Discriminants of Hermitian R[G]-modules and Brauer's Class Number Relation

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Abstract. The purpose of this paper is to lay the foundations for a quantitative theory of relations among discriminants of hermitian RG-modules which are induced by character relations. This is accomplished by introducing an invariant $\delta(M)$ attached to an RG-module M which plays the role of a correction term in such relations and to study its functorial properties such as localization and induction theorems, behaviour with respect to exact sequences, triviality etc. By means of this formalism it is shown that this invariant may be computed in many cases.

An application of this invariant is the class number relation of R. Brauer (1951) and, by using the formalism mentioned above, also that of Dirichlet (1842).

1. Introduction

A hundred and fifty years ago Dirichlet[Di2] proved the following theorem which he considered to be "one of the most beautiful theorems of the theory of complex numbers" ([Di1], p. 508).

Theorem 1.1 (Dirichlet, 1842) *The class number $h(K)$ of Dirichlet's biquadratic field $K = \mathbb{Q}(\sqrt{d}, \sqrt{-d})$, where $d > 1$ is square-free, is given by the formula*

$$h(K) = \tfrac{1}{2} Q h(d) h(-d). \tag{1.1}$$

Here $h(\pm d)$ denotes the class number of the quadratic field $\mathbb{Q}(\sqrt{\pm d})$ and

$$Q = \begin{cases} 1 & \text{if } N(\varepsilon_K) = \pm 1 \\ 2 & \text{if } N(\varepsilon_K) = \pm i, \end{cases} \tag{1.2}$$

where ε_K denotes a fundamental unit of K, except in the case that $K = \mathbb{Q}(\zeta_8)$ where $\varepsilon_K = \zeta_8$ denotes an eighth root of unity, and $N(\varepsilon_K) = N_{K/\mathbb{Q}(i)}(\varepsilon_K)$ denotes its "partial norm".

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Dirichlet's own proof of this theorem was by analytic techniques, but in 1896 Hilbert[Hi] gave an algebraic proof of this result. In 1922 Herglotz[He] extended this theorem to arbitrary biquadratic fields, and this was further generalized by Kuroda[Ku] in 1950 (see also Kubota[Kb] and Walter[Wa2]).

For a general Galois number field, Nehrkorn[Neh] obtained in 1933 some interesting partial results. The first significant progress, however, was achieved in 1951 by R. Brauer[Br] who proved the following theorem.

Theorem 1.2 (Brauer, 1951) *Let K be a Galois extension of $\mathbb{Q}$ with Galois group G. Suppose G is non-cyclic, so that we have a non-trivial character relation*

$$\sum_{H\le G} n_H 1_H^* = 0, \tag{1.3}$$

where the sum extends over all subgroups of G and 1_H^* denotes the permutation character attached to G/H. Then the class numbers $h(K^H)$ of the fixed fields K^H, $H \le G$, satisfy the relation

$$\prod_H h(K^H)^{n_H} = \prod_H (w(K^H)[G:H])^{n_H}\cdot J = \prod_H (w_2(K^H)[G:H])^{n_H}\cdot J, \tag{1.4}$$

where $w(L) = |U(L)_{tor}|$ denotes the number of roots of unity in the field L and $w_2(L) = |U(L)_{tor}^{(2)}|$ the number of those of 2-power order, and J is an "invariant" which depends only on the $\mathbb{Z}[G]$-module structure of the group of units $U(K)$ of K (and on $\{n_H\}$).

By using the Jordan–Zassenhaus Theorem that there are up to isomorphism only a finite number of $\mathbb{Z}G$-lattices of bounded rank, Brauer derived the following corollary which may be viewed as a qualitative generalization of Dirichlet's theorem:

Corollary 1.3 *Fix a group G and a character relation (1.3). Then, as K runs over all Galois extensions of $\mathbb{Q}$ with group G, the product*

$$\prod_H h(K^H)^{n_H}$$

assumes only finitely many values.

Brauer's proof consists of two main steps. In the first he uses the Artin formalism to derive from the character relation (1.3) the relation

$$\prod_H \zeta_{K^H}(s)^{n_H} = 1 \tag{1.5}$$

among the Dedekind zeta-functions, and from this he obtains the relation

$$\prod_H h(K^H)^{n_H} = \prod_H (w(K^H) reg(K^H)^{-1})^{n_H} \tag{1.6}$$

by comparing residues at $s = 1$. Here, $reg(L)$ denotes the regulator of L.

His second step consists of a detailed study of the right hand side of (1.6), and culminates in the proof of the *regulator relation*

$$J = \prod_H ([G : H] reg(K^H))^{-n_H}, \tag{1.7}$$

where J is a certain invariant of the $\mathbb{Z}G$-module $U(K)$ which is defined as the product of suitable indices of unit groups. Brauer's exposition is somewhat difficult to follow in this step, but a much more transparent treatment with considerable simplifications was given by C. Walter[Wa1] in 1979. In particular, Walter clarified Brauer's use of the Minkowski units by showing that they play the role of a "comparison sublattice".

In this paper we shall follow in the second step a different approach, one that was in part inspired by *Arakelov theory* for number fields (cf. Neukirch[Neu] or Szpiro[Sz]). The underlying idea here is that the regulator of a number field K may be viewed as the discriminant of the group of units $U(K)$ with respect to a suitable bilinear form (or metric), a fact that is implicit in the Arakelovian point of view of number theory. In the case that $K/\mathbb{Q}$ is Galois with group G, then $M = U(K)$ is a $\mathbb{Z}[G]$-module with a G-equivariant metric ρ; following the usage of Scharlau[Sch], such a pair (M, ρ) will be called a *hermitian* $\mathbb{Z}[G]$-module.

Viewed in this light, it is natural to expect that Brauer's theorem may be deduced from a very general theorem on relations among discriminants of hermitian $\mathbb{Z}G$-modules, and this turns out to be indeed the case.

The theorem in question is based on the observation that if $\{n_H\}$ defines a character relation (1.3), then the expression

$$\delta^*(M) = \prod_H disc(M^H, h_{|M^H})^{n_H} \tag{1.8}$$

does not depend on the choice of the hermitian structure h on the $\mathbb{Z}G$-module M and hence defines an invariant $\delta^*(M) = \delta^*(\{n_H\}, M)$ of the $\mathbb{Z}G$-module M. On the other hand, we can also view equation (1.8) as a *discriminant relation*, one that includes Brauer's regulator relation (1.7) as a special case, once we have interpreted regulators as discriminants (cf. Theorem 2.7).

Here it should be emphasized that the invariant $\delta^*(M)$, although closely related to Brauer's invariant J, is nevertheless substantially different from that of Brauer since its definition does not require the existence of a "comparison sublattice". As a result, the entire theory becomes not only more flexible and more functorial but also much more general in that the general discriminant theorem (Theorem 2.5) also applies to hermitian $\mathbb{Z}[G]$-modules which have no obvious comparison submodules. A typical example here is the *Mordell–Weil lattice* of an elliptic curve for which one can therefore obtain an analogue of Brauer's theorem, as will be shown in a subsequent paper.

While the above discriminant relation (1.8) suffices to prove Brauer's class number relation and its corollary, it sheds little light on the value of the invariant $\delta^*(M)$. The *main purpose* of this paper, therefore, is to introduce a formalism with which the invariant $\delta(M)$ may be computed in many cases and thereby to lay the foundations for a *quantitative theory* of discriminant relations induced by character relations. This is accomplished by studying some functorial properties of the δ-invariant such as localization, reciprocity theorems, behaviour with respect to exact sequences etc. In particular, it will be shown how to deduce Dirichlet's theorem directly by examining the $\mathbb{Z}[G]$-module structure of $U(K)$ (cf. Proposition 2.18d)).

In developing this formalism, it is advantageous to modify the above invariant somewhat. First of all, since the unit groups invariably involve torsion — indeed, the case distinction in Dirichlet's Theorem is precisely due to the fact that the $\mathbb{Z}G$-module structure on $U(K)$ cannot be read off from that of its torsion-free part $\overline{U(K)} = U(K)/U(K)_{tor}$ — it is useful to redefine the definition of a discriminant in such a way as to be sensitive to its torsion subgroup. Moreover, by introducing these modified discriminants, many proofs actually become simpler since these new discriminants satisfy better functorial properties.

The second modification is to introduce a normalization factor when restricting a hermitian pairing h on M to its invariant submodule M^H (cf. Notation 2.4). Although the resulting modified invariant $\delta(M)$ is just a minor variation of the original one (cf. Remark 2.6 for the precise connection), this modification is absolutely essential for the validity of many of the functorial properties such as Frobenius reciprocity (Theorem 2.12) and the Triviality Theorem 2.9. Moreover, it also turns out to be the natural choice for the hermitian pairings on unit groups (cf. Proposition 8.2).

The basic method for proving the discriminant relation (1.8) and the other results is to translate the character relation (1.3) into an isomorphism

$$M^+ \otimes \mathbb{Q} \simeq M^- \otimes \mathbb{Q} \tag{1.9}$$

of $\mathbb{Q}G$-modules, where M^+ and M^- are certain permutation modules attached to the character relation (1.3). Viewed in this way, the entire theory can generalized as follows.

Given a non-degenerate $\mathbb{Z}G$-module M and two $\mathbb{Z}G$-modules M_1 and M_2 which are symmetric (or self-dual), i.e. $M_i^* \simeq M_i$, and which satisfy (1.9), the expression

$$\delta(M_1, M_2; M) = d_G(M_1 \otimes M, h_1 \otimes h) d_G(M_2 \otimes M, h_2 \otimes h)^{-1} \tag{1.10}$$

does not depend on hermitian structure h on M nor on the *unimodular* hermitian structures h_i on M_i, and hence defines an invariant of M; here,

$$d_G(M, h) = disc(M^G, \tfrac{1}{|G|} h_{|M^G}).$$

Moreover, virtually all the functorial properties of the invariant $\delta^*(\{n_H\}, M)$ can be derived from the analogous properties of the invariant $\delta(M_1, M_2; M)$, as will be shown in sections 6 and 7.

This paper is organized as follows.

For the convenience of the reader we present in section 2 an overview of the main definitions and results of the paper. In this overview we focus only on the results for the invariant $\delta(\{n_H\}, M)$, but it should be emphasized that virtually all the theorems here are special cases of the theorems of section 6 which apply more generally to the invariant $\delta(M_1, M_2; M)$.

In section 3 we define and study the functorial properties of the (modified) discriminant of a bilinear R-module. For this purpose we first introduce the *content ideal* $\chi(f)$ attached to a linear map $f : M \to N$, which extends the notion of the content ideal $\chi(T)$ of a torsion module (cf. Bourbaki[BCA]) and also that of the "Herbrand quotient" of Fulton[Fu], Appendix A.2, and which reduces to the "q-invariant" of Tate[Ta] in the case that $R = \mathbb{Z}$. This will then be used to define the *relative invariant* $\chi(f, g)$ of two R-linear maps $f, g : M \to N$ which refines and generalizes the relative invariant of Bourbaki attached to submodules. Finally, the discriminant will be defined as a suitable relative invariant.

In section 4 we define hermitian RG-modules and study their functoriality properties via the process of (co)induction and restriction. In particular, we shall formulate several versions of a generalized Frobenius reciprocity theorem for hermitian RG-modules. While the presentation here is much more elaborate than is necessary for our purposes, it should be viewed as a contribution towards an *equivariant Arakelov theory.*

In section 5 we construct the fundamental invariant $\delta(M_1, M_2; M)$. By introducing suitable Grothendieck rings, it is possible to view this invariant as a pairing and thereby work with it more efficiently.

In section 6 we formulate and prove the main properties of the invariant $\delta(M_1, M_2; M)$. As was mentioned above, most of these properties are natural generalizations of the properties of the invariant $\delta(\{n_H\}, M)$ which are presented in section 2: the theorem on discriminant relations, the Uniform Boundedness Theorem, the theorems on induction and inflation, behaviour with respect to exact sequences, and the Triviality Theorem.

In section 7 we interpret the results of section 6 in terms of character relations and thereby prove the theorems stated in section 2.

Finally, in section 8 we apply the above theory to study relations among S-regulators and S-class numbers of number fields by calculating the δ-invariant of the group $U_S(K)$ of S-units. As an application, we prove Dirichlet's Theorem and a generalization of Brauer's Theorem.

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2. Main results

Throughout this paper, the following notations and assumptions are in effect. Let

R	be a principal ideal domain,
$K = Quot(R)$	denote its quotient field,
$L \supset K$	be an extension field,
G	a finite group with $\mathrm{char}(K) \not| \; n = \|G\|$,
$RG = R[G]$	the group ring of G with coefficients in R.

Actually, virtually all of the results below are valid if R is a Dedekind domain or even an arbitrary integrally closed noetherian domain, but for simplicity we restrict to the case of a principal ideal domain since it is the most general ring required for the applications which use mainly the case $R = \mathbb{Z}$ and $L = \mathbb{R}$.

The following definitions are fundamental for the entire paper and will be discussed in more detail in the next sections.

Definition 2.1 An L-valued *hermitian RG-module* is a pair (M, h), where M is a finitely generated RG-module and

$$h : M \to M^* \otimes L = Hom_R(M, L)$$

is an RG-linear map. Here $M^* = Hom_R(M, R)$ denotes the contragredient RG-module. We call (M, h) *non-degenerate* if the induced map

$$h \otimes L : M \otimes L \to M^* \otimes L$$

is an isomorphism.

If $G = \{1\}$ is the trivial group, then a hermitian RG-module (M, h) will be called a *bilinear R-module.*

Remark 2.2 It follows from the definition that an RG-module M which admits a non-degenerate L-valued RG-module structure $h : M \to M^* \otimes L$ satisfies the symmetry condition $M \otimes L \simeq M^* \otimes L$, or equivalently , $M \otimes K \simeq M^* \otimes K$, which says that the KG-module $M \otimes K$ is isomorphic to its contragredient module (as KG-modules). An RG-module M with this property will be called *non-degenerate.*

Definition 2.3 The *discriminant* of an L-valued bilinear R-module (M, h) is the (principal) R-submodule of L defined by

$$disc(M, h) = \chi(M_{tor})^{-1} \det(h(x_i, x_j)) \cdot R,$$

where $x_1, \ldots x_n \in M$ is a basis of $\overline{M} = M/M_{tor}$, with M_{tor} denoting the torsion submodule of M. Furthermore, $h(x_i, x_j) = h(x_i)(x_j)$, and $\chi(\cdots)$ denotes the *content ideal* in the sense of Bourbaki [BCA]. (In particular, if $R = \mathbb{Z}$ then $\chi(M_{tor}) = |M_{tor}| \cdot \mathbb{Z}$.)

To define the invariant $\delta(\{n_H\}, M)$, we introduce the following notation.

Notation 2.4 For a hermitian RG-module (M, h) and a subgroup $H \leq G$, define the bilinear R-module $Inv_H(M, h)$ by

$$Inv_H(M, h) = (Inv_H(M), inv_H(h)),$$

where $Inv_H(M) = M^H = \{m \in M : hm = m\}$ denotes the R-submodule of H-invariant elements and

$$inv_H(h) = \tfrac{1}{|H|} h^H : M^H \to (M^* \otimes L)^H = (M^*)^H \otimes L$$

denotes the induced map on the invariant spaces multiplied by the factor $\frac{1}{|H|}$. The reason for introducing this normalization factor will be explained below; cf. Theorems 2.9 and 2.12, and also section 4.

With these definitions and notations we have:

Theorem 2.5 (Discriminant Relation) *Let $\{n_H\}_{H \leq G}$ define a character relation (1.3). Then we have:*

a) (Existence) *For any non-degenerate RG-module M there is a unique principal fractional R-ideal*

$$\delta(M) = \delta_R(\{n_H\}, M)$$

such that for any non-degenerate L-valued hermitian RG-structure on M we have:

$$\delta(M) = \prod_H disc_R(Inv_H(M, h))^{n_H}. \tag{2.1}$$

b) (Additivity) *The invariant $\delta_R(\{n_H\}, M)$ is bi-additive:*

$$\begin{aligned} \delta_R(\{n_H\}, M_1 \oplus M_2) &= \delta_R(\{n_H\}, M_1) \cdot \delta_R(\{n_H\}, M_2), \\ \delta_R(\{n_H + m_H\}, M) &= \delta_R(\{n_H\}, M) \cdot \delta_R(\{m_H\}, M). \end{aligned}$$

c) (Base change) *If $R' \supset R$ is a principal ideal domain then we have:*

$$\delta_{R'}(M \otimes_R R') = \delta_R(M) \cdot R'.$$

d) (Support) *We have $\delta_R(M) \cdot R[\frac{1}{n}] = R[\frac{1}{n}]$, so $\delta_R(M)$ is supported on the prime ideals dividing $n = |G|$.*

e) (Localization) *The following localization formula is valid:*

$$\delta_R(M) = \bigcap_{\mathfrak{p}|n} \delta_{R_\mathfrak{p}}(M_\mathfrak{p}); \tag{2.2}$$

here, the intersection runs over all prime ideals $\mathfrak{p}$ of R which contain $n = |G|$. In particular, if M and M' are in the same genus (i.e. $M_\mathfrak{p} \simeq M'_\mathfrak{p}$, for all $\mathfrak{p}|n$), then

$$\delta_R(M) = \delta_R(M'). \tag{2.3}$$

Remark 2.6 In the applications one is often interested in the (unnormalized) product

$$\delta^*_R(\{n_X\}, M) = \prod_X disc_R(M^H, h_{|M^H})^{n_H} \tag{2.4}$$

in place of the product (2.1) which is calculated using the normalization introduced in Notation 2.4. However, it is easy to relate this quantity to the invariant δ:

$$\delta^*(\{n_H\}, M) = \varepsilon(\{n_H\}, M)\delta(\{n_H\}, M), \tag{2.5}$$

where

$$\varepsilon(\{n_H\}, M) = \prod_H |H|^{n_H rk(M^H)}. \tag{2.6}$$

In particular, it follows that all the assertions of Theorem 2.5 remain true when we replace δ by δ^*.

We note that it follows from (2.5) that the invariant of the trivial module $M = R$ is given by

$$\delta_R(\{n_H\}, R) = \prod_H |H|^{-n_H} \cdot R \tag{2.7}$$

because it is immediate that $\delta^*(\{n_H\}, R) = R$.

