $E \times E$*, where* E *is given by the equation* $y^2 = x^3 + 4x^2 + 2x$ *and has* j*-invariant* $j(E) = 2^6 \cdot 3^5$*.*

Proof. We first note that $q/2$ is an *improperly primitive* form (in the sense of Dickson [Di]) because $q/4$ is a primitive form and the coefficient of the term yz (or of xy, or of xz) of $q/4$ is odd. Thus, q satisfies condition (1) in Theorem 2 of [Ki], and also condition (2) holds because $q(1, 0, 0) = 4$ is a square. Since $\mathrm{char}(K) = 0$, it thus follows from that theorem that there exists a principally polarized abelian surface (A, θ) over K such that $q_{(A,\theta)} \sim q$.

It is clear that q cannot represent 1, so it follows from (24) that $(A, \theta) \simeq (J_C, \theta_C)$, for some genus 2 curve C/K. Note that since A/K is a CM abelian product surface by Theorem 4, it follows that C is a curve of CM product type, so $J_C \simeq E_1 \times E_2$, for some isogeneous CM elliptic curves E_i/K.

In order to apply Proposition 14, we need to show that C satisfies the required conditions. For this, we first compute the discriminant $d(q)$. By definition (cf. [Wa], p. 2) we have that $d(q) = -\frac{1}{2}\det(A(q))$, where $A(q)$ is the coefficient matrix of q. In our case we have that

$$A(q) = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix},$$

so we see that $d(q) = -8 \cdot 16$. Thus, by Proposition 5 we obtain that $d(q_{E_1,E_2}) = -8$. Since -8 is a fundamental discriminant, it follows that q_{E_1,E_2} is primitive, so $\kappa_q = \mathrm{cont}(q_{E_1,E_2}) = 1$. Thus, C satisfies the hypotheses of Proposition 14 because $h(-8) = 1$ (cf. Theorem 7.30 of [Co])), and so $J_C \simeq E \times E$, where E is a CM elliptic curve such that $\mathrm{End}(E)$ has discriminant -8. Since this determines E/K uniquely up to isomorphism (cf. Proposition 14), we see from the first table on p. 483 of [Si] that $j(E) = 2^6 \cdot 5^3$ and from the second table there that E/K is given by the equation $y^2 = x^3 + 4x^2 + 2x$.

To determine $\mathrm{Aut}(C)$, we can use either of two methods. The first consists of computing $r_4^*(q)$. For this, we observe that clearly

$$\{\pm(1,0,0), \pm(0,1,0), \pm(0,0,1), \pm(1,-1,0), \pm(1,0,-1), \pm(0,1,-1)\} \subset R_4(q),$$

so $r_4^*(q) \geq 12$. We thus see from (2) that $a(q) = 3r_4^* - 12 \geq 24$, so by (25) we obtain that $|\mathrm{Aut}(C)| = 2a(q) \geq 48$. But since $|\mathrm{Aut}(C)| \leq 48$, for all such curves C/K (cf. Theorem 25 of [K7]), it follows that $|\mathrm{Aut}(C)| = 48$ (and that $r_4^*(q) = 12$).

Alternately, we can determine $|\mathrm{Aut}(C)|$ by calculating $|\mathrm{Aut}(q)| = |\mathrm{Aut}(q/2)|$, and for this we can use the method and tables of Dickson[Di], §82-83. Indeed, we see that $q/2$ is the first entry of Table II on p. 185 of [Di], so we have by that table that $|\mathrm{Aut}^+(q/2)| = 24$. (The method for computing this table is given in Theorem 105 of [Di].) We thus have that $|\mathrm{Aut}(q)| = 2|\mathrm{Aut}^+(q/2)| = 48$, and hence also $|\mathrm{Aut}(C)| = 48$ by Proposition 14.

Remark 17 On can say more about the curve C/K which was constructed in Example 16. Indeed, since here $\mathrm{char}(K) = 0$, it is known that up to isomorphism there is only one curve C/K with $|\mathrm{Aut}(C)| = 48$, and so it follows that C is given by the equation $y^2 = x(x^4 - 1)$; cf. [Ig], §8. Furthermore, we have that $\mathrm{Aut}(C) \simeq \mathrm{GL}_2(3)$ by Theorem 2 of [SV]. In [AP], this curve is called the *Burnside curve.*

The next Example 18 uses again Proposition 14 to deduce the uniqueness of a curve C/K for a given ternary quadratic form. This example also illustrates that there may be two non-isomorphic curves of genus 2 on the same abelian surface.