In order to be able to deduce Brauer's results from Theorem 2.5, we need to relate the δ-invariant of the unit group $U(K)$ to the regulators of the subfields of K. As a consequence of the formalism of δ-invariants introduced below, this can be done more generally for the group $U_S(K)$ of S-units of K:

Theorem 2.7 (Class Number Relation) *Let K be a Galois extension of a number field K_0 with Galois group G, and let S be a finite set of places of K which is G-invariant and contains the set S_∞ of infinite places. Then the invariant of the the S-unit group $U_S(K)$ of K, viewed as a $\mathbb{Z}[G]$-module, is given by*

$$\delta_{\mathbb{Z}}(\{n_H\}, U_S(K)) = \prod_H \left(\frac{reg_{H\backslash S}(K^H)^2}{w(K^H)}\right)^{n_H} \delta_{\mathbb{Z}}(\{n_H\}, \mathbb{Z})\delta_{\mathbb{Z}}(\{n_H\}, \mathbb{Z}[S])^{-1},$$

where $reg_{H\backslash S}(K^H)$ denotes the $\bar{S}$-regulator of K^H with respect to the set $\bar{S} = H\backslash S$ of places of K^H, and the δ-invariants $\delta_{\mathbb{Z}}(\{n_H\}, \mathbb{Z})$ and $\delta_{\mathbb{Z}}(\{n_H\}, \mathbb{Z}[S])$ are given by (2.7) *and* (2.19)*, respectively.*

We thus obtain the following relation among the S-class numbers $h_{H\backslash S}(K^H)$ of the intermediate fields K^H of K/K_0:

$$\prod_H h_{H\backslash S}(K^H)^{2n_H} = \delta(\mathbb{Z})\delta(U_S(K))^{-1}\delta(U(K)_{tor})^{-1}\delta(\mathbb{Z}[S])^{-1}. \tag{2.8}$$

Moreover, if the (finite) places of S are tamely ramified over K_0 (e.g. if $S = S_\infty$), then $\delta_{\mathbb{Z}}(\{n_H\}, \mathbb{Z}[S]) = \mathbb{Z}$, and so the above class number relation reduces to

$$\prod_H h_{H\backslash S}(K^H)^{2n_H} = \delta(\mathbb{Z})\delta(U_S(K))^{-1}\delta(U(K)_{tor})^{-1}. \tag{2.9}$$

If G, $\{n_H\}$ and $\#S$ are fixed, then, as was explained in the introduction, it follows immediately from the Jordan-Zassenhaus Theorem that the expression (2.8) assumes only finitely many values as K varies over all Galois extensions of $\mathbb{Q}$ with Galois group G. However, this result also follows from the following much more general and more precise theorem:

Theorem 2.8 (Uniform Boundedness) *For each character relation (1.3) there is an element $r = r(\{n_H\}) \in R\backslash\{0\}$ such that for any non-degenerate RG-module M we have*

$$[M_{tor}]^N r^{rk(M)} \delta_R(\{n_H\}, M) \subset R, \quad [M_{tor}]^N r^{rk(M)} \delta_R(\{n_H\}, M)^{-1} \subset R,$$

where $N = \sum |n_H|$.

Note that this theorem is considerably stronger than Brauer's result. Not only is the bound here uniform in the rank of M, but the result is also valid for arbitrary principal ideal domains R, not only for those for which the Jordan–Zassenhaus theorem holds. Furthermore, as the proof shows, it is possible to give fairly explicit interpretation of the "denominator" r above: its essential constituent is the "genus defect" attached to a character relation, which measures the failure of the character relation to be trivial as a genus class (cf. sections 6 and 7 for further explanations).

In view of the above class number relation (2.8), it would be desirable to be able to compute the invariant $\delta_R(M)$ for all (non-degenerate) *RG*-modules M. While this task seems to be quite difficult in general, the following formalism can be used to compute this invariant in many cases.

Theorem 2.9 (Triviality) *If M is a non-degenerate RG-module which is cohomologically trivial then its invariant is also trivial:*

$$\delta_R(\{n_H\}, M) = R. \tag{2.10}$$

The above theorem may be viewed as a justification for the normalization factor introduced in the definition of δ, as well as for the unusual definition of the discriminant of *RG*-modules M which may have R-torsion. A further justification of the latter is also given by the validity of the following result:

Theorem 2.10 (Exact Sequence Formula) *Let*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be an exact sequence of RG-modules. If any two of the modules M, M' and M'' are non-degenerate, then so is the third, and for any character relation we have

$$\delta(\{n_H\}, M) = \delta(\{n_H\}, M')\delta(\{n_H\}, M'')\psi(\{n_H\}, g)^2, \tag{2.11}$$

where

$$\begin{aligned}\psi(\{n_H\},g) &= \prod_X \chi(Coker(g^X : H^0(X,M) \to H^0(X,M'')))^{n_X} \\ &= \prod_X \chi(Ker(f : H^1(X,M') \to H^1(X,M)))^{n_X}.\end{aligned}$$

Combining these two theorems yields the following remarkable consequence:

Corollary 2.11 *Let*

$$0 \to M' \to P_r \to \cdots \to P_1 \to M'' \to 0$$

be an exact sequence of non-degenerate RG-modules. If the P_i are cohomologically trivial, then for any character relation $\{n_H\}$ the δ-invariants of M' and M'' are connected by the formula

$$\delta(\{n_H\},M') = \delta(\{n_H\},M'')^{(-1)^r} \cdot \prod_H \chi^r(H,M')^{2n_H}, \tag{2.12}$$

where the truncated Euler characteristic $\chi^r(G,M)$ is defined by

$$\chi^r(G,M') = \prod_{q=1}^{r} \chi(H^q(G,M'))^{(-1)^{q+1}} \tag{2.13}$$

The above corollary allows us to compute the δ-invariant of one *RG*-module in terms of another one. The following induction/restriction formalism, on the other hand, relates the δ-invariant of (certain) *RG*-modules to those of modules belonging to subgroups of G.

Theorem 2.12 (Induction) *Let $X \le G$ be a subgroup. If M, respectively N, is a (non-degenerate) RG-module, respectively RX-module, and $\{n_H\}_{H\le G}$, respectively $\{m_Y\}_{Y\le X}$, defines a character relation of G, respectively of X, then we have*

$$\delta_R(ind_X^G(\{m_Y\}),M) = \delta(\{m_Y\},res_X^G(M)), \tag{2.14}$$

$$\delta_R(res_X^G(\{n_H\}),N) = \delta(\{n_H\},ind_X^G(N)), \tag{2.15}$$

where the induced character relation $ind_X^G(\{m_Y\}) = \{m_H^*\}_{H\le G}$ is defined by extension by 0 (i.e. $m_H^* = m_H$, if $H \le X$ and $m_H^* = 0$ otherwise), and the restricted character relation $res_X^G(\{n_H\}) = \{n_Y^*\}_{Y\le X}$ is defined by the formula

$$n_Y^* = \sum_H n_H m(H,Y) \tag{2.16}$$

where $m(H,Y) = \#\{g \in G : gHg^{-1} \cap X = Y\}/(|H|[X:Y])$.

Note that although the above coefficients $m(H,Y)$ of res_H^G are actually rational numbers, the denominators disappear if one collects those terms together which belong to conjugate subgroups; this is possible since the δ-invariant does not change if we replace in a character relation a subgroup by a conjugate.

The above notation suggests that Theorem 2.12 is closely connected to a "Frobenius induction theorem". This is indeed the case, for the result is deduced from the "Frobenius induction theorem for hermitian RG-modules" (Theorem 4.16).

As an illustration of the above theorem we present the following examples. The first two follow from the fact that cyclic groups have no non-trivial character relations and that the process of restriction and induction preserves character relations.

Example 2.13 a) If $M = ind_X^G(N)$ is induced from a cyclic subgroup $X \leq G$, then for every character relation $\{n_H\}$ we have

$$\delta_R(\{n_H\}, M) = R. \tag{2.17}$$

b) If G acts *tamely* on the G-set S in the sense that all stabilizers are cyclic, then for the associated permutation module $R[S]$ we have

$$\delta_R(\{n_H\}, R[S]) = R. \tag{2.18}$$

c) For any G-set S we have

$$\delta_R(\{n_H\}, R[S]) = \prod_{s \in G\backslash S} \prod_{H} \prod_{g \in H\backslash G/G_s} |H \cap gG_sg^{-1}|^{-n_H} \cdot R, \tag{2.19}$$

where G_s denotes the stabilizer of a point $s \in S$.

Another useful induction-type theorem is the following result which relates the δ-invariants of modules which are lifted (or "inflated") from quotient groups.

Theorem 2.14 (Inflation) *Let $X \trianglelefteq G$ be a normal subgroup of G and let $Q = G/X$ denote the quotient group.*

a) Let M be a non-degenerate RG-module and $\{n_Y\}_{Y \leq Q}$ be a character relation of Q. Then, viewing the invariant module M^X as an RQ-module, we have

$$\delta_R(inf_G^Q(\{n_Y\}), M) = \delta_R(\{n_Y\}, M^X), \tag{2.20}$$

where the lifted character relation $inf_G^Q(\{n_Y\}) = \{n_H^*\}$ is given by $n_H^* = 0$ if $H \not\geq X$ and by

$$n_H^* = \sum_{Y \sim H/X} n_Y/[G : N_G(H)],$$

if $H \geq X$; here the sum extends over all $Y \leq Q$ which are conjugate to H/X.

b) For every non-degenerate RQ-module M and character relation $\{n_H\}$ of G we have

$$\delta_R^*(\{n_H\}, Res_G^Q(M)) = \delta_R^*(ind_G^Q(\{n_H\}), M), \tag{2.21}$$

where the induced character relation $ind_G^Q(\{n_H\}) = \{\overline{n}_{Y/X}\}_{Y/X \leq Q}$ is defined by

$$\overline{n}_{Y/X} = \sum_{\substack{H \\ H\cdot X = Y}} n_H.$$

In terms of the δ-invariant, equation (2.21) may be written as

$$\delta_R(ind_G^Q(\{n_H\}), M) = \varepsilon_X(\{n_H\}, M)\delta_R(\{n_H\}, Res_G^Q(M)), \tag{2.22}$$

where ε_X is defined by

$$\begin{aligned} \varepsilon_X(\{n_H\}, M) &= \varepsilon(ind_G^Q(\{n_H\}), M)\varepsilon(\{n_H\}, Res_G^Q(M))^{-1} \\ &= \prod_H |H \cap X|^{n_H rk(M^H)}. \end{aligned} \tag{2.23}$$

Actually, Theorems 2.12 and 2.14 are just two special cases of a theorem which relates the δ-invariants of modules which are restricted and/or (co)induced via an arbitrary group homomorphism $f : G_1 \to G_2$; cf. Theorem 7.8 for a precise statement.

As with restriction to subgroups, the process of inflation and quotient-induction preserves character relations. Thus, if we use once more the fact that cyclic groups have no non-trivial character relations, then we obtain from the above theorem the following examples.

Example 2.15 a) If M is an R-module on which RG acts trivially, then we have

$$\delta_R(\{n_H\}, M) = \delta_R(\{n_H\}, R)^{rk(M)} = \prod_H |H|^{-n_H rk(M)} R; \tag{2.24}$$

this generalizes and gives another proof of (2.7).

b) If $M = M^X$ for some normal subgroup $X \trianglelefteq G$ with cyclic quotient $Q = G/X$, then

$$\delta_R(\{n_H\}, M) = \varepsilon_X(\{n_H\}, M)^{-1} R. \tag{2.25}$$

Corollary 2.16 *If M is an RG-module which is a torsion module then*

$$\varepsilon_X(\{n_H\}, M) = 1,$$

and so equation (2.21) holds with δ^* replaced by δ.

In particular, if $M = M^X$ is a torsion module which is invariant under a normal subgroup $X \trianglelefteq G$ with a cyclic quotient G/X, then

$$\delta_R(\{n_H\}, M) = R. \tag{2.26}$$

Remark 2.17 This corollary applies in particular to the case that $M = U(K)^{(p)}$, the p-primary component of the torsion subgroup of the unit group of a galois extension K/K_0 as in Theorem 2.7, provided that $p > 2$. Thus, in the notation of Theorem 1.2 we obtain Brauer's relation (i.e. the second equality in (1.4)) as a special case:

$$\prod w_p(K^H)^{n_H} = 1, \quad \text{if } p > 2. \tag{2.27}$$

It is perhaps illuminating to compute the δ-invariant in some easy non-trivial cases. Specifically, we shall prove the following result from which Dirichlet's Theorem follows immediately:

Proposition 2.18 *Let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ be the elementary abelian group of order p^2, and put $n_{\{1\}} = -1, n_G = -p$ and $n_H = 1$ for $|H| = p$. Then $\{n_H\}$ is a character relation of G which is the only one up to integral multiples. Moreover:*

a) Suppose M is a $\mathbb{Z}G$-module which is a finite cyclic abelian group. Then $\delta(M) := \delta_{\mathbb{Z}}(\{n_H\}, M) = \mathbb{Z}$ except in the case when G acts faithfully on the p-primary subgroup $M^{(p)} \subset M$. Moreover, in the exceptional case we have $p = 2$, $8 \,|\, m = |M|$ and $\delta(M) = 2\mathbb{Z}$.

b) If M is an RG-module such that G acts trivially on $\overline{M} = M/M_{tor}$, then

$$\delta(M) = p^{(p-1)rk(M)} \prod_H \chi(M/M^H)^{2n_H} \delta(M_{tor})^{-1}. \tag{2.28}$$

In particular, $\delta(R) = p^{(p-1)}R$.

c) Suppose M is a non-degenerate RG-module of rank 1 such that G acts non-trivially on $\overline{M}$. Then $p = 2$ and there is a unique subgroup $X = \langle\sigma\rangle \leq G$ of order 2 such that $\overline{M}^X = \overline{M}$. Moreover,

$$\begin{aligned} \delta(M) &= \tfrac{1}{2}\delta(M_{tor})\chi(M/(M_{tor} + M^X))^2 \\ &= 2\delta(M_{tor})\chi(\hat{H}^0(X, M_{tor}))^2\chi(\hat{H}^0(X, M))^{-2}. \end{aligned} \tag{2.29}$$

d) Suppose M is a non-degenerate $\mathbb{Z}G$-module such that $\overline{M}$ is as in c) and M_{tor} is as in a). In addition, assume that σ acts on M_{tor} by multiplication by -1 and that $4 \,|\, m = |M_{tor}|$. Then there is a unique subgroup $Y \leq G$ such that $M_4^Y = M_4$, where $M_4 = \{x \in M : 4x = 0\}$, and we have

$$\delta(M_{tor})\delta(M) = 2|\hat{H}^0(Y, M)|^{-2}. \tag{2.30}$$

Moreover, $|\hat{H}^0(Y, M)| = 1$ if there is an element $x \in M$ such that $N_Y(x)$ generates M^Y, and $|\hat{H}^0(Y, M)| = 2$ otherwise.

3. Content ideals, relative invariants and discriminants

The purpose of this section is to define and study the *content ideal* $\chi(f)$ attached to an R-module homomorphism $f : M \to N$. This invariant extends the notion of a content ideal $\chi(T)$ attached to a torsion module (cf. Bourbaki[BCA], ch. VII.4.5) and reduces to the "q-invariant" $q(f)$ of Tate[Ta] in the case that $R = \mathbb{Z}$. We then define the *relative invariant* $\chi(f, g)$ of two R-module homomorphisms $f, g : M \to V \otimes_K L$ which is a slight refinement and generalization of the relative invariant $\chi(M_1, M_2)$ of Bourbaki[BCA], ch. VII.4.6, which is attached to two submodules M_1, M_2 of a K-vector space V. This refinement will then be used to define the *discriminant* of an L-valued bilinear R-module (M, h).

3.1. Content ideals. As before, R is a principal ideal domain, and all R-modules are tacitly assumed to be finitely generated. If T is any torsion R-module, then its *content ideal* $\chi(T)$ is defined as the product

$$\chi(T) = \mu_1 \cdots \mu_r R \tag{3.1}$$

of its invariant factors $\mu_1|\mu_2|\cdots|\mu_r$. Alternatively, by the well-known formula relating invariant factors to determinants, we can also define $\chi(T)$ as the determinant

$$\chi(T) = \det(f) \cdot R, \tag{3.2}$$

where

$$0 \to R^n \xrightarrow{f} R^n \to T \to 0, \tag{3.3}$$

is a presentation of T. From the latter definition it is immediate that we have the formula

$$\chi(Coker(f)) = det(f) \cdot R \tag{3.4}$$

for every $f \in End_R(R^n)$, provided we set $\chi(T) = 0$, if T is not a torsion R-module. Furthermore, we see that the invariant $\chi(T)$ agrees with that of Bourbaki[BCA], ch. VII.4.5. In particular, it follows from [BCA] that $\chi(T)$ is additive on exact sequences, i.e. if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of R-modules then we have

$$\chi(M_2) = \chi(M_1) \cdot \chi(M_3). \tag{3.5}$$

We next want to define the *content ideal* $\chi(f)$ of an R-linear map $f : M \to N$. If f is an *R-isogeny* in the sense that $Ker(f)$ and $Coker(f)$ are torsion modules, then its *content ideal* is the (fractional) R-ideal defined by

$$\chi(f) = \chi_R(f; M, N) = \chi(Coker(f)) \cdot \chi(Ker(f))^{-1}. \tag{3.6}$$

As above, it is useful to extend the symbol $\chi(f)$ to all $f \in Hom_R(M,N)$ by setting $\chi(f) = 0$ whenever f is not an R-isogeny. We thus have by (3.2) that

$$\chi(f) = \det(f)R, \tag{3.7}$$

if $f \in End_R(M)$ and M is a free R-module. More generally, if M and N are free R-modules with bases $x_1, \dots, x_m$ and $y_1, \dots, y_n$, respectively, then we have

$$\chi(f) = \det(A)R, \tag{3.8}$$

where A denotes the matrix of f with respect to these bases, and $\det(A) = 0$ if the matrix A is not square.

Remark 3.1 If $R' \supset R$ is a principal ideal domain which contains R and M is an R-module, then the content ideal of the extended module $M \otimes_R R'$ is given by the formula

$$\chi_{R'}(M \otimes_R R') = \chi_R(M) \cdot R' \tag{3.9}$$

because on the one hand M is a torsion R-module if and only if $M \otimes R'$ is a torsion R'-module and on the other hand the sequence (3.3) remains exact after tensoring with R'. Thus, if $f : M \to N$ is an R-linear map, then the content ideal of the extended map $f \otimes R' : M \otimes_R R' \to N \otimes_R R'$ is given by

$$\chi_{R'}(f \otimes R') = \chi_R(f) \cdot R'. \tag{3.10}$$

The following "functorial" properties of the content ideal are easily verified (cf. Tate[Ta], p. 58).

Proposition 3.2 *If $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are two R-module homomorphisms, then*

$$\chi(g \circ f) = \chi(g) \cdot \chi(f). \tag{3.11}$$

Corollary 3.3 *Let $f : M_1 \to M_2$ be an R-module homomorphism and $M_i' \subset M_i$, $(i = 1, 2)$ two R-submodules such that $f(M_1') \subset M_2'$. Then, if $f' = f_{|M_1'} : M_1' \to M_2'$ denotes the restriction of f to M_1', we have*

$$\chi(f)\chi(M_1/M_1') = \chi(f')\chi(M_2/M_2'). \tag{3.12}$$

Proposition 3.4 *Let*

$$\begin{array}{ccccccccc} 0 & \to & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 & \to & 0 \\ & & \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \\ 0 & \to & M_1' & \xrightarrow{f'} & M_2' & \xrightarrow{g'} & M_3' & \to & 0 \end{array} \tag{3.13}$$

be a commutative diagram of R-homomorphisms with exact rows. Then we have

$$\chi(h_2) = \chi(h_1)\chi(h_3), \tag{3.14}$$

except possibly in the case that neither $Ker(h_3)$ nor $Coker(h_1)$ are torsion modules. Furthermore, if any two of the h_i's are isogenies, then so is the third and (3.14) holds.

In the sequel it will often be convenient to treat the torsion and free part of a module separately. To this end we introduce the following notation.

Notation 3.5 If M is an R-module, then $M_{tor} \subset M$ denotes its torsion submodule and

$$\pi_M : M \to \overline{M} = M/M_{tor}$$

the quotient map onto the quotient module $\overline{M} = M/M_{tor}$. Furthermore, if $f : M \to N$ is a homomorphism, then we let

$$f_{tor} : M_{tor} \to N_{tor}$$
$$\overline{f} : \overline{M} \to \overline{N}$$

denote the induced maps.

By applying Proposition 3.4 to the exact sequences induced by π_M and π_N, we easily obtain the following formulae:

$$\chi(f) = \chi(f_{tor})\chi(\overline{f}) \tag{3.15}$$
$$\chi(f_{tor}) = \chi(M_{tor})^{-1}\chi(N_{tor}) \tag{3.16}$$
$$\chi(\overline{f}) = \chi(f)\chi(M_{tor})\chi(N_{tor})^{-1} \tag{3.17}$$

In particular, if $M \simeq N$, then we have

$$\chi(f_{tor}) = R \ \text{ and } \ \chi(\overline{f}) = \chi(f). \tag{3.18}$$

In what follows we will need to know the relation of the content ideal $\chi(f)$ of a homomorphism $f : M \to N$ to that of its dual map $f^* : M^* \to N^*$, where, as before, $M^* = Hom_R(M, R)$.

Proposition 3.6 *The content ideal $\chi(f^*)$ of the dual map is related to the content ideal $\chi(f)$ of $f : M \to N$ by the formula*

$$\chi(f^*) = \chi(f)\chi(f_{tor})^{-1} = \chi(f)\chi(M_{tor})\chi(N_{tor})^{-1}. \tag{3.19}$$

Proof. If M and N are free modules with bases $x_1, \dots x_m$, and $y_1, \dots y_n$, respectively, and A denotes the matrix of f with respect to these bases, then the matrix of f^* with respect to the dual bases x_i^* and y_j^* is the transpose A^t. We thus have by (3.8) that

$$\chi(f) = \det(A)R = \det(A^t)R = \chi(f^*),$$

which is (3.19) in this case.

In the general case we note that $\overline{M}^* = M^*, \overline{N}^* = N^*$ and $\overline{f}^* = f^*$. Applying the case just treated to $\overline{f}$, we therefore have

$$\chi(f^*) = \chi(\overline{f^*}) = \chi(\overline{f}) = \chi(f)\chi(M_{tor})\chi(N_{tor})^{-1},$$

the latter equality resulting from (3.17). In view of (3.16), this proves (3.19).

3.2. Relative invariants. As in the previous section, let $L \supset K$ be an extension field of $K = Quot(R)$. As was indicated there, we shall be interested in R-homomorphisms $h : M \to V$, where V is a (finite dimensional) L-vector space.

Definition 3.7 An R-submodule $M \subset V$ is called *non-degenerate with respect to* L (or L-*non-degenerate*) if M has an R-module basis which is also an L-basis of V. Since every (finitely generated) R-submodule $M \subset V$ is free (because R is a principal ideal domain), it follows that M is non-degenerate if and only if $LM = V$ and $rank_R(M) = dim_L(V)$. In particular, if $L = K$, then a non-degenerate submodule $M \subset V$ is just a *lattice* in the sense of Bourbaki[BCA], VII.4.1.

If M is an R-module and $f : M \to V$ is R-linear, then f is said to be *non-degenerate* (with respect to L) if $f(M)$ is a non-degenerate submodule of V and $Ker(f)$ is a torsion module (and hence $Ker(f) = M_{tor}$).

Definition 3.8 If M_1 and M_2 are two non-degenerate submodules of V, then their *relative invariant* is defined by

$$\chi(M_1, M_2) = \chi_{R,L}(M_1, M_2; V) = \det(\alpha)R, \tag{3.20}$$

where $\alpha \in Aut_L(V)$ is any automorphism of V such that

$$\alpha(M_2) = M_1. \tag{3.21}$$

Furthermore, if $f_i : M_i \to V$, $i = 1, 2$ are two R-linear maps which are non-degenerate, then their *relative invariant* is defined by

$$\chi(f_1, f_2) = \chi(f_1, f_2; V) = \chi((M_1)_{tor})^{-1}\chi((M_2)_{tor})\chi(f_1(M_1), f_2(M_2)). \tag{3.22}$$

Remark 3.9 a) Note that in the above definition the right hand side of (3.20) does not depend on the choice of α, for if β is another choice satisfying (3.21), then $\beta = \gamma_1 \circ \alpha \circ \gamma_2$ with $\gamma_i \in Aut_R(M_i)$, and so $\det_L(\beta)R = \det_R(\gamma_1)\det_L(\alpha)\det_R(\gamma_2)R = \det(\alpha)R$ since $\det(\gamma_i) \in R^\times$. We thus see by [BCA], VII.4.6, equation (6), that in the case that $L = K$ the relative invariant $\chi(M_1, M_2)$ coincides with that of Bourbaki.

b) The above Definition 3.8 may be extended to the case that M_1 is degenerate by observing that since M_2 is non-degenerate, we can always find $\alpha \in End_L(V)$ such that (3.21) holds, at least if $rank_R(M_1) \leq dim(V)$ (see c) below). Thus Definition 3.20 still makes sense (provided we set we set $\chi(M_1, M_2) = 0$ when $rank(M_1) > dim(V)$), and we obtain

$$\chi(M_1, M_2) = 0 \iff M_1 \text{ is degenerate.} \tag{3.23}$$

In a similar manner we can extend the symbol $\chi(f_1, f_2)$ to the case that f_2 is non-degenerate and f_1 is arbitrary by setting

$$\chi(f_1, f_2) = 0 \tag{3.24}$$

whenever f_1 is degenerate.

c) More explicitly, Definition 3.8 may be written as follows. Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ be R-bases of the submodules M_1 and M_2, respectively. Since M_2 is L-non-degenerate by hypothesis, we can find $c_{ij} \in L$ such that

$$x_i = \sum_{j=1}^{n} c_{ij} y_j, \text{ for } 1 \le i \le m, \tag{3.25}$$

and so we have

$$\chi(M_1, M_2) = \det(c_{ij})R, \tag{3.26}$$

where we set $\det(C) = 0$ if the matrix C is not square.

The following properties of the symbol $\chi(f_1, f_2)$ follow immediately from the definitions:

$$\chi(f_2, f_1) = \chi(f_1, f_2)^{-1}; \tag{3.27}$$

$$\chi(f_1, f_3) = \chi(f_1, f_2)\chi(f_2, f_3); \tag{3.28}$$

$$\chi(\alpha \circ f_1, f_2)) = \det(\alpha)\chi(f_1, f_2); \tag{3.29}$$

here, as before, the f_i are non-degenerate R-linear maps to V and $\alpha \in Aut_L(V)$ is an automorphism. There are analogous properties for the symbol $\chi(M_1, M_2)$, but we do not need these here.

The relative invariants are closely connected to the content ideals defined above. To see this, suppose first that M_1 and M_2 are two non-degenerate submodules of V with $M_1 \subset M_2$. Since both have the same rank, the quotient module M_2/M_1 is a torsion R-module and we have by comparing (3.4) with (3.20):

$$\chi(M_1, M_2) = \chi(M_2/M_1). \tag{3.30}$$

More generally, if M_1 and M_2 are two non-degenerate submodules of V which are *commensurable* in the sense that there exists a non-degenerate submodule $M \supset M_1 + M_2$ containing M_1 and M_2, then we have by (3.30) and (3.28):

$$\chi(M_1, M_2) = \chi(M/M_1) \cdot \chi(M/M_2)^{-1}. \tag{3.31}$$

Note that in this case $M_3 = M_1 \cap M_2$ (as well as $M_1 + M_2$) is also non-degenerate, and so we also have

$$\chi(M_1, M_2) = \chi(M_1/M_3)^{-1} \cdot \chi(M_2/M_3). \tag{3.32}$$

From these formulae we see that if $f_i : M_i \to V$, $i = 1, 2$ are two non-degenerate R-linear maps which are *commensurable* in the sense that $M = f_1(M_1) + f_2(M_2)$ is a non-degenerate submodule of V, then we have

$$\chi(f_1, f_2) = \chi(f_1; M_1, M) \cdot \chi(f_2; M_2, M)^{-1}, \tag{3.33}$$

where we regard the f_i as R-linear maps $f_i : M_i \to M \subset V$. In view of this formula, which follows immediately from the definition (3.22) and from (3.31), many of the properties of content ideals can be transferred to the relative invariants. For example, if $g_i : N_i \to M_i$ are R-isogenies, then by (3.11) and (3.33) we obtain the formula

$$\chi(f_1 \circ g_1, f_2 \circ g_2) = \chi(g_1)\chi(g_2)^{-1}\chi(f_1, f_2), \tag{3.34}$$

at least when f_1 and f_2 are commensurable. However, this formula is easily verified to be correct in the general case. Indeed, since f_i and $f_i \circ g_i$ are clearly commensurable, we obtain by (3.28) and the proven case of (3.34) that

$$\begin{aligned} \chi(f_1 \circ g_1, f_2 \circ g_2) &= \chi(f_1 \circ g_1, f_1)\chi(f_1, f_2)\chi(f_2, f_2 \circ g_2) \\ &= \chi(g_1)\chi(f_1, f_1)\chi(f_1, f_2)\chi(g_2)^{-1}\chi(f_2, f_2) \\ &= \chi(g_1)\chi(g_2)^{-1}\chi(f_1, f_2), \end{aligned}$$

which proves (3.34) in general. Note that this formula continues to be true if we drop the condition that f_1 be an isogeny.

3.3. Discriminants of bilinear R-modules. Recall from the previous section that an L-valued, bilinear R-module is a pair (M, h) consisting of an R-module M and an R-linear map

$$h : M \to M^* \otimes L = Hom_R(M, L).$$

We can also describe h by its associated *bilinear map*

$$\beta_h : M \times M \to L$$

which is given by the formula

$$\beta_h(x, y) = h(x)(y), \tag{3.35}$$

for β_h and h determine each other via the formula (3.35). (Usually $h = \hat{\beta}$ is called the *adjoint map* of $\beta = \beta_h$.) If no confusion results, then we will just write

$$h(x, y) = \beta_h(x, y).$$

Definition 3.10 The *discriminant* of an L-valued bilinear R-module (M, h) is defined as the relative invariant

$$disc_{R,L}(M, h) = \chi(h, j_{M,L}; V) \tag{3.36}$$

where

$$j_{M,L} = id_{M^*} \otimes L : M^* \to V = M^* \otimes L$$

denotes the canonical map $j_{M,L}(m) = m \otimes 1$. Note that j is trivially non-degenerate (in the sense of Definition 3.7), so this relative invariant is always defined.

Remark 3.11 a) If $M = \overline{M}$ is a free R-module with basis $x_1, \ldots, x_n$, then its discriminant is given by the determinant of the associated *Gram matrix*, i.e. we have the formula

$$disc(M, h) = \det(h(x_i, x_j)) \cdot R, \tag{3.37}$$

which is the usual definition of the discriminant. To see this, note that if we let $x_1^*, \ldots, x_n^* \in M^*$ denote the dual basis of $x_1, \ldots, x_n$, then $x_1^* \otimes 1, \ldots, x_n^* \otimes 1$ is an R-module basis of $M_2 = j(M^*) \subset V = M^* \otimes L$ and we have

$$h(x_i) = \sum_{j=1}^{n} h(x_i, x_j) x_j^* \otimes 1.$$

Thus, if $h(x_1), \ldots, h(x_n)$ is a basis of $M_1 = h(M)$, then (3.37) follows from (3.26). On the other hand, if $h(x_1), \ldots, h(x_n)$ are linearly dependent, then we have

$$\det(h(x_i, x_j)) = 0 = disc(M, h),$$

where the latter equality holds by definition since M_1 is degenerate, and so (3.37) holds in all cases.

b) For an arbitrary bilinear R-module (M, h) we have

$$disc(M, h) = \chi(M_{tor})^{-1} disc(\overline{M}, \overline{h}), \tag{3.38}$$

where $\overline{h} : \overline{M} \to \overline{M}^* \otimes L = M^* \otimes L$ denotes the induced map (cf. Notation 3.5). This is true because we have $h = \overline{h} \circ \pi_M$ and $j_{M,L} = j_{\overline{M},L}$ and so we obtain from (3.34) that $disc(M, h) = \chi(h, j_{M,L}) = \chi(\pi_M)\chi(\overline{h}, j_{\overline{M},L}) = \chi(M_{tor})^{-1} disc(\overline{M}, \overline{h})$.

c) It is natural to call (M, h) *non-degenerate* if $h : M \to M^* \otimes L$ has this property. We then have by Remark 3.9b):

$$(M, h) \quad \text{is non-degenerate} \quad \iff \quad disc(M, h) \neq 0.$$

For future reference, let us note that discriminants are compatible with base-change:

Proposition 3.12 *Let $L' \supset L$ be an extension field of L and let R' be a principal ideal domain with $R \subset R' \subset L'$. Then for any bilinear R-module (M, h) we have*

$$disc_{R',L'}(M \otimes_R R', h \otimes R') = disc_{R,L}(M, h) \cdot R'. \tag{3.39}$$

Proof. Write $M' = M \otimes R'$. By flatness we have that $M'_{tor} = M_{tor} \otimes R'$, and so we obtain from Remark 3.1 that

$$\chi(M'_{tor}) = \chi(M) \cdot R'.$$

Furthermore, since $\overline{M}' = \overline{M} \otimes R'$, it follows from (3.37) that

$$disc_{R'L'}(\overline{M'}, \overline{h} \otimes R') = disc_{R,L}(\overline{M}, \overline{h}) \cdot R',$$

and so the assertion follows in view of (3.38).

Corollary 3.13 *For any bilinear R-module (M, h) we have*

$$disc_{R,L}(M, h) = \bigcap_{\mathfrak{p}} disc_{R_{\mathfrak{p}},L}(M_{\mathfrak{p}}, h_{\mathfrak{p}}), \tag{3.40}$$

where the intersection is over all non-zero prime ideals $\mathfrak{p}$ of R.

Proof. We have

$$disc_{R,L}(M, h) = \bigcap disc_{R,L}(M, h) \cdot R_{\mathfrak{p}} = \bigcap disc_{R_{\mathfrak{p}},L}(M_{\mathfrak{p}}, h_{\mathfrak{p}}),$$

where the latter follows from (3.39).

There are various operations that can be performed on bilinear R-modules (M, h). These will now be considered in turn and their effect on discriminants determined.

The first of these is *twisting* an L-valued bilinear module (M, h) by an automorphism $\alpha \in Aut_L(M^* \otimes L)$:

$$\alpha \cdot (M, h) = (M, \alpha h) = (M, \alpha \circ h). \tag{3.41}$$

A special case of this is the case that $\alpha = c \cdot id$ is multiplication by an element $c \in L$; in this case we shall also write

$$(M, h)(c) = (c \cdot id)(M, h).$$

By formula (3.29) we have:

$$disc(M, \alpha h) = \det(\alpha) disc(M, h); \tag{3.42}$$

in particular,

$$disc((M, h)(c)) = disc((M, ch)) = c^{rk(M)} disc(M, h), \quad \text{if} \quad c \in L. \tag{3.43}$$

Another operation is the *pullback* of (M,h) with respect to two R-linear maps $f,g \in Hom_R(N,M)$; it is defined by

$$(f,g)^*(M,h) = (N, (g^* \otimes L) \circ h \circ f). \tag{3.44}$$

Thus, the associated bilinear maps are related by the formula

$$\beta_{h'} = \beta_h \circ (f \times g), \tag{3.45}$$

where $h' = (g^* \otimes L) \circ h \circ f$. If $f = g$ then we shall write

$$f^{\#}(M,h) = (N, f^{\#}h) = (f,f)^*(M,h). \tag{3.46}$$

Proposition 3.14 *If $dim(M \otimes K) = dim(N \otimes K)$, then we have*

$$\begin{aligned} disc((f,g)^*(M,h)) &= \chi(f)\chi(g^*)disc(M,h) \\ &= \chi(f)\chi(g)\chi(N_{tor})\chi(M_{tor})^{-1}disc(M,h). \end{aligned} \tag{3.47}$$

Proof. Since the second formula follows directly from Proposition 3.6, it is enough to verify the first. Here we have by definition and (3.34)

$$disc((f,g)^*(M,h)) = \chi((g^* \otimes L) \circ h \circ f, j_{N,L}) = \chi(f)\chi((g^* \otimes L) \circ h, j_{N,L}).$$

Now if g is not an isogeny, then neither is g^*, and hence $(g^* \otimes L) \circ h$ is degenerate because $dim(M^* \otimes L) = dim(N^* \otimes L)$. Thus, both sides of (3.47) are zero in this case, and so we may assume henceforth that g is an isogeny. Then $(g^* \otimes L) \circ j_{M,L} = j_{N,L} \circ g^* : M^* \to N^* \otimes L$ is non-degenerate, so we have by (3.28) and (3.34)

$$\begin{aligned} \chi((g^* \otimes L) \circ h, j_{N,L}) &= \chi((g^* \otimes L) \circ h, (g^* \otimes L) \circ j_{M,L})\chi(j_{N,L} \circ g^*, j_{N,L}) \\ &= \chi(h, j_{M,L})\chi(g^*), \end{aligned}$$

and hence the formula follows.