Example 18 *Consider the following two ternary quadratic forms*

$$q_1(x,y,z) \;=\; 4x^2 + 5y^2 + 8z^2 - 4yz - 4xy \text{ and } q_2(x,y,z) \;=\; 4x^2 + 4y^2 + 8z^2 - 4xz.$$

If $\mathrm{char}(K) = 0$*, then up to isomorphism there is a unique genus* 2 *curve* C_i/K *such that* $q_{C_i} \sim q_i$*, for* $i = 1, 2$*. Moreover,* $|\mathrm{Aut}(C_1)| = 4$ *and* $|\mathrm{Aut}(C_2)| = 8$*, and the Jacobian* J_{C_i} *of* C_i *is isomorphic to* $E \times E$*, where* E *is given by the equation* $y^2 = x^3 - 595x + 5586$ *and has* j*-invariant* $j(E) = 3^3 \cdot 5^3 \cdot 17^3$*. Furthermore,* $C_1 \not\simeq C_2$*.*

Proof. For the first form q_1, note first that $q_1(x,y,z) = (2x-y)^2 + 4(y^2 - yz + 2z^2)$ is the sum of two positive forms, and so q_1 is a positive form. In fact, we easily see that $q_1 \geq 4$, if $(x,y,z) \neq (0,0,0)$, so q_1 cannot represent 1.

As in Example 16, we can calculate the discriminants $d(q_i)$, for $i = 1, 2$ by computing the determinant of $A(q_i)$. This gives $d(q_i) = -16 \cdot 28$, for $i = 1, 2$. Since $q_1(1,1,1) = 3^2$, the form q_1 represents a square which is relatively prime to its discriminant. Furthermore, since $q_1 \equiv y^2 \pmod 4$, we see that $q_1 \equiv 0, 1 \pmod 4$. If $\mathrm{char}(K) = 0$, then it follows from Theorem 1 of [K6] that $q_1 \sim q_{(A,\theta)}$, for some principally polarized abelian surface (A,θ) over K. Moreover, as was mentioned above,

q_1 cannot represent 1, so it follows from (24) that $(A, \theta) \simeq (J_{C_1}, \theta_{C_1})$, for some genus 2 curve C_1/K. As in Example 16, we conclude that C_1 is of CM product type.

For the second form q_2, note that $q_2/2$ is an improperly primitive form because $q_2/4$ is a primitive form and the coefficient of the term xz of $q_2/4$ is odd. Moreover,q_2 is a positive form because $q_2(x,y,z) = 4(x^2 - xz + 2z^2) + 4y^2$ is the sum of two positive forms. Thus, q_2 satisfies condition (1) in Theorem 2 of [Ki], and also condition (2) holds because $q_2(1,0,0) = 2^2$. If $\mathrm{char}(K) = 0$, then it follows from that theorem that there exists a principally polarized abelian surface (A', θ') over K such that $q_2 \sim q_{(A',\theta')}$. Furthermore, it is clear that q_2 cannot represent 1, so it again follows from (24) that $(A', \theta') \simeq (J_{C_2}, \theta_{C_2})$, for some genus 2 curve C_2/K. Similarly, we conclude that C_2 is of CM product type.

In order to apply Proposition 14, we need to show that the C_i satisfy the required conditions.

To begin, we have that $\Delta_{C_i} = d(q_i)/16 = -28$, and so $h(\Delta_{C_i}) = 1$ by Theorem 7.30 of [Co]. Next, to compute κ_{C_i}, recall from [B1] that the genus invariant $|I_1(q)|$ of a primitive form q is the content of the *adjoint form* $\mathrm{adj}(q)$, which is defined by the formula