Corollary 3.15 *If (M,h) is a bilinear R-module and $M' \subset M$ is a submodule of finite index (i.e. $M' \otimes K = M \otimes K$), then*

$$disc(M', h_{|M'}) = \chi(M/M')^2 \chi(M_{tor})^{-1}\chi(M'_{tor})disc(M,h). \tag{3.48}$$

Proof. This follows from (3.47) by taking $f = g : M' \to M$ as the inclusion map; recall that then $\chi(f) = \chi(M/M')$ by definition.

The last operations which we consider here are the *direct sum* and *tensor product* of two bilinear R-modules (M_1,h_1) and (M_2,h_2). These are defined by the formulae

$$(M_1,h_1) \oplus (M_2,h_2) = (M_1 \oplus M_2, h_1 \oplus h_2) \tag{3.49}$$

$$(M_1,h_1) \otimes (M_2,h_2) = (M_1 \otimes M_2, h_1 \otimes h_2), \tag{3.50}$$

where the maps $h_1 \oplus h_2$ and $h_1 \otimes h_2$ are the compositions

$$M_1 \oplus M_2 \stackrel{h_1 \oplus h_2}{\to} (M_1^* \otimes L) \oplus (M_2^* \otimes L) \stackrel{\rho}{\to} (M_1 \oplus M_2)^* \otimes L, \tag{3.51}$$

$$M_1 \otimes M_2 \stackrel{h_1 \otimes h_2}{\to} (M_1^* \otimes L) \otimes (M_2^* \otimes L) \stackrel{\mu}{\to} (M_1 \otimes M_2)^* \otimes L, \tag{3.52}$$

where $\rho : M_1^* \oplus M_2^* \to (M_1 \oplus M_2)^*$ and $\mu : M_1^* \otimes M_2^* \to (M_1 \otimes M_2)^*$ are the canonical maps (cf. Bourbaki[BA], II.2.6 and II.4.4). Thus, for their associated bilinear forms we have:

$$\beta_{h_1 \oplus h_2}(x_1 \oplus x_2, y_1 \oplus y_2) = \beta_{h_1}(x_1, y_1) + \beta_{h_2}(x_2, y_2) \tag{3.53}$$

$$\beta_{h_1 \otimes h_2}(x_1 \otimes x_2, y_1 \otimes y_2) = \beta_{h_1}(x_1, y_1) \cdot \beta_{h_2}(x_2, y_2). \tag{3.54}$$

Proposition 3.16 *We have*

$$disc((M_1, h_1) \oplus (M_2, h_2)) = disc(M_1, h_1) \cdot disc(M_2, h_2), \tag{3.55}$$

$$disc((M_1, h_1) \otimes (M_2, h_2)) = \chi((M_1)_{tor} \otimes (M_2)_{tor})^{-1} disc(M_1, h_1)^{d_2} \cdot disc(M_2, h_2)^{d_1}, \tag{3.56}$$

where $d_i = dim_K(M_i \otimes K)$.

Proof. If M_1 and M_2 are free R-modules, then the discriminants are the determinants of the corresponding Gram matrices and hence these formulae follow from the usual identities for determinant of a direct sum (respectively, of a tensor product) of matrices.

Now let M_1 and M_2 be arbitrary. Then $(M_1 \oplus M_2)_{tor} = (M_1)_{tor} \oplus (M_2)_{tor}$ so $\chi((M_1 \oplus M_2)_{tor}) = \chi((M_1)_{tor}) \cdot \chi((M_2)_{tor})$ and hence the first equation follows.

Next, since $M_i \simeq (M_i)_{tor} \oplus \overline{M}_i$ we have

$$(M_1 \otimes M_2)_{tor} = ((M_1)_{tor} \otimes (M_2)_{tor}) \oplus (\overline{M}_1 \otimes (M_2)_{tor}) \oplus ((M_1)_{tor} \otimes \overline{M}_2)$$
$$\simeq ((M_1)_{tor} \otimes (M_2)_{tor}) \oplus ((M_1)_{tor})^{rkM_2} \oplus ((M_2)_{tor})^{rkM_1},$$

and so

$$disc(M_1 \otimes M_2) = \chi((M_1 \otimes M_2)_{tor})^{-1} disc(\overline{M}_1 \otimes \overline{M}_2)$$
$$= \chi((M_1)_{tor} \otimes (M_2)_{tor})^{-1} \cdot \chi((M_1)_{tor}))^{-d_2} \cdot \chi((M_2)_{tor})^{-d_1} \cdot disc(\overline{M}_1)^{d_2} \cdot disc(\overline{M}_2)^{d_1}$$
$$= \chi((M_1)_{tor} \otimes (M_2)_{tor})^{-1} disc(M_1)^{d_2} disc(M_2)^{d_1},$$

which proves the second equation.

Corollary 3.17 *If* (M_1, h_1) *and* (M_2, h_2) *are non-degenerate bilinear* R*-modules, then so are* $(M_1, h_1) \oplus (M_2, h_2)$ *and* $(M_1, h_1) \otimes (M_2, h_2)$.

Proof. In view of Remark 3.11c), the hypothesis means that $disc(M_i, h_i) \neq 0$, for $i = 1, 2$. By formula (3.55), respectively (3.56), we see that $disc((M_1, h_1) \oplus (M_2, h_2)) \neq 0$, respectively $disc((M_1, h_1) \otimes (M_2, h_2)) \neq 0$, and so $(M_1, h_1) \oplus (M_2, h_2)$ and $(M_1, h_1) \otimes (M_2, h_2)$ are non-degenerate by Remark 3.11c) again.

4. Induced hermitian R[G]-modules

In this section we shall review the definition of a hermitian RG-module and study the processes of (co)induction and restriction. In particular, we shall formulate and prove several versions of a (generalized) Frobenius reciprocity theorem for hermitian RG-modules.

4.1. Hermitian RG-modules. Recall from section 2 that an L-valued, *hermitian* RG-module is a bilinear R-module (M, h) such that M is a left RG-module and $h : M \to M^* \otimes L$ is RG-linear. In terms of the associated bilinear map β_h, the RG-linearity is equivalent to the *G-equivariance* condition

$$\beta_h(gm, gm') = \beta_h(m, m'), \forall g \in G,\ m, m' \in M. \tag{4.1}$$

The above use of the adjective "hermitian", which follows in part that of Scharlau[Sch], p. 244, is justified by the fact that h is essentially a sesquilinear RG-module homomorphism with respect to the canonical involution * on RG given by

$$(\sum_{g\in G} a_g g)^* = \sum_{g\in G} a_g g^{-1}.$$

More generally, if * is any involution on R which extends to one on L, then it can be extended to one on RG and we can define hermitian RG-modules relative to this involution. It is easily checked that all the results below readily extend to this more general situation. However, since all the applications only use the case $^* = id_R$ (and since the general case is more awkward to state) we restrict attention to this situation.

The following example will play a major role in the applications to character relations (cf. section 7).

Example 4.1 *Permutation modules.*
Let S be a finite set on which G acts from the left, and let

$$R[S] = \bigoplus_{s\in S} Rs$$

denote the associated permutation module. Then $M = R[S]$ comes equipped with a natural hermitian RG-module structure $h_S : M \to M^*$ defined by

$$h_S(s) = s^*, \text{ for all } s \in S, \tag{4.2}$$

where $\{s^*\}$ denotes the dual basis of M^* with respect to $\{s\}_{s\in S}$. It is clear that $(R[S], h_S)$ is a *unimodular* hermitian RG-module in the sense that $h_S : R[S] \to R[S]^*$ is an RG-isomorphism. Note that if we take $S = \{s\}$, a singleton, then we obtain the *trivial* hermitian RG-module $(R, id) = (R[S], h_S)$ as a special case.

It should also be remarked that the hermitian RG-module $(R[S], h_S)$ is uniquely determined by the G-set S, for each G-set isomorphism $S \simeq S'$ extends uniquely to an RG-isometry $(R[S], h_S) \simeq (R[S'], h_{S'})$.

It is immediate that the operations on bilinear R-modules which were introduced in the previous section readily extend to RG-modules. For the twisting, direct sum and tensor product operations no further assumptions are necessary but for the pullback we need to assume that the homomorphisms $f, g \in Hom_R(N, M)$ are actually RG-linear so that $(f, g)^*(M, h)$ is again a hermitian RG-module.

4.2. Restriction and Induction. We now turn to operations on hermitian RG-modules which result from changing the group.

Definition 4.2 Let $f : G_1 \to G_2$ be a group homomorphism.

a) If (M_2, h_2) is a hermitian RG_2-module, then its *restriction* is defined as

$$Res_f(M_2, h_2) = (M_2, h_2),$$

where we view M_2 as an RG_1-module via f. Then h_2 is automatically RG_1-linear, so $Res_f(M_2, h_2)$ is a hermitian RG_1-module.

b) If (M_1, h_1) is a hermitian RG_1-module, then the *induced hermitian* RG_2-module

$$Ind_f(M_1, h_1) = (Ind_f(M_1), ind_f(h_1))$$

is defined as follows. The underlying RG_2-module is

$$Ind_f(M_1) = RG_2 \otimes_{RG_1} M_1,$$

which is endowed with the hermitian structure $ind_f(h_1)$ defined by the formula

$$ind_f(h_1)(g_2 \otimes m, g_2' \otimes m') = h_1(\rho_f(g_2, g_2')m, m'), \tag{4.3}$$

where $g_2, g_2' \in G_2, m, m' \in M_1$ and where $\rho_f : RG_2 \times RG_2 \to RG_1$ is the R-bilinear map defined by

$$\rho_f(g_2, g_2') = \sum_{g_1 \in f^{-1}(g_2'^{-1} g_2)} g_1, \quad \text{for } g_2, g_2' \in G_2. \tag{4.4}$$

If the homomorphism $f : G_1 \to G_2$ is clear from the context, then we shall also write $Res_{G_1}^{G_2}(M_2, h_2) = Res_f(M_2, h_2)$ and $Ind_{G_1}^{G_2}(M_1, h_1) = Ind_f(M_1, h_1)$.

Remark 4.3 a) The fact that the above formula (4.3) actually defines a (unique) hermitian structure $h_2 = ind_f(h_1)$ on $M_2 = Ind_f(M_1)$ requires some justification. To this end, we first observe that the function ρ_f satisfies the following properties:

$$\rho_f(g_2 f(g_1), g_2') = \rho_f(g_2, g_2')g_1, \tag{4.5}$$

$$\rho_f(g_2, g_2' f(g_1)) = g_1^{-1} \rho_f(g_2, g_2'), \tag{4.6}$$

$$\rho_f(gg_2, gg_2') = \rho_f(g_2, g_2'), \tag{4.7}$$

where $g_1 \in G_1$ and $g, g_2, g_2' \in G_2$.

From the first two properties and the fact that h_1 is RG_1-linear it follows that the function

$$\tilde{h}_2(g_2, m, g_2', m') = h_1(\rho_f(g_2, g_2')m, m')$$

on $RG_2 \times M \times RG_2 \times M$ satisfies

$$\tilde{h}_2(g_2 f(g_1), m, g_2', m') = \tilde{h}_2(g_2, g_1 m, g_2', m'), \tag{4.8}$$

$$\tilde{h}_2(g_2, m, g_2' f(g_1), m') = \tilde{h}_2(g_2, m, g_2', g_1 m'), \tag{4.9}$$

and hence induces a unique function h_2 on $(RG_2 \otimes_{RG_1} M) \times (RG_2 \otimes_{RG_1} M)$ which satisfies (4.3). Furthermore, the third property shows that h_2 is G_2-equivariant.

b) The above formula (4.3) defines the hermitian structure on $Ind_f(M)$ in terms of its associated bilinear form. However, it is not difficult to write down the hermitian structure $h_2 = ind_f(h_1) : M_2 = RG_2 \otimes_{RG_1} M_1 \to M_2^* \otimes L$ directly; in fact, we clearly have

$$ind_f(h_1) = \rho_{f,M_1} \otimes L \circ id_{RG_2} \otimes h_1, \tag{4.10}$$

where the map

$$\rho_{f,M_1} : RG_2 \otimes M_1^* \to (RG_2 \otimes M)^*$$

is defined by the rule

$$\rho_{f,M_1}(g_2 \otimes m^*)(g_2' \otimes m') = \rho_f(g_2, g_2')m^*(m') = \sum_{g_1 \in f^{-1}(g_2'^{-1} g_2)} m^*(g_1^{-1} m'); \tag{4.11}$$

here, $g_2, g_2' \in G_2, m' \in M$ and $m^* \in M^*$. Note that properties (4.5) and (4.6) guarantee that ρ_{f,M_1} is well-defined and (4.7) shows that it is RG_2-linear.

For later reference, let us also observe that the rule

$$\phi(g' \otimes m)(g) = \rho_f(g', g^{-1})m, \tag{4.12}$$

where $g, g' \in G_2, m \in M$, defines by properties (4.5) – (4.7) an RG_2-linear map

$$\phi = \phi_{f,M} : RG_2 \otimes_{RG_1} M \to Hom_{RG_1}(RG_2, M).$$

Example 4.4 a) Let $H \leq G$ be a subgroup, and let $f : H \to G$ denote the inclusion map. If (M, h) is a hermitian RH-module, then its induced hermitian RG-module is $Ind_H^G(M, h) = (Ind_H^G(M), ind_H^G(h))$, where $Ind_H^G(M) = RG \otimes_{RH} M$ denotes the usual induced module and $h' = ind_H^G(h)$ is given by the formula

$$h'(\textstyle\sum g \otimes m_g, \sum g \otimes m_g') = \sum h(m_g, m_g'), \tag{4.13}$$

where all sums are over a system $\mathcal{T} = \{g_t\}_{t \in T} \subset G$ of left coset representatives of G/H. This follows immediately from (4.3) since we have

$$\rho_f(g_t, g_s) = \delta(g_t, g_s)$$

for $g_t, g_s \in \mathcal{T}$. Note that in this case the map

$$\rho_{f,M} : Ind(M^*) \to (Ind(M))^*$$

is the same as that defined by [CR1], Proposition (10.28), and is therefor an isomorphism if M is torsionfree.

b) Let $S = G/H$ denote the left coset space of G with respect to $H \le G$. Then with the notation of Example 4.1 we have by (4.13) that

$$(R[S], h_S) = Ind_H^G(R, id). \tag{4.14}$$

More generally, for any G-set S we have

$$(R[S], h_S) = \bigoplus_{s \in G\backslash S} Ind_{G_s}^G(R, id), \tag{4.15}$$

where $G_s = \{g \in G : gs = s\}$ denotes the stabilizer of $s \in S$.

c) Let $f : G_1 = G \to G_2 = \{1\}$ denote the trivial map, and let (M, h) be a hermitian RG-module. Then

$$Ind_f(M) = R \otimes_{RG} M = H_0(G, M) = M_G = M/D_G M \tag{4.16}$$

is the *0-th homology module* of M, where $D_G M$ is the submodule of M generated by the elements $gm - m,\ g \in G$; cf. Serre[Se], ch. VII.4. The hermitian structure on $H_0(G, M) = M/D_G M$ is given by

$$h_G(m, m') = h(N_G m, m'), \text{ for } m, m' \in M, \tag{4.17}$$

where $N_G = \sum_{g \in G} g = \rho_f(1)$.

In the sequel we shall denote this example by

$$H_0(G, (M, h)) = (M_G, h_G) = Ind_f(M, h). \tag{4.18}$$

Proposition 4.5 *Let $f : G_1 \to G_2$ be an injective group homomorphism, and let (M_i, h_i) be a hermitian RG_i-module for $i = 1, 2$. Then there is a natural isometry*

$$Ind_f((M_1, h_1) \otimes_R Res_f(M_2, h_2)) \simeq (Ind_f(M_1, h_1)) \otimes_R (M_2, h_2). \tag{4.19}$$

Proof. Fix a system $\mathcal{T}$ of left coset representatives of $G/Im(f)$, and consider the R-linear map

$$\varphi = \varphi_{\mathcal{T}} : RG_2 \otimes_{RG_1} (M_1 \otimes_R M_2) \to (RG_2 \otimes_{RG_1} M_1) \otimes_R M_2$$

defined by $\varphi(t \otimes (m_1 \otimes m_2)) = (t \otimes m_1) \otimes tm_2$, where $t \in \mathcal{T},\ m_i \in M_i,\ i = 1, 2$; such a map exists and is in fact an isomorphism of R-modules because

$$\begin{aligned} RG_2 \otimes_{RG_1} (M_1 \otimes_R M_2) &= \bigoplus_{t \in \mathcal{T}} t \otimes (M_1 \otimes_R M_2), \\ (RG_2 \otimes_{RG_1} M_1) \otimes_R M_2 &= \bigoplus_{t \in \mathcal{T}} (t \otimes M_1) \otimes_R M_2. \end{aligned}$$

By a calculation as in [Hu], p. 60, one checks that φ is RG_2-linear, and hence is an isomorphism of RG_2-modules. To check that it is also an isometry, let $t, t' \in \mathcal{T},\ m_i, m_i' \in M_i$. Then, using the fact that $\rho_f(t,t') = \delta(t,t') = 0$ or 1, we obtain:

$$\begin{aligned}(ind_f(h_1) \otimes h_2)&(\varphi(t \otimes (m_1 \otimes m_2)), \varphi(t' \otimes (m_1' \otimes m_2')))\\ &= (ind_f(h_1) \otimes h_2)((t \otimes m_1) \otimes tm_2, (t' \otimes m_1') \otimes t'm_2')\\ &= ind_f(t \otimes m_1, t' \otimes m_1') \cdot h_2(tm_2, t'm_2')\\ &= h_1(\rho_f(g,g')m_1, m_1') \cdot h_2(m_2, m_2')\\ &= (h_1 \otimes h_2)(\rho_f(t,t')(m_1 \otimes m_2), m_1' \otimes m_2')\\ &= ind_f(h_1 \otimes h_2)(t \otimes (m_1 \otimes m_2), t' \otimes (m_1' \otimes m_2')),\end{aligned}$$

which proves that φ is an isometry.

The process of induction is functorial in the following sense.

Proposition 4.6 *Let $f_1 : G_1 \to G_2$ and $f_2 : G_2 \to G_3$ be two group homomorphisms, and let (M_1, h_1) be a hermitian RG_1-module. Then there is a natural isometry*

$$Ind_{f_2 \circ f_1}(M_1, h_1) \simeq Ind_{f_2}(Ind_{f_1}(M_1, h_1)).$$

Proof. Consider the map

$$\psi : RG_3 \otimes_{RG_2} (RG_2 \otimes_{RG_1} M) \to RG_3 \otimes_{RG_1} M$$

defined by $\psi(g_3 \otimes (g_2 \otimes m)) = g_3 f_2(g_2) \otimes m$, which is clearly an isomorphism of RG_3-modules. To see that it is an isometry, we first note that the following identity holds:

$$\rho_{f_2 \circ f_1}(g_3 f_2(g_2), g_3' f_2(g_2')) = \rho_{f_1}(\rho_{f_2}(g_3, g_3')g_2, g_2'). \tag{4.20}$$

Using this, we obtain

$$\begin{aligned}ind_{f_2 \circ f_1}(h_1)&(\psi(g_3 \otimes (g_2 \otimes m)), \psi(g_3' \otimes (g_2' \otimes m')))\\ &= ind_{f_2 \circ f_1}(h_1)(g_3 f_2(g_2) \otimes m, g_3' f_2(g_2') \otimes m')\\ &= h_1(\rho(g_3 f_2(g_2), g_3' f_2(g_2'))m, m')\\ &= h_1(\rho_{f_1}(\rho_{f_2}(g_3, g_3')g_2, g_2')m, m')\\ &= ind_{f_1}(h_1)(\rho_{f_1}(g_3, g_3')(g_2 \otimes m), g_2' \otimes m')\\ &= ind_{f_2}(ind_{f_1}(h_1))(g_3 \otimes (g_2 \otimes m), g_3' \otimes (g_2' \otimes m')),\end{aligned}$$

which shows that ψ is an isometry.

Corollary 4.7 *Let $f : G_1 \to G_2$ be an injective group homomorphism and let (M_i, h_i) be a hermitian RG_i-module. Then there is an isometry*

$$H_0(G_2, Ind_f(M_1, h_1) \otimes (M_2, h_2)) \simeq H_0(G_1, (M_1, h_1) \otimes Res_f(M_2, h_2)). \tag{4.21}$$

Proof. Let $f' : G_2 \to G_3 = \{1\}$ denote the trivial map of G_2. Then $f' \circ f : G_1 \to G_3$ is the trivial map of G_1, so we have by Propositions 4.5 and 4.6 (and Example 4.4c))

$$\begin{aligned} H_0(G_1, (M_1, h_1) \otimes Res_f(M_2, h_2)) &= Ind_{f' \circ f}((M_1, h_1) \otimes Res_f(M_2, h_2)) \\ &\simeq Ind_{f'}(Ind_f((M_1, h_1) \otimes Res_f(M_2, h_2)) \\ &\simeq Ind_{f'}(Ind_f(M_1, h_1) \otimes (M_2, h_2)) \\ &= H_0(G_2, (Ind_f(M_1, h_1) \otimes (M_2, h_2))). \end{aligned}$$

The above Corollary 4.7 may be viewed as a (primitive) version of *Frobenius Reciprocity*, for over a field $R = K$ one can easily deduce the usual reciprocity theorem by taking dimensions of the underlying vector spaces. For our purposes, however, this version is not so useful because the applications require the module

$$H^0(G, M) = M^G = inv_G(M) \tag{4.22}$$

of G-invariants of M in place of $H_0(G, M)$. Over a field $R = K$ these two modules are the same, but in general they are only connected by the exact sequence

$$0 \to \hat{H}^{-1}(G, M) \to H_0(G, M) \overset{\bar{N}_G}{\to} H^0(G, M) \to \hat{H}^0(G, M) \to 0, \tag{4.23}$$

where $\bar{N}_G$ is the map induced by the endomorphism $m \to N_G m$ of M, and the $\hat{H}^q$ denote the Tate cohomology groups.

Notation 4.8 If (M, h) is a hermitian RG-module, then, as in section 2, we shall endow the module $H^0(G, M) = Inv_G(M)$ with the hermitian structure

$$inv_G(h)(m, m') = \tfrac{1}{|G|} h(m, m'). \tag{4.24}$$

The resulting bilinear R-module is denoted by

$$H^0(G, (M, h)) = Inv_G(M, h) = (Inv_G(M), inv_G(h)).$$

Remark 4.9 The normalization in (4.24) has been chosen so that the hermitian structure on $H^0(G, M)$ pulls back to that on $H_0(G, M)$ via $\bar{N}_G$:

$$\bar{N}_G^{\#}(H^0(G, (M, h))) = H_0(G, (M, h)). \tag{4.25}$$

This follows immediately from formula (4.17) because by the G-equivariance of h we have

$$inv_G(h)(N_G m, N_G m') = \frac{1}{|G|} \sum_{g \in G} h(N_G m, g m') = h(N_G m, m').$$

4.3. Coinduction. We had seen above that the bilinear module $H_0(G, (M, h))$ is a special case of the general construction of induced modules. In a similar manner, the "dual" module $H^0(G, (M, h))$ is a special case of the dual construction of *coinduced* modules which we introduce now.

Definition 4.10 Let $f : G_1 \to G_2$ be a homomorphism of groups, and let (M_1, h_1) be a hermitian RG_1-module. Then the *coinduced hermitian* RG_2-module

$$Coind_f(M_1, h_1) = (Coind_f(M_1), coind_f(h_1))$$

is defined as follows. The underlying RG_2-module is

$$Coind_f(M_1) = Hom_{RG_1}(RG_2, M_1),$$

which is endowed with the hermitian structure $coind_f(h_1)$ defined by the formula

$$coind_f(h_1)(f_1, f_2) = \frac{1}{|G_1|} \sum_{g_2 \in G_2} h_1(f_1(g_2), f_2(g_2)), \tag{4.26}$$

where $f_1, f_2 \in Hom_{RG_1}(RG_2, M_1)$.

Remark 4.11 a) The above bilinear form $coind_f(h_1)$ is G_2-equivariant because

$$\begin{aligned} coind_f(h_1)(g_2 f_1, g_2 f_2) &= \frac{1}{|G_1|} \sum_{g \in G_2} h_1((g_2 f_1)(g), (g_2 f_2)(g)) \\ &= \frac{1}{|G_1|} \sum_{g \in G_2} h_1(f_1(g g_2), f_2(g g_2)) \\ &= \frac{1}{|G_1|} \sum_{g' \in G_2} h_1(f_1(g'), f_2(g')) \\ &= coind_f(h_1)(f_1, f_2). \end{aligned}$$

b) Let $G_2 = \bigcup_{i \in I} Im(f) g_i$ be a decomposition of G_2 into (left) cosets of $Im(f) \backslash G_2$. Then by the G_1-equivariance of h_1 we have

$$coind_f(h_1)(f_1, f_2) = \frac{1}{|Ker(f)|} \sum_{i \in I} h_1(f_1(g_i), f_2(g_i)). \tag{4.27}$$

The coinduced modules are related to the induced RG-modules $Ind_f(M, h)$ and to the invariant modules $H^0(G, (M, h))$ in the following way.

Proposition 4.12 *a) If $f : G_1 \to G_2$ is injective, then there is an isometry of RG_2-modules:*

$$Coind_f(M_1, h_1) \simeq Ind_f(M_1, h_1). \tag{4.28}$$

b) If $f : G_1 \to G_2$ is surjective, then there is an isometry of RG_2-modules:

$$Coind_f(M_1, h_1) \simeq H^0(Ker(f), (M_1, h_1)). \tag{4.29}$$

Proof. a) Fix a system of (right) coset representatives of $G_2/Im(f)$, say $G_2 = \bigcup_{i\in I} g_i Im(f)$, and define

$$\phi' : Hom_{RG_1}(RG_2, M_1) \to RG_2 \otimes_{RG_1} M_1$$

by $\phi'(t) = \sum_{i\in I} g_i \otimes t(g_i^{-1})$. It is easy to see that $\phi' = \phi_{f,M_1}^{-1}$ is the inverse of the map constructed in Remark 4.3b) and hence is an isomorphism of RG_2-modules. It thus remains to show that ϕ' is an isometry. If $t, s \in Hom_{RG_1}(RG_2, M_1)$ then

$$\begin{aligned} ind_f(h_1)(\phi'(t), \phi'(s)) &= ind_f(h_1)\left(\sum_{i\in I} g_i \otimes t(g_i^{-1}), \sum_{i\in I} g_i \otimes s(g_i^{-1})\right) \\ &= \sum_{i\in I} h_1(t(g_i^{-1}), s(g_i^{-1})) = coind_f(h_1)(t, s) \end{aligned}$$

by (4.13) and by formula (4.27) above since $\{g_i^{-1}\}_{i\in I}$ is system of left coset representatives of $Im(f)\backslash G_2$. Thus ϕ' is an isometry.

b) Put $N = Ker(f)$, and define

$$\psi : Hom_{RG_1}(RG_2, M_1) \to Inv_N(M_1) = H^0(N, M_1)$$

by $\psi(t) = t(1)$. If we view $Inv_N(M_1)$ as a G_2-module via the rule $f(g_1)m = g_1 m$ for $g_1 \in G_1$ and $m \in Inv_N(M_1)$, then ψ is an RG_2-module homomorphism because $\psi(f(g_1)t) = (f(g_1)t)(1) = t(f(g_1)) = g_1 t(1) = g_1\psi(t) = f(g_1)\cdot\psi(t)$.

It is immediate that ψ is injective because if $\psi(t) = 0$, then $t(1) = 0$ and then $t(f(g_1)) = g_1 t(1) = 0$, so $t = 0$. Moreover, ψ is surjective: if $m \in Inv_N(M_1)$, then define $t \in Hom_R(RG_2, M_1)$ by $t(f(g_1)) = g_1 m$. Note that since M is N-invariant, t is well-defined. Now $t(f(g_1)f(g_1')) = g_1 g_1' m = g_1 t(f(g_1'))$, so $t \in Hom_{RG_1}(RG_2, M_1)$, and hence ψ is surjective.

It remains to prove that ψ is an isometry. If $s, t \in Hom_{RG_1}(RG_2, M_1)$, then

$$inv_N(h_1)(\psi(s), \psi(t)) = \tfrac{1}{|N|} h_1(\psi(s), \psi(t)) = \tfrac{1}{|N|} h_1(s(1), t(1)) = coind_f(h_1)(s, t)$$

by formula (4.27), and so ψ is an isometry.

Just like induction, the process of coinduction enjoys the following functoriality property.

Proposition 4.13 *Let $f_1 : G_1 \to G_2$ and $f_2 : G_2 \to G_3$ be two group homomorphisms, and let (M_1, h_1) be an hermitian RG_1-module. Then there is an isometry of hermitian RG_3-modules:*

$$Coind_{f_2}(Coind_{f_1}(M_1, h_1)) \simeq Coind_{f_2\circ f_1}(M_1, h_1).$$

Proof. Define

$$\sigma : Hom_{RG_2}(RG_3, Hom_{RG_1}(RG_2, M_1)) \to Hom_{RG_1}(RG_3, M_1)$$

by $\sigma(t)(g_3) = t(g_3)(1)$. It is easy to see that σ is an isomorphism of RG_3-modules; cf. Cartan-Eilenberg[CE], pp. 28-29, using the identification $RG_3 \otimes_{RG_2} RG_2 \simeq RG_3$.

To prove that σ is an isometry, let $s, t \in Hom_{RG_2}(RG_3, Hom_{RG_1}(RG_2, M_1))$. Then the RG_2-linearity means that $t(f_2(g_2')g_3) = g_2' t(g_3)$, if $g_2' \in G_2$ and $g_3 \in RG_3$, so

$$t(f_2(g_2')g_3)(g_2) = g_2' t(g_3)(g_2) = t(g_3)(g_2 g_2'),$$

and similarly for s in place of t. Thus we have

$$\begin{aligned}
coind_{f_2}(coind_{f_1}(h_1))(s,t) &= \frac{1}{|G_2|} \sum_{g_3 \in G_3} coind_{f_1}(h_1)(s(g_3), t(g_3)) \\
&= \frac{1}{|G_2|} \sum_{g_3 \in G_3} \frac{1}{|G_1|} \sum_{g_2 \in G_2} h_1(s(g_3)(g_2), t(g_3)(g_2)) \\
&= \frac{1}{|G_2|} \sum_{g_3 \in G_3} \frac{1}{|G_1|} \sum_{g_2 \in G_2} h_1(s(f_2(g_2)g_3)(1), t(f_2(g_2)g_3)(1)) \\
&= \frac{1}{|G_2|} \sum_{g_2 \in G_2} \frac{1}{|G_1|} \sum_{g_3 \in G_3} h_1(s(f_2(g_2)g_3)(1), t(f_2(g_2)g_3)(1)) \\
&= \frac{1}{|G_2|} \sum_{g_2 \in G_2} \frac{1}{|G_1|} \sum_{g_3' \in G_3} h_1(s(g_3')(1), t(g_3')(1)) \\
&= \frac{1}{|G_1|} \sum_{g_3 \in G_3} h_1(s(g_3)(1), t(g_3)(1)) \\
&= coind_{f_2 \circ f_1}(h_1)(\sigma(s), \sigma(t)),
\end{aligned}$$

which proves that σ is an isometry.

Corollary 4.14 *If $f : G_1 \to G_2$ is a group homomorphism and (M_1, h_1) is a hermitian RG_1-module, then there is an isometry*

$$H^0(G_2, Coind_f(M_1, h_1)) \simeq H^0(G_1, (M_1, h_1)).$$

Furthermore, if f is injective then we also have an isometry

$$H^0(G_2, Ind_f(M_1, h_1)) \simeq H^0(G_1, (M_1, h_1)).$$

Proof. The first assertion follows by taking $f_1 = f$ and $G_3 = \{1\}$ in Proposition 4.13 (cf. proof of Corollary 4.7), and the second follows from the first by Proposition 4.12a).

As we shall see below, the following result will be useful when working with permutation modules:

Corollary 4.15 *Let $f : G_1 \to G_2$ be a group homomorphism and let $X \le G_1$ be a subgroup of G_1. If $k = |X \cap Ker(f)|^{-1} \cdot 1_R$, then there is an isometry between the coinduced hermitian permutation module attached to G_1/X and the k-twist of the hermitian permutation module attached to $G_2/f(X)$:*

$$Coind_f(R[G_1/X], h_{G_1/X}) \simeq (R[G_2/f(X)], h_{G_2/f(X)})(k) \tag{4.30}$$

Proof. Let $g = f_{|X} : X \to Y = f(X)$ denote the restriction of f to X. Since $f \circ i_X = i_Y \circ g$, where $i_X : X \to G_1$ and $i_Y : Y \to G_2$ denote the inclusion maps, we obtain by Proposition 4.13 an isometry

$$Coind_f(Coind_X^{G_1}(R, id)) \simeq Coind_Y^{G_2}(Coind_g(R, id)). \tag{4.31}$$

By Proposition 4.12a) and Example 4.4b) we have on the one hand that

$$Coind_X^{G_1}(R, id) \simeq Ind_X^{G_1}(R, id) = (R[G_1/X], h_{G_1/X}); \tag{4.32}$$

on the other hand we obtain from Proposition 4.12b) that

$$Coind_g(R, id) = H^0(Ker(g), (R, id)) = (R, id)(k),$$

because $|Ker(g)| = \frac{1}{k}$. It thus follows that

$$Coind_Y^{G_2}(Coind_g(R, id)) \simeq Ind_Y^{G_2}(R, id)(k) = (R[G_2/f(X)], h_{G_2/f(X)})(k).$$

Substituting this and (4.32) in (4.31) yields the isometry (4.30).

4.4. Frobenius Reciprocity. We are now ready to prove the following version of *Frobenius Reciprocity* for hermitian RG-modules.

Theorem 4.16 (Frobenius Reciprocity) *Let $X \le G$ be a subgroup of G and let (M', h') be a hermitian RX-module. Then for any hermitian RG-module (M, h) there is an isometry of bilinear R-modules*

$$H^0(G, Ind_X^G(M', h') \otimes (M, h)) \simeq H^0(X, (M', h') \otimes Res_X^G(M, h)).$$

Proof. By Proposition 4.5 and Corollary 4.14 we have

$$\begin{aligned} H^0(G, Ind_X^G(M', h') \otimes (M, h)) &\simeq H^0(G, Ind_X^G((M', h') \otimes Res_X^G(M, h))) \\ &\simeq H^0(X, (M', h') \otimes Res_X^G(M, h)). \end{aligned}$$

Corollary 4.17 *If S is a finite G-set and (M, h) is a hermitian RG-module, then we have an R-isometry*

$$H^0(G, (R[S], h_S) \otimes (M, h)) \simeq \bigoplus_{s \in G\backslash S} H^0(G_s, (M, h)),$$

where $G_s = \{g \in G : gs = s\}$ denotes the stabilizer subgroup of $s \in S$.

Proof. By additivity it is enough to consider the case $S = G/X$, where $X \le G$ is a subgroup. By Example 4.4b) we have in this case $(R[S], h_S) = Ind_X^G(R, id)$, and so the assertion follows by taking $(M', h') = (R, id)$ in Theorem 4.14.

For hermitian *RG-lattices*, there is a more general Frobenius reciprocity theorem available. This is based on the following analogue of Proposition 4.5:

Proposition 4.18 *Let $f : G_1 \to G_2$ be a group homomorphism, and let (M_i, h_i) be a hermitian RG_i-module for $i = 1, 2$. If M_2 is R-torsionfree (i.e. an RG_2-lattice) or if $|Ker(f)|$ is invertible in R, then there is a canonical isometry*

$$(4.33) \quad Coind_f(M_1, h_1) \otimes (M_2, h_2) \simeq Coind_f((M_1, h_1) \otimes Res_f(M_2, h_2)).$$

In order to prove this proposition it will be convenient to isolate the construction of the desired RG_2-module homomorphism which is compatible with the hermitian structures. Subsequently we shall investigate when it is an isometry.

Lemma 4.19 *If $f : G_1 \to G_2$ is a group homomorphism and (M_i, h_i) is a hermitian RG_i-module for $i = 1, 2$, then there is a unique RG_2-module homomorphism*

$$\nu : Hom_{RG_1}(RG_2, M_1) \otimes_R M_2 \to Hom_{RG_1}(RG_2, M_1 \otimes_R M_2)$$

such that for $t \in M := Hom_{RG_1}(R_2, M_1)$, $g_2 \in G_2$, $m_2 \in M_2$, we have

$$(4.34) \qquad \nu(t \otimes m_2)(g_2) = t(g_2) \otimes g_2 m_2.$$

Moreover, ν is compatible with the hermitian structures is the sense that

$$(4.35) \qquad \nu^{\#}(coind_f(h_1 \otimes h_2) = coind_f(h_1) \otimes h_2.$$

In addition, ν is an isomorphism if and only if the "cup-product" map

$$(4.36) \qquad u_M : M^{G_1} \otimes_R M_2^{triv} \to (M \otimes M_2^{triv})^{G_1}$$

is an isomorphism where M_2^{triv} denotes the R-module M_2 endowed with the trivial G_1-module structure and $M = Hom_R(RG_2, M_1)$ is viewed as a G_1-module in the usual way.