$$A(\mathrm{adj}(q)) = -2\mathrm{adj}(A(q)) = -2\det(A(q))A(q)^{-1}.$$

Hence, we see that $|I_1(q_1)| = \gcd(16\cdot 9, 16\cdot 8, 16\cdot 4, 16\cdot 4, 16\cdot 2, 16\cdot 8) = 16$ and that $|I_1(q_2/4)| = \gcd(8,7,4,0,4,0) = 1$, and thus $\kappa_{C_i} = 1$, for $i = 1,2$ (by definition). This shows that the C_i satisfy the hypotheses of Proposition 14, and so $J_{C_i} \simeq E_i \times E_i$, where E_i is a CM elliptic curve such that $\mathrm{End}(E_i)$ has discriminant -28, for $i = 1,2$. Since this determines E_i/K uniquely up to isomorphism (cf. Proposition 14), we obtain that $E_1 \simeq E_2 := E$, and hence $J_{C_i} \simeq E \times E$, for $i = 1,2$. Furthermore, we see from the first table on p. 483 of [Si] that $j(E) = 3^3 \cdot 5^3 \cdot 17^3$ and from the second table there that E/K is given by the equation $y^2 = x^3 - 595x + 5586$, as asserted.

To determine $|\mathrm{Aut}(C_i)|$, we can use two different methods as was mentioned in Example 16. We use the first method to find $|\mathrm{Aut}(C_1)|$, and the second one to calculate $|\mathrm{Aut}(C_2)|$.

The first method consists of computing $a(q_1)$, and this can be achieved by calculating $r_4^*(q_1)$. But, this is already done in Corollary 30 of [K7], for primitive ternary forms q_C. Indeed, since q_1 satisfies the inequalities of Theorem 103 of [Di], q_1 is an *Eisenstein-reduced* ternary form. Since q_1 is primitive, we are in the situation of Corollary 30 of [K7], and it follows from there that $a(q_1) = 2$. Thus, $|\mathrm{Aut}(C_1)| = 2a(q_1) = 4$ by equation (25).

To apply the second method to the form q_2, we determine $|\mathrm{Aut}(q_2)| = |\mathrm{Aut}(q_2/2)|$. Note that $q_2/2$ is the sixth entry of Table II on p. 185 of [Di], and so we have by that table that $|\mathrm{Aut}^+(q_2/2)| = 4$, and thus $|\mathrm{Aut}(q_2)| = 2|\mathrm{Aut}^+(q_2/2)| = 8$. Therefore, it follows that $|\mathrm{Aut}(C_2)| = |\mathrm{Aut}(q_2)| = 8$ by Proposition 14.

Finally, we observe that $C_1 \not\simeq C_2$ because $q_{C_1} \not\sim q_{C_2}$. Indeed, $q_{C_1} \sim q_1$ is primitive, whereas $q_{C_2} \sim q_2$ is not, so $q_{C_1} \not\sim q_{C_2}$.

The following example shows that it is possible to have $N_C = 1$ in a situation where Proposition 14 does not apply.

Example 19 *Let q be the primitive ternary quadratic form defined by*

$$q(x, y, z) \ = \ 4x^2 + 4y^2 + 9z^2 - 4xy.$$

If $\mathrm{char}(K) = 0$ *or if* $\mathrm{char}(K) \equiv 1 \pmod 3$*, then there exists (up to isomorphism) a unique curve C/K such that $q_C \sim q$. Furthermore, $|\,\mathrm{Aut}(C)| = 12$ and $J_C \simeq E_1 \times E_2$, where E_1 is the elliptic curve given by $y^2 + y = x^3$ and E_2 is the elliptic curve given by $y^2 + y = x^3 - 30x + 63$. Moreover, $j(E_1) = 0$ and $j(E_2) = -2^{15} \cdot 3 \cdot 5^3$.*

Proof. Note first that $q(x, y, z) = 4(x^2 - xy + y^2) + 9z^2$ is the sum of two positive forms, so q is also positive. In fact, we see that $q(x, y, z) \geq 4$, if $(x, y, z) \neq (0, 0, 0)$, so q cannot represent 1.