Proof. First note that since ν is R-bilinear in t and m_2, the above rule (4.34) defines a unique R-homomorphism

$$\nu : Hom_{RG_1}(RG_2, M_1) \otimes_R M_2 \to Hom_R(RG_2, M_1 \otimes_R M_2).$$

Next we observe that $\nu(t \otimes m_2) \in Hom_R(RG_2, M_1 \otimes M_2)$ is G_1-linear because

$$\begin{aligned} g_1 \cdot \nu(t \otimes m_2)(g_2) &= g_1(t(g_2) \otimes g_2 m_2) &&= g_1 t(g_2) \otimes f(g_1) g_2 m_2 \\ &= t(f(g_1)g_2) \otimes f(g_1) g_2 m_2 &&= \nu(t \otimes m_2)(f(g_1)g_2), \end{aligned}$$

and so $Im(\nu) \subset Hom_{RG_1}(RG_2, M_1 \otimes M_2)$. Moreover, ν is G_2-linear because

$$\begin{array}{rclcl} \nu(g_2(t \otimes m_2))(g_2') & = & \nu((g_2 t) \otimes (g_2 m_2))(g_2') & = & (g_2 t)(g_2') \otimes g_2' g_2 m_2 \\ & = & t(g_2' g_2) \otimes g_2' g_2 m_2 & = & \nu(t \otimes m_2)(g_2' g_2) \\ & = & g_2 \cdot \nu(t \otimes m_2)(g_2'). & & \end{array}$$

We have thus verified the existence of the RG_2-linear map ν. The next step is to show that equation (4.35) is valid. For this, let $m_2, m_2' \in M_2$ and $t, t' \in Hom_{RG_1}(RG_2, M_1)$. Then

$$\begin{array}{l} coind_f(h_1 \otimes h_2)(\nu(t \otimes m_2), \nu(t' \otimes m_2')) \\ \quad = \dfrac{1}{|G_1|} \sum_{g \in G_2} h_1 \otimes h_2(\nu(t \otimes m_2)(g), \nu(t' \otimes m_2')(g)) \\ \quad = \dfrac{1}{|G_1|} \sum_{g \in G_2} h_1 \otimes h_2(t(g) \otimes g m_2, t'(g) \otimes g m_2') \\ \quad = \dfrac{1}{|G_1|} \sum_{g \in G_2} h_1(t(g), t'(g)) h_2(g m_2, g m_2') \\ \quad = \dfrac{1}{|G_1|} \sum_{g \in G_2} h_1(t(g), t'(g)) h_2(m_2, m_2') \\ \quad = coind_f(h_1)(t, t') h_2(m_2, m_2') \\ \quad = (coind_f(h_1) \otimes h_2)(t \otimes m_2, t' \otimes m_2'). \end{array}$$

It remains to analyze the bijectivity of ν. For this, consider the R-linear map

$$\nu_1 : M \otimes M_2 \to Hom_R(RG_2, M_1 \otimes M_2)$$

defined by (4.34). We note that ν_1 is clearly an isomorphism of R-modules since RG_2 is a free R-module and multiplication by g_2 is an R-automorphism of M_2. Next we observe that

$$\begin{array}{rcl} g_1 \cdot \nu_1(t \otimes m_2)(g_2) & = & g_1 \nu_1(t \otimes m_2)(f(g_1)^{-1} g_2) \\ & = & g_1(t(f(g_1)^{-1} g_2) \otimes f(g_1)^{-1} g_2 m_2) \\ & = & g_1 t(f(g_1)^{-1} g_2) \otimes f(g_1) f(g_1)^{-1} g_2 m_2 \\ & = & (g_1 t)(g_2) \otimes g_2 m_2 \\ & = & \nu_1(g_1 t \otimes m_2)(g_2), \end{array}$$

which means that ν' is G_1-linear, if we view M_2 as a trivial G_1-module when it appears in $M \otimes M_2$ (but continue to view the M_2 appearing in $M_1 \otimes M_2$ as a G_1-module via f). Thus ν_1 is an isomorphism of RG_1-modules, and hence induces an R-isomorphism on the invariant spaces:

$$\nu' = \nu_1^G : (M \otimes M_2^{triv})^G \xrightarrow{\sim} (Hom_R(RG_2, M_1 \otimes M_2)^G = Hom_{RG_1}(RG_2, M_1 \otimes M_2).$$

On the other hand, we clearly have the factorization

$$\nu = \nu' \circ u_M,$$

and so it follows that ν is an isomorphism if and only if this is true for u_M, as asserted.

Proof of Proposition 4.19. By the above lemma, we only need to verify that ν is an isomorphism under the hypotheses of the proposition.

By the lemma we see that the fact whether or not ν is an isomorphism depends only on the R-module structure of M_2, so it is enough to investigate the situation when $M_2 = M_2^{triv}$ has the trivial G_2-module structure. In that case the $A = RG_1$-module structure on $M_1 \otimes M_2$ coincides with that of Bourbaki[BA], II.4.2, and also the above map ν coincides with the map ν defined there (cf. loc. cit., equation (7)), if we take $B = R, E = RG_2, G = M_1, F = M_2$ in the notation there. We thus obtain by [BA], II.4.2, Proposition 2, that ν is an isomorphism if $G = M_2$ is R-projective (or, equivalently, M_2 is R-torsionfree since R is a principal ideal domain), or if if RG_2 is RG_1-projective. By the lemma below, this is the case here, and so the assertion follows.

Lemma 4.20 *If $f : G_1 \to G_2$ is a group homomorphism and R a ring in which $|Ker(f)|$ is invertible, then RG_2 is a projective RG_1-module.*

Proof. By factoring $f = f_2 \circ f_1$ as a surjection followed by an injection, we see that it is enough to consider the case that f is surjective since RG_2 is a free RG_1-module if f is injective. Thus, assume that f is surjective.

To prove that $M := RG_2$ is RG_1-projective, it is enough to show that M is G_1-cohomological trivial, i.e. $H^q(G, M) = 0$ for all $q \geq 1$ and all subgroups $G \leq G_1$. Fix $G \leq G_1$ and put $Q = f(G), N = Ker(f) \cap G$. Since $|N|$ is invertible in R, we have $H^q(N, M) = 0$ for $q > 0$. Thus, by the Hochschild-Serre (spectral) sequence (cf. Serre[Se], VII.6), we have $H^q(G, M) \simeq H^q(Q, M^N) = H^q(Q, M)$. But $H^q(Q, M) = H^q(Q, RQ^n) = 0$, where $n = [G_2 : Q]$, so the assertion follows.

In view of Proposition 4.12a), the following theorem generalizes the Frobenius Reciprocity Theorem 4.16:

Theorem 4.21 (Generalized Frobenius Reciprocity) *Let $f : G_1 \to G_2$ be a group homomorphism, and let (M_i, h_i) be a hermitian RG_i-module for $i = 1, 2$. If M_2 is an RG_2-lattice or if $|Ker(f)|$ is invertible in R, then there is an R-isometry*

$$H^0(G_2, Coind_f(M_1, h_1) \otimes (M_2, h_2)) \simeq H^0(G_1, (M_1, h_1) \otimes Res_f(M_2, h_2)).$$

Proof. By Proposition 4.18 and Corollary 4.14 we have

$$\begin{aligned} & H^0(G_2, Coind_f(M_1, h_1) \otimes (M_2, h_2)) \\ & \quad \simeq \quad H^0(G_2, Coind_f((M_1, h_1) \otimes Res_f(M_2, h_2))) \\ & \quad \simeq \quad H^0(G_1, (M_1, h_1) \otimes Res_f(M_2, h_2)). \end{aligned}$$

Although we do not need this in the sequel, it would be interesting to know if the restrictive hypotheses (i.e. M_2 torsionfree or $|Ker(f)|$ invertible in R) in the above theorem can be removed. For our purposes, however, the following strengthening of the theorem to the case that $(M_1, h_1) = (R[S], h_S)$ is a permutation module is sufficient.

Theorem 4.22 (Frobenius Reciprocity for Permutation Modules) *Let $f : G_1 \to G_2$ be a group homomorphism, and let $(M_1, h_1) = (R[S], h_S)$ be a hermitian RG_1-permutation module. Then for any hermitian RG_2-module (M_2, h_2) there is an R-isometry*

$$H^0(G_2, Coind_f(M_1, h_1) \otimes (M_2, h_2)) \simeq H^0(G_1, (M_1, h_1) \otimes Res_f(M_2, h_2)).$$

Proof. By additivity it is enough to consider the case $S = G_1/X$, where $X \leq G_1$ is a subgroup. Then by Corollary 4.17 we have

$$(4.37)\quad H^0(G_1, (R[G_1/X], h_{G_1/X}) \otimes Res_f(M_2, h_2)) \simeq H^0(X, Res_f(M_2, h_2)).$$

On the other hand, if we put $k = |X \cap Ker(f)|^{-1}$ and $Y = f(X)$, then by Corollary 4.15 and Corollary 4.17 we obtain

$$\begin{aligned} & H^0(G_2, Coind_f(R[G_1/X], h_{G_1/X}) \otimes (M_2, h_2)) \\ \simeq \quad & H^0(G_2, (R[G_2/Y], h_{G_2/Y})(k) \otimes (M_2, h_2)) \simeq H^0(Y, (M_2, h_2))(k). \end{aligned}$$

By comparing this with (4.37) we thus see that the assertion follows from the following lemma:

Lemma 4.23 *Let $f : G_1 \to G_2$ be a group homomorphism and (M_2, h_2) a hermitian RG_2-module. For a subgroup $X \leq G_1$ put $Y = f(X)$ and*

$$k = |Y|/|X| = |X \cap Ker(f)|^{-1}.$$

Then we have

$$H^0(X, res_f(M_2, h_2)) = H^0(Y, (M_2, h_2))(k).$$

Proof. Clearly $M := Inv_X(res_f(M_2)) = Inv_Y(M_2)$ as submodules of M_2. Thus, if we let h denote the restriction of h_2 to this submodule, then we have

$$H^0(X, res_f(M_2, h_2)) = (M, \tfrac{1}{|X|}h) = (M, \tfrac{1}{|Y|}h)(k) = H^0(Y, (M_2, h_2))(k),$$

and so the lemma follows.

5. The construction of the invariant δ

In this section we shall construct the fundamental invariant $\delta(M_1, M_2; M)$ in three steps. First we introduce the invariant $d_{RG,L}(M,h)$ which is attached to any hermitian RG-module (M,h). This will then be used to construct the preliminary invariant $\Delta((M_1,h_1),(M_2,h_2);M)$. By introducing the Grothendieck ring $\mathcal{H}(RG,L)$ of hermitian RG-modules, this invariant may also be viewed as a pairing of a certain ideal of this ring with the Grothendieck ring $Mod(RG)_{non-deg}$ of non-degenerate RG-modules. Finally we show that Δ induces the desired invariant δ. In the next section we shall study the main properties of this invariant, and then in a subsequent section relate this invariant to the one discussed in the introduction.

5.1. The invariant $d_{RG}(M,h)$. In the previous section (cf. Notation 4.8) we had defined the "0-th cohomology module" $H^0(G,(M,h)) = Inv_G(M,h)$; taking its discriminant (cf. Definition 3.10) yields the invariant $d_{RG,L}(M,h)$:

Notation 5.1 If (M,h) is an L-valued hermitian RG-module then we let

$$d_{RG,L}(M,h) = disc_R(Inv_G(M,h)) = disc_R(H^0(G,(M,h)))$$

denote the discriminant of the bilinear module of invariants of (M,h). If the ring R and the field L are clear from the context, then we shall write d_G in place of $d_{RG,L}$.

Remark 5.2 Following the line of thought of section 4, we could also consider the invariant

$$d'_{RG,L}(M,h) = disc_R(M_G, h_G) = disc_R(H_0(G,(M,h))).$$

However, this invariant is of secondary importance because all the applications use only the invariant $d_{RG,L}$. Moreover, by (4.25) and (4.23) (and Prop. 3.14), these two invariants are related by the formula

$$(5.1)\quad d'_{RG,L}(M,h) = \chi(H_0(G,(M,h))_{tor})\chi(H^0(G,(M,h))_{tor})^{-1}h(G,M)^2 \cdot d_{RG,L}(M,h)$$

where

$$(5.2)\qquad h(G,M) = \chi(\bar{N}_G) = \chi(\hat{H}^0(G,M)) \cdot \chi(\hat{H}^{-1}(G,M))^{-1}$$

denotes the "Herbrand quotient" of M. (The latter is a natural generalization of the usual Herbrand quotient which is commonly only defined for cyclic groups G and $R = \mathbb{Z}$; cf. Serre[Se], ch. VIII.4.)

We first verify that $d_{RG,L}(M,h) \neq 0$ whenever (M,h) is non-degenerate, i.e. whenever $disc_R(M,h) \neq 0$ (cf. Remark 3.11c)).

Lemma 5.3 *Let* $N'_G = |G| - N_G = \sum_{g \in G}(1-g) \in RG$, *where, as before,* $N_G = \sum_{g\in G} g$. *Then for any hermitian RG-module* (M, h) *we have*

a) $M^G \perp N'_G M$;

b) $M \otimes K = (M^G \otimes K) \oplus (N'_G M \otimes K) = (N_G M \otimes K) \oplus (N'_G M \otimes K)$;

c) if $disc_R(M, h) \neq 0$ *then also* $d_{RG}(M, h) \neq 0$.

Proof. a) Since $(1-g)m = 0$ for all $g \in G$, and $m \in M^G$, we have

$$h(m, N'_G m') = \sum_{g\in G} h(m, (1-g)m') = \sum_{g \in G} h((1-g^{-1})m, m') = 0,$$

if $m \in M^G$ and $m' \in M$, and so a) holds.

b) Since $N_G(M \otimes K) \subset (M^G \otimes K)$, it is enough to prove that $N_G(M\otimes K) \oplus N'_G(M \otimes K) = M \otimes K$ and this follows since $N_G + N'_G = n \cdot 1$ and $n = |G|$ is invertible in K.

c) Put $M' = \overline{M}^G \oplus N'_G\overline{M}$. Since the canonical map $f : M' \to M$ is an isogeny by b), it follows from Proposition 3.14 that $disc_R(M', f^\# h) \neq 0$. Furthermore, by a) we have that $(M', f^\# h) = (\overline{M}^G, \overline{h}_{\overline{M}^G}) \oplus (N'_G\overline{M}, \overline{h}_{N'_G\overline{M}})$ and so by Proposition 3.16 we obtain

$$disc_R(M', f^\# h) = disc_R(\overline{M}^G, \overline{h}_{|\overline{M}^G}) disc_R(N'_G\overline{M}, \overline{h}_{|N'_G\overline{M}}),$$

which shows that $disc_R(M^G, h_{|M^G}) = \chi((M^G)_{tor})^{-1} disc_R(\overline{M}^G, \overline{h}_{|\overline{M}^G}) \neq 0$.

Some elementary properties of the invariant d_{RG} are summarized below.

Proposition 5.4 *a) The invariant* d_{RG} *is additive:*

$$d_{RG}((M_1, h_1) \oplus (M_2, h_2)) = d_{RG}(M_1, h_1) d_{RG}(M_2, h_2).$$

b) In the situation of Proposition 3.12 *we have*

$$d_{R'G,L'}(M \otimes R', h \otimes R') = d_{RG,L}(M, h) \cdot R'.$$

c) The following localization formula is valid:

$$d_{RG,L}(M, h) = \bigcap d_{R_\mathfrak{p},L}(M_\mathfrak{p}, h_\mathfrak{p}).$$

d) The invariant of a twist by $r \in R$ *is given by the formula*

$$d_{RG,L}(M, rh) = r^{rk(H^0(G,M))} d_{RG,L}(M, h).$$

Proof. a) Since $H^0(G, (M_1, h_1) \oplus (M_2, h_2)) = H^0(G, (M_1, h_1)) \oplus H^0(G, (M_2, h_2))$, this follows directly from Proposition 3.16.

b) Since R' is flat over R we have $H^0(G, (M' \otimes R', h \otimes R')) = H^0(G, (M, h)) \otimes R'$, and hence the assertion follows from Proposition 3.12.

c) As in the proof of Corollary 3.13, this follows immediately from b).

d) This follows immediately from formula (3.43).

5.2. The invariant Δ. The construction of the invariant Δ is based on the following fundamental but perhaps somewhat technical fact.

Proposition 5.5 *Let (M_i, h_i), $i = 1, 2, 3$, be three non-degenerate hermitian RG-modules. If $M_1 \otimes K \simeq M_2 \otimes K$, then there is an RG-isogeny $f : M_1 \to M_2$ and for any such f there is an automorphism $\alpha \in Aut_{LG}(M_1^* \otimes L)$ such that $f^\# h_2 = \alpha \circ h_1$. We then have*

$$d_{RG}((M_1, h_1) \otimes (M_3, h_3)) = \Delta \cdot d_{RG}((M_2, h_2) \otimes (M_3, h_3)) \tag{5.3}$$

where

$$\Delta = \chi(inv_G(f \otimes id_{M_3}))^2 \det(inv_G(\alpha \otimes id_{M_3^* \otimes L}))^{-1} \cdot \chi(inv_G(M_1 \otimes M_3)_{tor})\chi(inv_G(M_2 \otimes M_3)_{tor})^{-1} \tag{5.4}$$

and $inv_G(f \otimes id_{M_3}) : inv_G(M_1 \otimes M_3) \to inv_G(M_2 \otimes M_3)$ denotes the induced map on the invariant spaces. In particular, the principal R-module

$$\Delta = d_G((M_1, h_1) \otimes (M_3, h_3)) d_G((M_2, h_2) \otimes (M_3, h_3))^{-1} \tag{5.5}$$

does not depend on h_3.

Proof. We first observe that the last assertion follows immediately from (5.3) since Δ does not depend on h_3. (Note that by Lemma 5.3c) it follows that $d_G((M_2, h_2) \otimes (M, h)) \neq 0$ because $(M_2, h_2) \otimes (M_3, h_3)$ is non-degenerate by Corollary 3.17.)

We now prove the first assertion. By hypothesis there is an isomorphism $\tilde{f} \in Hom_{KG}(M_1 \otimes K, M_2 \otimes K) \simeq Hom_{RG}(M_1, M_2) \otimes K$ (cf. [BCA], II.2.7). Thus $\tilde{f} = f \otimes (1/r)$ with $f \in Hom_{RG}(M_1, M_2)$ and $0 \neq r \in R$. Since $f : M_1 \to M_2$ is automatically an RG-isogeny, this proves the existence of the desired f.

Next, suppose that an RG-isogeny f is given. Since $h_1, f^\# h_2 : M_1 \to M_1^* \otimes L$ are non-degenerate, we see by Remark 3.9c) that there is a (unique) $\alpha \in Aut_L(M_1^* \otimes L)$ such that $f^\# h_2 = \alpha \circ h_1$. Since $f^\# h_2$ and h_1 are RG-linear, so is α, and so the second assertion follows.

It thus remains to prove equation (5.3). Here we first note that since $\alpha \otimes id$ is an LG-isomorphism of $(M_1 \otimes M_3)^* \otimes L = (M_1^* \otimes L) \otimes (M_3^* \otimes L)$, it induces the isomorphism $inv_G(\alpha \otimes id)$ on the invariant space $inv_G((M_1 \otimes M_3)^* \otimes L)$, and we have $inv_G(\alpha \otimes id) \circ inv_G(h_1 \otimes h_3) = inv_G((\alpha \circ h_1) \otimes h_3)$. Thus by equation (3.42)

$$\begin{aligned} d_G((M_1, h_1) \otimes (M_3, h_3)) &= disc(inv_G(M_1 \otimes M_3), inv_G(h_1 \otimes h_3)) \\ &= \det(inv_G(\alpha \otimes id))^{-1} disc(inv_G(M_1 \otimes M_3), inv_G(\alpha \circ h_1 \otimes h_3)) \\ &= \det(inv_G(\alpha \otimes id))^{-1} disc(inv_G(M_1 \otimes M_3), inv_G(f^\#(h_2) \otimes h_3)). \end{aligned}$$

Furthermore, since $inv_G(M_1 \otimes M_3 \otimes K) \simeq inv_G(M_2 \otimes M_3 \otimes K)$, it follows that the hypothesis of Proposition 3.14 is satisfied, and so we obtain

$$\begin{aligned} & disc(inv_G(M_1 \otimes M_3), inv_G((f^\#(h_2)) \otimes h_3)) \\ &\quad = disc(inv_G(f \otimes id)^\#(inv_G(M_2 \otimes M_3), inv_G(h_2 \otimes h_3))) \end{aligned}$$

$$= \chi((inv_G(M_1 \otimes M_3))_{tor})\chi((inv_G(M_2 \otimes M_3))_{tor})^{-1}$$
$$\cdot \chi(inv_G(f \otimes id))^2 disc(inv_G(M_2 \otimes M_3, h_2 \otimes h_3)),$$

from which formula (5.3) follows.

The following corollary will later play an important role in the construction of the invariant δ.

Corollary 5.6 *If, in the situation of Proposition* 5.5, *we have in addition that* $M_1 \simeq M_2$ *and that* h_1, h_2 *are unimodular (i.e. that* $h_i : M_i \xrightarrow{\sim} M_i^*$ *are isomorphisms), then we have that* $\Delta = R$.

Proof. We may assume that $M_1 = M_2 = M$. Then $f = id_M$ and $\alpha = \alpha_0 \otimes L$, where $\alpha_0 \in Aut(M_0)$, so $\chi(inv_G(id_M \otimes id_{M_3})) = R$ and $\det(inv_G(\alpha \otimes id_{M_3^* \otimes L})) = \det(inv_G(\alpha_0 \otimes id_{M_3^*})) = R$, and hence $\Delta = R$.

We now come to the definition of the invariant Δ, which is defined for RG-modules M which are *non-degenerate* in the following sense.

Definition 5.7 An RG-module M is *non-degenerate* if $M \otimes K \simeq M^* \otimes K$ (as KG-modules).

Remark 5.8 It is well-known (cf. [Hu], p. 29) that M is non-degenerate if and only if $M \otimes L \simeq M^* \otimes L$, and this is in turn equivalent to the existence of a non-degenerate L-valued hermitian RG-module structure $h : M \to M^* \otimes L$.

Notation 5.9 Let (M_1, h_1) and (M_2, h_2) be two non-degenerate hermitian RG-modules and let M be a non-degenerate RG-module. Then by Remark 5.8 M admits a non-degenerate hermitian RG-module structure $h : M \to M^* \otimes L$. We now put

$$\Delta_{RG,L}((M_1, h_1), (M_2, h_2), M) = d_G((M_1, h_1) \otimes (M, h)) d_G((M_2, h_2) \otimes (M, h))^{-1},$$

which, by Proposition 5.5, is independent of the choice of h.

5.3. The Grothendieck ring $\mathcal{H}(RG, L)$. In order to be able to work efficiently with the above defined invariant Δ, it advantageous to be able view it as a pairing of certain groups. To this end we introduce here the *Grothendieck ring* of hermitian RG-modules, which enables us to write everything in a much more compact and transparent form.

Notation 5.10 Let $\mathcal{H}(RG, L)^+$ denote the set of RG-isometry classes of L-valued non-degenerate hermitian RG-modules (M, h). It is easily checked (using Corollary 3.17) that the operations $\oplus$ and $\otimes$ make $\mathcal{H}(RG, L)^+$ into a commutative semi-ring with unit (R, id_R). We let $\mathcal{H}(RG, L)$ denote the associated *Grothendieck ring* (in the sense of Scharlau[Sch], p. 30), and

$$i_G : \mathcal{H}(RG, L)^+ \to \mathcal{H}(RG, L)$$

the canonical semi-ring homomorphism. Thus $\mathcal{H}(RG,L)$ is a (commutative) ring which is universal with respect to semi-ring homomorphisms $f:\mathcal{H}(RG,L)^+\to A$ to rings A in the sense that $f=f'\circ i_G$ factors uniquely over a ring homomorphism $f':\mathcal{H}(RG,L)\to A$. For future reference, note that by the construction of $\mathcal{H}(RG,L)$ given in Scharlau[Sch], p. 30, each $x\in\mathcal{H}(RG,L)$ has the form

$$x=[M_1,h_1]-[M_2,h_2], \tag{5.6}$$

where $[M_i,h_i]=i_G((M_1,h_i))$ denotes the image of (the isometry class) of (M_i,h_i) in $\mathcal{H}(RG,L)$.

As a first application of the universal property, we note that there is a canonical ring homomorphism

$$\kappa=\kappa_G:\mathcal{H}(RG,L)\to K_0(KG)=G_0(KG)$$

to the Grothendieck ring of (projective) KG-modules (cf. [CR1], p. 406) such that

$$\kappa([M,h])=[M\otimes_R K],$$

for the map $(M,h)\mapsto[M\otimes K]$ defines a semi-ring homomorphism $\mathcal{H}(RG,L)^+\to K_0(KG)$ which therefore factors over $\mathcal{H}(RG,L)$.

By Remark 5.8 we see that the image of κ is

$$Im(\kappa)=K_0(KG)_{sym}=\{[V_1]-[V_2]:V_i^*\simeq V_i\},$$

the subring of $K_0(KG)$ generated by the *symmetric* KG-modules. On the other hand, if we let

$$\mathcal{H}(RG,L)^0=Ker(\kappa)$$

denote the kernel, then by (5.6) we have

$$\mathcal{H}(RG,L)^0=\{[M_1,h_1]-[M_2,h_2]:M_1\otimes K\simeq M_2\otimes K\}. \tag{5.7}$$

In a similar manner, if $Mod(RG)$ denotes the Grothendieck ring associated to the semi-ring $Mod(RG)^+$ consisting of RG-isomorphism classes of (finitely generated) RG-modules, then we have a natural ring homomorphism

$$\rho=\rho_G:\mathcal{H}(RG,L)\to Mod(RG)$$

such that $\rho([M,h])=[M]$. Note that by Remark (5.8) its image is

$$Im(\rho)=Mod(RG)_{non-deg},$$

the subring generated by the non-degenerate RG-modules. Furthermore, the map κ factors over ρ:

$$\kappa=\overline{\kappa}\circ\rho.$$

By identifying the group $Id(R,L)$ of non-zero principal R-submodules of L with the quotient group $L^\times/R^\times$, we see by Lemma 5.3c) that $d_{RG,L}$ defines a homomorphism of semi-groups

$$d^+_{RG,L} : \mathcal{H}(RG,L)^+ \to Id(R,L) = L^\times/R^\times.$$

By the universal property of Grothendieck groups there is a unique induced group homomorphism

$$d_{RG} : \mathcal{H}(RG,L) \to Id(R,L) = L^\times/R^\times.$$

With the above notations we can now express the fundamental invariant Δ as follows.

Theorem 5.11 *There is a unique pairing*

$$\Delta_{RG,L} : \mathcal{H}(RG,L)^0 \times Mod(RG)_{non-deg} \to Id(R,L) = L^\times/R^\times$$

such that for any $x \in \mathcal{H}(RG,L)^0,\ y \in \mathcal{H}(RG,L)$ *we have*

$$\Delta_{RG,L}(x, \rho(y)) = d_{RG}(x \cdot y). \tag{5.8}$$

Proof. Since $\rho : \mathcal{H}(RG,L) \to Mod(RG)$ is surjective, there exists at most one map satisfying (5.8). To prove existence, we shall use the invariant Δ which was introduced in Notation 5.9. Using the notation of the following Lemma 5.10, we note that Δ defines a bi-additive map

$$\Delta : (\mathcal{H}(RG,L)^+ \oplus \mathcal{H}(RG,L)^+)^0 \times Mod(RG)^+_{non-deg} \to L^\times/R^\times.$$

Since it is immediate from the definition (cf. Notation 5.9) that

$$\Delta(M_1, h_1, M_1, h_1, M) = 0,$$

the following lemma shows that Δ induces the desired bi-additive map.

Lemma 5.12 *Let* A^+ *and* B^+ *be two commutative semigroups with associated Grothendieck groups* A *and* B *and canonical maps* $i_A : A^+ \to A$ *and* $i_B : B^+ \to B$, *respectively. Suppose* $f : A \to C$ *is a homomorphism to an abelian group* C *and put* $A^0 = Ker(f)$ *and*

$$(A^+ \oplus A^+)^0 = \{(a_1, a_2) \in A^+ \oplus A^+ : i_A(a_1) - i_A(a_2) \in A^0\}.$$

Furthermore, let D *be another abelian group and suppose that*

$$g : (A^+ \oplus A^+)^0 \times B^+ \to D$$

is a bi-additive map of semigroups such that

$$g((a,a),b) = 0 \tag{5.9}$$

for all $a \in A^+, b \in B^+$. *Then* g *induces a unique bi-additive map*

$$\overline{g} : A^0 \times B \to D$$

such that

$$\overline{g}(i_A(a_1) - i_A(a_2), i_B(b)) = g((a_1, a_2), b), \tag{5.10}$$

for all $(a_1, a_2) \in (A^+ \times A^+)^0$ *and* $b \in B^+$.

Proof. Since A^0 consists of expressions of the form $i_A(a_1) - i_A(a_2)$ and B is generated by the elements $i_B(b)$, it is clear that $\overline{g}$ is uniquely determined by (5.10).

To define $\overline{g}$, fix $(a_1, a_2) \in (A^+ \oplus A^+)^0$ and define $g_{a_1,a_2} : B^+ \to D$ by

$$g_{a_1,a_2}(b) = g((a_1, a_2), b).$$

Since g is additive in b, it follows that g_{a_1,a_2} induces a unique homomorphism $\overline{g}_{a_1,a_2} : B \to D$ such that

$$\overline{g}_{a_1,a_2}(i_B(b^+)) = g((a_1, a_2), b^+), \quad \text{for} \quad b^+ \in B^+.$$

We note that if $(a_1', a_2') \in (A^+ \oplus A^+)^0$ is another element, then we have

$$\overline{g}_{a_1,a_2}(b) + \overline{g}_{a_1',a_2'}(b) = \overline{g}_{a_1+a_1',a_2+a_2'}(b), \tag{5.11}$$

for every $b \in B$. Furthermore, it follows from (5.9) that

$$\overline{g}_{a,a} = 0, \quad \text{for all} \quad a \in A^+. \tag{5.12}$$

From this we obtain that

$$\overline{g}_{a_1,a_2} + \overline{g}_{a_2,a_1} = \overline{g}_{a_1+a_2,a_1+a_2} = 0,$$

or, equivalently, that

$$\overline{g}_{a_2,a_1} = -\overline{g}_{a_1,a_2}. \tag{5.13}$$

We next show that if $(a_1, a_2), (a_1', a_2') \in (A^+ \oplus A^+)^0$ are such that

$$i_A(a_1) - i_A(a_2) = i_A(a_1') - i_A(a_2'), \tag{5.14}$$

then

$$\overline{g}_{a_1,a_2} = \overline{g}_{a_1',a_2'} \tag{5.15}$$

Indeed, by (5.13) we have $i_A(a_1 + a_2') = i_A(a_2 + a_1')$, so there exits an $a_3 \in A^+$ such that $a_1 + a_2' + a_3 = a_1 + a_1' + a_3$. We thus obtain by (5.13), (5.12) and (5.11) that

$$\overline{g}_{a_1,a_2} - \overline{g}_{a_1',a_2'} = \overline{g}_{a_1,a_2} + \overline{g}_{a_2',a_1'} + \overline{g}_{a_3,a_3} = \overline{g}_{a_1+a_2'+a_3,a_2+a_1'+a_3} = 0,$$

which proves (5.15). Thus, the rule

$$\overline{g}(a,b) = g_{a_1,a_2}(b), \quad \text{where} \quad a = i_A(a_1) - i_A(a_2),$$

defines a map $\overline{g} : A^0 \times B \to D$ which is bi-additive by (5.11) and (5.13).

We now investigate some of the properties of the pairing Δ. Its elementary properties are summarized in the next proposition.

Proposition 5.13 *a) For any non-degenerate hermitian RG-module* (M,h) *and any* $x = [M_1, h_1] - [M_2, h_2] \in \mathcal{H}(RG, L)^0$ *we have*

(5.16) $$d_G((M_1,h_1) \otimes (M,h)) = \Delta(x,M) d_G((M_2,h_2) \otimes (M,h)).$$

b) If $x, y \in \mathcal{H}(RG, L)^0$ *and* M, M' *are non-degenerate RG-modules, then*

$$\begin{aligned} \Delta(x, M \oplus M') &= \Delta(x,M) \cdot \Delta(x,M') \\ \Delta(x+y, M) &= \Delta(x,M) \cdot \Delta(y,M) \end{aligned}$$

c) If $x \in \mathcal{H}(RG, L)^0$ *and* M, M' *are non-degenerate RG-modules, then for any non-degenerate hermitian RG-module structure* h *on* M *we have*

(5.17) $$\Delta(x, M \otimes M') = \Delta(x \cdot [M,h], M').$$

In particular,

(5.18) $$\Delta(x, M) = \Delta(x \cdot [M,h], R).$$

d) Let R' *be a principal ideal domain which is an extension ring of* R *contained in an extension field* L' *of* L*. Then there is an induced "base change" map*

$$\mu : \mathcal{H}(RG, L) \to \mathcal{H}(R'G, L')$$

and we have for each non-degenerate RG-module M *and* $x \in \mathcal{H}(RG, L)^0$ *the compatibility relation*

(5.19) $$\Delta_{R'G,L'}(\mu(x), M \otimes R') = \Delta_{RG,L}(x, M) \cdot R'.$$

Proof. a), b) are clear from the construction.

c) Write $y = [M,h], z = [M', h']$, where h' is some non-degenerate hermitian *RG*-module structure on M'. Then by definition

$$\Delta(x, M \otimes M') = d_{RG}(x \cdot (y \cdot z)) = d_{RG}((x \cdot y) \cdot z) = \Delta(x \cdot y, M'),$$

which proves the first assertion. The second follows by taking $(M', h') = (R, id)$.

d) Since the map $\mu^+(M,h) = (M \otimes R', h \otimes R')$ defines a homomorphism of semi-rings $\mu^+ : \mathcal{H}(RG, L)^+ \to \mathcal{H}(R'G, L')^+$, we have by the universal property of Grothendieck rings an induced ring homomorphism $\mu : \mathcal{H}(RG, L) \to \mathcal{H}(R'G, L')$. From Proposition 5.4b) (and (5.16)) it follows that formula (5.19) is valid.

5.4. The pairing δ. We now come to the definition of the invariant δ. For this, it is useful to introduce the following additional Grothendieck rings.

Notation 5.14 Recall from Example 4.1 that a hermitian RG-module (M, h) is called *unimodular* if $h : M \xrightarrow{\sim} M^*$ is an RG-module isomorphism. Note that in this case M is automatically R-torsionfree, or an RG-lattice in the terminology of Curtis-Reiner[CR1].

Let $\mathcal{H}(RG)^+_{uni}$ denote the semi-ring consisting of isometry classes of *unimodular* hermitian RG-modules, and let $\mathcal{H}(RG)^+_{uni}$ denote the associated Grothendieck ring:

$$\mathcal{H}(RG)_{uni} = \{[M_1, h_1] - [M_2, h_2] : (M_i, h_i) \text{ is unimodular for } i = 1, 2\}.$$

Then by the universal property of Grothendieck rings, the inclusion $\mathcal{H}(RG)^+_{uni} \subset \mathcal{H}(RG, L)^+$ induces a ring homomorphism

$$\eta_{RG,L} : \mathcal{H}(RG)_{uni} \to \mathcal{H}(RG, L).$$

Furthermore, if we let $Mod(RG)_{sym}$ denote the Grothendieck ring associated to the semi-ring $Mod(RG)^+_{sym}$ consisting of isomorphism classes of *symmetric* (or *self-dual*) RG-modules $M \simeq M^*$, then we also have an induced map

$$\rho_{sym} : \mathcal{H}(RG)_{uni} \to Mod(RG)_{sym}$$

which is clearly surjective. We thus have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{H}(RG)_{uni} & \xrightarrow{\eta} & \mathcal{H}(RG, L) & & \\ \downarrow \rho_{sym} & & \downarrow \rho & \searrow \kappa & \\ Mod(RG)_{sym} & \xrightarrow{\overline{\eta}} & Mod(RG)_{non-deg} & \xrightarrow{\overline{\kappa}} & K_0(KG) \end{array} \tag{5.20}$$

Keeping in harmony with earlier notation, we let

$$\begin{aligned} Mod(RG)^0_{sym} &= \{[M_1] - [M_2] \in Mod(RG)_{sym} : M_1 \otimes K \simeq M_2 \otimes K\} \\ &= Ker(\overline{\kappa} \circ \overline{\eta}). \end{aligned}$$

Theorem 5.15 *There is a unique (bi-additive) pairing*

$$\delta_{RG} : Mod(RG)^0_{sym} \times Mod(RG)_{non-deg} \to Id(R, K)$$

such that

$$\delta(\rho_{sym}(x), y) = \Delta(\eta(x), y) \tag{5.21}$$

for $x \in Mod(RG)^0_{sym}$ *and* $y \in Mod(RG)_{non-deg}$.