As in Example 16, we can calculate its discriminant by computing the determinant of $A(q)$. This gives $d(q) = -16{\cdot}27$. Since $q(2, 2, 1) = 5^2$, the form q represents a square which is relatively prime to its discriminant. Furthermore, since $q \equiv z^2 \pmod 4$, we see that $q \equiv 0, 1 \pmod 4$. If $\mathrm{char}(K) = 0$, then it follows from Theorem 1 of [K6] that $q \sim q_{(A,\theta)}$, for some principally polarized abelian surface (A, θ) over K. Similarly, if $p := \mathrm{char}(K) > 0$, then $(\frac{d(q)/16}{p}) = (\frac{-3}{p}) = 1$ by our hypothesis, and so it follows from Theorem 28 of [K6] that $q \sim q_{(A,\theta)}$, for some principally polarized abelian surface (A, θ) over K.

As was mentioned above, q cannot represent 1, so it follows from (24) that $(A, \theta) \simeq (J_C, \theta_C)$, for some genus 2 curve C/K. As in Example 16, we conclude that C is of CM product type.

To show that C/K is uniquely determined by q, it suffices to show that $N_C = 1$, and for this we will use the formula (1). Thus, we need to compute the quantities on the right hand side of (1).

To begin, we have that $\Delta_C = d(q)/16 = -27$, and so $h(\Delta_C) = 1$ (by Theorem 7.30 of [Co]). Next, we see that $|I_1(q)| = \gcd(16 \cdot 9, 16 \cdot 9, 48, 0, 0, 16 \cdot 9) = 48$ as was discussed in Example 18, and thus $\kappa_C = 3$.

It remains to compute $|\,\mathrm{Aut}(C)|$ and $|\,\mathrm{Aut}(q)|$. For this, observe that q is an Eisenstein-reduced primitive ternary form, and we are thus in the situation of Corollary 30 of [K7], and so it follows from there that $a(q) = 6$. Thus, $|\,\mathrm{Aut}(C)| = 2a(q) = 12$ by equation (25).

Next, to compute $|\,\mathrm{Aut}(q)|$, we will use Theorem 105 of [Di]. Since q is Eisenstein-reduced, we see that the table on p. 180 of [Di] is applicable. We observe that the cases of lines 2, 5 and 9 hold and that the others do not. These cases give us five automorphs, and hence we have six automorphs including the identity. Since the coefficients of the terms yz and xz are 0, we conclude that $|\,\mathrm{Aut}^+(q)| = 6 \cdot 2 = 12$ by the footnote on p. 180 of [Di], and so $|\,\mathrm{Aut}(q)| = 24$.

It therefore follows from (1) that $N_C = 2^{\omega(3)} h(-27) \frac{|\mathrm{Aut}(C)|}{|\mathrm{Aut}(q_C)|} = \frac{2\cdot 12}{24} = 1$, so C/K is uniquely determined by q (up to isomorphism).

It remains to determine the structure of J_C. Since C is of CM product type, we have that $J_C \simeq E_1 \times E_2$, for some isogeneous CM elliptic curves E_1 and E_2. By (6) and (7) and the above computations, we see that q_{E_1,E_2} is a binary quadratic form of content 3 and discriminant -27. Moreover, by (17) we have that $\mathrm{End}^0(E_1) \simeq \mathrm{End}^0(E_2) = \mathbb{Q}(\sqrt{-27}) =: F$, and so $\Delta_F = -3$.

As in the proof of Proposition 9, let $f_i = f_{E_i}$ denote the conductor of the order $\mathrm{End}(E_i)$ of F. Then by the first formula of (79) in [K2] we see that $\mathrm{lcm}(f_1, f_2)^2 = \frac{d(q_{E_1,E_2})}{\Delta_F} = \frac{-27}{-3} = 9$, and so it follows from the second formula of (79) in [K2] that $\gcd(f_1, f_2) = \frac{\mathrm{lcm}(f_1,f_2)}{\mathrm{cont}(q_{E_1,E_2})} = \frac{3}{3} = 1$. Thus, by interchanging E_1 and E_2, if necessary, we may assume without loss of generality that $f_1 = 1$ and $f_2 = 3$. This means that $\Delta_{E_1} = -3$ and $\Delta_{E_2} = -27$. Since $h(-3) = h(-27) = 1$, the curves E_i/K are uniquely determined (up to isomorphism) by their endomorphism discriminants, and so we see from the first table of [Si] on p. 483 that $j(E_1) = 0$ and that $j(E_2) = -2^{15} \cdot 3 \cdot 5^3$, and from the second table that E_1 and E_2 are given by the asserted equations.

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