Proof. Since ρ_{sym} is surjective, the uniqueness assertion is clear. To prove existence, we shall apply Lemma 5.12 again. Here we take $A^+ = Mod(RG)^+_{sym}$, $B^+ = Mod(RG)^+_{non-deg}$, $f = \overline{\kappa} \circ \overline{\eta}$ and $D = Id(R, K)$. The map

$$g : (A^+ \oplus A^+)^0 \to D$$

is defined by

$$g((M_1, M_2), M) = \Delta([M_1, h_1] - [M_2, h_2], M),$$

where $h_i : M_i \xrightarrow{\sim} M_i^*$ is any unimodular hermitian RG-module structure on M_i, for $i = 1, 2$; note that by Corollary 5.6 the right hand side does not depend on the choice of h_i. Since g clearly satisfies (5.9), we obtain by the lemma the desired map δ.

Notation 5.16 Let $M_1 \simeq M_1^*$ and $M_2 \simeq M_2^*$ be two symmetric (or self-dual) RG-modules such that $M_1 \otimes K \simeq M_2 \otimes K$, and let M be a non- degenerate RG-module (i.e. $M \otimes K \simeq M^* \otimes K$). Then $x = [M_1] - [M_2] \in Mod(RG)^0_{sym}$ and $y = [M] \in Mod(RG)_{non-deg}$, and hence we can define the fundamental invariant

$$\delta(M_1, M_2; M) = \delta(x, y), \tag{5.22}$$

whose main properties will be studied in the next section.

Remark 5.17 Although there is no easy method for actually computing the invariant $\delta(M_1, M_2; M)$, Proposition 5.5 does give us an explicit formula which turns out to be useful in many cases. The drawback is that the formula depends a number of choices, as follows.

To be precise, there are three (independent) choices to be made: we need to choose a unimodular structure $h_i : M_i \xrightarrow{\sim} M_i^*$ on M_i for $i = 1, 2$ and also an RG-isogeny $f : M_1 \to M_2$, whose existence is guaranteed by Proposition 5.5. Once these choices have been made, there is a unique $\alpha \in Aut_{KG}(M_1^* \otimes K)$ such that $f^\# h_2 = \alpha \circ h_1$, and for any (non-degenerate) M the associated δ-invariant is given by the formula

$$\begin{aligned} \delta(M_1, M_2; M) \quad = \quad & \chi(inv_G(f \otimes id_M))^2 \det(inv_G(\alpha \otimes id_{M^* \otimes K}))^{-1} \\ & \cdot \chi(inv_G(M_1 \otimes M_{tor}))\chi(inv_G(M_2 \otimes M_{tor}))^{-1}; \end{aligned} \tag{5.23}$$

this follows from Proposition 5.5 and equation (5.21), combined with the fact that

$$(inv_G(M_i \otimes M))_{tor} = inv_G((M_i \otimes M)_{tor}) = inv_G(M_i \otimes M_{tor}),$$

because M_i is torsionfree.

6. The main properties of invariant δ

In this section we shall derive the main properties of the fundamental invariant $\delta(M_1, M_2; M)$ which was constructed in the previous section. For the most part, these properties are generalizations of the main theorems presented in section 2: the theorem on discriminant relations (Theorem 2.5), the Uniform Boundedness Theorem (Theorem 2.8), the theorems on induction and inflation (Theorems 2.12 and 2.14), behaviour with respect to exact sequences (Theorem 2.10), and the Triviality Theorem (Theorem 2.9).

6.1. Elementary properties. Recall from the previous section that the invariant $\delta(M_1, M_2; M)$ was constructed via the pairing

$$\delta : Mod(RG)^0_{sym} \times Mod(RG)_{non-deg} \to Id(R, K),$$

where the Grothendieck rings $Mod(RG)_{sym}$ and $Mod(RG)_{non-deg}$ are as defined in Notations 5.14 and 5.10, respectively, and $Mod(RG)^0_{sym}$ denotes the kernel of the map $\overline{\kappa} \circ \overline{\eta} : Mod(RG)_{sym} \to K_0(KG)$ (cf. Notation 5.14). Furthermore, $Id(R, K)$ denotes as in Notation 5.10 the group of fractional R-ideals. Then we have:

Theorem 6.1 (Discriminant Relations) *Let M_1 and M_2 be two symmetric (or self-dual) RG-modules with $M_1 \otimes K \simeq M_2 \otimes K$, and let M be a non-degenerate RG-module (i.e. $M \otimes K \simeq M^* \otimes K$). Then $x = [M_1] - [M_2] \in Mod(RG)^0_{sym}$ and $y = [M] \in Mod(RG)_{non-deg}$, and for the invariant $\delta(M_1, M_2; M) = \delta(x, y)$ the following properties are valid:*

a) (Discriminant relation) *For any unimodular hermitian RG-module structure h_i on M_i $(i = 1, 2)$ and any L-valued hermitian RG-module structure h on M we have*

$$disc(Inv_G(M_1 \otimes M, h_1 \otimes h)) = \delta(x, y) \cdot disc(Inv_G(M_2 \otimes M, h_2 \otimes h)). \tag{6.1}$$

b) (Additivity) *δ is bi-additive: if M_1', M_2' and M' satisfy the same hypotheses as M_1, M_2 and M respectively, then we have:*

$$\begin{aligned} \delta(M_1 \oplus M_1', M_2 \oplus M_2'; M) &= \delta(M_1, M_2; M) \cdot \delta(M_1', M_2'; M); \\ \delta(M_1, M_2; M \oplus M') &= \delta(M_1, M_2; M) \cdot \delta(M_1, M_2; M'). \end{aligned}$$

c) (Symmetry) *If $M_3 \simeq (M_3)^*$ is symmetric then*

$$\delta(M_1, M_2; M_3 \otimes M) = \delta(M_1 \otimes M_3, M_2 \otimes M_3; M). \tag{6.2}$$

d) (Base change) *If $R' \supset R$ is a principal ideal domain which contains R then we have*

$$\delta_{R'}(M_1 \otimes R', M_2 \otimes R'; M \otimes R') = \delta(M_1, M_2; M) \cdot R'. \tag{6.3}$$

e) (Localization) *We have*

$$\delta_R(M_1, M_2; M) = \bigcap_{\mathfrak{p}} \delta_{R_\mathfrak{p}}((M_1)_\mathfrak{p}, (M_2)_\mathfrak{p}; M_\mathfrak{p}). \tag{6.4}$$

In particular, if M_1 and M_2 are in the same genus class (i.e. $(M_1)_\mathfrak{p} \simeq (M_2)_\mathfrak{p}$, for all $\mathfrak{p} \in Spec(R)$), then

$$\delta(M_1, M_2; M) = R. \tag{6.5}$$

Similarly, if M' is in the same genus class as M then

$$\delta(M_1, M_2; M) = \delta(M_1, M_2; M'). \tag{6.6}$$

f) (Support) *Let $n = |G|$. Then*

$$\delta(M_1, M_2; M) \cdot R[\frac{1}{n}] = R[\frac{1}{n}]. \tag{6.7}$$

Thus $\delta(M_1, M_2; M)$ is supported on the prime ideals dividing n.

Proof. a) This follows immediately from the construction of δ, in particular from equations (5.21) and (5.16).

b) δ is bi-additive by construction.

c) Write $z = [M_3] \in Mod(RG)_{sym}$. Then $\delta(x \cdot z, y) = \delta(x, \overline{\eta}(z) \cdot y)$ by associativity of the tensor product, which proves (6.2).

d) In view of (5.21), this follows directly from Proposition 5.13d).

e) As in the proof of Corollary 3.13, this follows immediately from d).

f) By d) we may assume that $R = R[\frac{1}{n}]$. Since by hypothesis $M_1 \otimes K \simeq M_2 \otimes K$, it follows from Curtis-Reiner[CR1], p. 642-643 (and p. 582) that M_1 and M_2 are in the same genus class, and so the assertion follows from e).

Remark 6.2 a) In the sequel it will be convenient to have a name for *RG*-modules M_1, M_2 satisfying the hypotheses of Theorem 6.1. Thus, we shall call a pair (M_1, M_2) of *RG*-modules *admissible* if each M_i is symmetric (i.e. $M_i \simeq M_i^*$) and if $M_1 \otimes K \simeq M_2 \otimes K$.

b) The fact that the invariant $\delta(M_1, M_2; M)$ is defined via the pairing δ on the Grothendieck rings implies a number of other relations as well; for example, if (M_1, M_2) and (M_2, M_3) are admissible pairs, then so is (M_1, M_3) and we have

$$\delta(M_1, M_2; M)\delta(M_2, M_3; M) = \delta(M_1, M_3; M). \tag{6.8}$$

As was mentioned in the introduction, most of the interesting examples of *RG*-modules M for which we want to compute $\delta(M)$ are those involving torsion. Since *RG*-lattices are technically easier to handle, it is therefore often useful to know the relation between the invariant $\delta(M_1, M_2; M)$ attached to M and between the invariant $\delta(M_1, M_2; \overline{M})$ attached to the torsionfree part $\overline{M} = M/M_{tor}$ of M. While no simple relation seems to exist, we do have the following result:

Proposition 6.3 *Let M_1, M_2 and M be as above. Then we have*

$$\delta(M_1, M_2; M) = c_{RG}(M_1, M)c_{RG}(M_2, M)^{-1}\delta(M_1, M_2; \overline{M}), \tag{6.9}$$

where for $i = 1, 2$ *the correction term* $c_{RG}(M_i, M)$ *is defined by*

$$\begin{aligned} c_{RG}(M_i, M) &= \chi(inv_G(M_i \otimes M_{tor}))\chi(inv_G(id_{M_i} \otimes \pi_M))^2 \\ &= \chi(inv_G(M_i \otimes M_{tor}))^{-1}\chi(Coker(inv_G(id_{M_i} \otimes \pi_M)))^2. \end{aligned} \tag{6.10}$$

Proof. We shall apply formula (5.23) of Remark 5.17. With the notation there we obtain

$$\begin{aligned} \delta(M_1, M_2; M)\delta(M_1, M_2; \overline{M})^{-1} &= \chi(inv_G(f \otimes id_M))^2\chi(inv_G(f \otimes id_{\overline{M}}))^{-2} \\ &\quad \cdot\chi(inv_G(M_1 \otimes M_{tor}))\chi(inv_G(M_2 \otimes M_{tor}))^{-1} \\ &= \chi(inv_G(id_{M_1} \otimes \pi_M))^2\chi(inv_G(id_{M_2} \otimes \pi_M))^{-2} \\ &\quad \cdot\chi(inv_G(M_1 \otimes M_{tor}))\chi(inv_G(M_2 \otimes M_{tor}))^{-1} \\ &= c_{RG}(M_1, M)c_{RG}(M_2, M)^{-1}. \end{aligned}$$

Here the first equation follows from (5.23) together with the fact that $\overline{M}^* = M^*$, so the two determinant factors cancel each other out, and the second follows from the identity

$$\chi(inv_G(id_{M_2} \otimes \pi_M))\chi(inv_G(f \otimes id_M)) = \chi(inv_G(f \otimes id_{\overline{M}}))\chi(inv_G(id_{M_1} \otimes \pi_M))$$

which in turn follows by applying Proposition 3.2 to the commutativity formula

$$inv_G(id_{M_2} \otimes \pi_M) \circ inv_G(f \otimes id_M) = inv_G(f \otimes id_{\overline{M}}) \circ inv_G(id_{M_1} \otimes \pi_M).$$

6.2. The Uniform Boundedness Theorem. We now turn to the Uniform Boundedness Theorem which, as was explained in section 2, may be viewed as a *quantitative* generalization of Brauer's Finiteness Theorem (Corollary 1.3).

In order to be able to state as precise a statement as possible, we introduce the following interesting invariant.

Notation 6.4 Let M and M' be two RG-lattices with $M \otimes K \simeq M' \otimes K$. Then the *genus defect* of M relative to M' is defined as the R-ideal

$$\gamma_{RG}(M : M') = \sum_{M'' \in \Gamma} \chi(M/M''), \tag{6.11}$$

where the sum extends over the set $\Gamma = \Gamma(M, M') = \{M'' \subset M : M'' \vee M'\}$ of all RG-sublattices $M'' of M$ which are in the same genus class as M' (i.e. $M'' \vee M'$ in the notation of [CR1]).

Remark 6.5 It is useful to observe that there always exists an RG-submodule $M_{max} \in \Gamma(M, M')$ such that

$$\chi(M/M_{max}) = \gamma(M : M'); \tag{6.12}$$

any such M_{max} is clearly maximal among the RG-submodules in Γ, but there may be more than one such submodule.

To see that at least one such submodule M_{max} exists, note first that this is clear if $R = R_{\mathfrak{p}}$ is a local ring (hence a discrete valuation ring) because in that case the R-ideals are linearly ordered. In the general case we thus have for each $\mathfrak{p} \in Spec(R)$ an $R_{\mathfrak{p}}G$-submodule $M''(\mathfrak{p}) \subset M_{\mathfrak{p}}$ with $M''(\mathfrak{p}) \simeq M'_{\mathfrak{p}}$. Put $M'' = \cap\, M''(\mathfrak{p})$; then $M''_{\mathfrak{p}} = M''(\mathfrak{p})$ (cf. [BCA], VII.4.3) and

$$\chi(M/M'') = \bigcap \chi(M_{\mathfrak{p}}/M''_{\mathfrak{p}}) = \bigcap \gamma_{R_{\mathfrak{p}}}(M_{\mathfrak{p}} : M'_{\mathfrak{p}}) \supset \gamma_R(M : M').$$

On the other hand, since by construction $M'' \vee M'$, it follows that $\chi(M/M'') \subset \gamma(M : M')$ and hence we see that (6.12) holds for $M_{max} = M''$.

The same argument also shows that we have the *localization formula*

$$\gamma_{RG}(M : M') = \bigcap_{\mathfrak{p}} \gamma_{R_{\mathfrak{p}}G}(M_{\mathfrak{p}} : M'_{\mathfrak{p}}). \tag{6.13}$$

Note that as a special case of (6.12) we see that M and M' are in the same genus class (i.e. $M \vee M'$) if and only if $\gamma(M : M') = R$; this justifies the name “genus defect”.

Theorem 6.6 (Uniform Boundedness) *If M_1 and M_2 are symmetric RG-modules of rank d with $M_1 \otimes K \simeq M_2 \otimes K$, define $r \in R$ and $N \in \mathbb{N}$ by the formulae*

$$rR = n^{2d}\gamma(M_2 : M_1)^4, \quad N = (2n+1)d,$$

where, as before, $n = |G|$. Then for any non-degenerate RG-module M the ideals

$$\chi(M_{tor})^N r^{rk(M)} \delta(M_1, M_2; M) \quad \text{and} \quad \chi(M_{tor})^N r^{rk(M)} \delta(M_1, M_2; M)^{-1}$$

are R-integral. In other words, for any prime ideal $\mathfrak{p}$ of R, the $\mathfrak{p}$-exponent of $\delta(M_1, M_2; M)$ is uniformly bounded:

$$|v_{\mathfrak{p}}(\delta(M_1, M_2; M))| \leq rk(M) v_{\mathfrak{p}}(r) + N v_{\mathfrak{p}}(\chi(M_{tor})).$$

Proof. In view of the localization formula (6.13) we may assume that $R = R_{\mathfrak{p}}$ is local. In this case

$$\gamma(M_2 : M_1) = \sum_f \chi(f),$$

where the sum extends over all RG-isogenies $f : M_1 \to M_2$. It is thus enough to show that

$$(6.14)\quad |v_{\mathfrak{p}}(\delta(M_1, M_2; M))| \leq (2n+1)dv_{\mathfrak{p}}(\chi(M_{tor})) + 2drv_{\mathfrak{p}}(n) + 4rv_{\mathfrak{p}}(\chi(f)),$$

for every RG-isogeny $f : M_1 \to M_2$, where $r = dim(M \otimes K)$.

Fix such an isogeny f. To compute $v_{\mathfrak{p}}(\delta(M_1, M_2; M))$, we shall use formula (5.23) of Remark 5.17. Thus, if $h_i : M_i \xrightarrow{\sim} M_i^*$ is a unimodular structure on M_i for $i = 1, 2$, then $\alpha = f^{\#}h_2 \circ h_1^{-1} = f^* \circ h_2 \circ f \circ h_1^{-1} \in End_{RG}(M_1)$, and we have by equations (3.7), (3.11) and (3.19) that

$$(6.15)\qquad \det(\alpha) \cdot R = \chi(f^*)\chi(f) = \chi(f)^2.$$

Now by Proposition 6.3 and formula (5.23) applied to $\overline{M}$ we have

$$\begin{aligned}(6.16)\quad \delta(M_1, M_2; M) &= c_{RG}(M_1, M)c_{RG}(M_2, M)^{-1}\delta(M_1, M_2; \overline{M}) \\ &= c_{RG}(M_1, M)c_{RG}(M_2, M)^{-1} \\ &\quad \cdot\chi(inv_G(f \otimes id_{\overline{M}}))^2 \det(inv_G(\alpha \otimes id_{\overline{M}^* \otimes K}))^{-1}.\end{aligned}$$

Applying the results from the Lemmata 6.7, 6.9 and 6.10 below we therefore deduce from (6.16) that

$$\begin{aligned}|v_{\mathfrak{p}}(\delta(M_1, M_2; M))| &\leq |v_{\mathfrak{p}}(c_{RG}(M_1, M)) - v_{\mathfrak{p}}(c_{RG}(M_2, M))| \\ &\quad + 2|v_{\mathfrak{p}}(\chi(inv_G(f \otimes id_{\overline{M}})))| + |v_{\mathfrak{p}}(\det(inv_G(\alpha \otimes id_{\overline{M}^* \otimes K})))| \\ &\leq (2n+1)dv_{\mathfrak{p}}(\chi(M_{tor})) + 2drv_{\mathfrak{p}}(n) + 2rv_{\mathfrak{p}}(\chi(f)) + rv_{\mathfrak{p}}(\det(\alpha)) \\ &= (2n+1)dv_{\mathfrak{p}}(\chi(M_{tor})) + 2drv_{\mathfrak{p}}(n) + 4rv_{\mathfrak{p}}(\chi(f)),\end{aligned}$$

where the last equality follows from (6.15).

We thus see that the inequality (6.14) and hence Theorem 6.6 are proved once we have established the validity of the following lemmata.

Lemma 6.7 *For any prime ideal* $\mathfrak{p}$ *of* R *we have*

$$-d \cdot v_{\mathfrak{p}}(\chi(M_{tor})) \leq v_{\mathfrak{p}}(c_{RG}(M_i, M)) \leq 2nd \cdot v_{\mathfrak{p}}(\chi(M_{tor})).$$

Proof. By definition

$$(6.17)\quad c_{RG}(M_i, M)\chi(inv_G(M_i \otimes M_{tor})) = \chi(Coker(inv_G(id_{M_i} \otimes \pi_M)))^2$$

is R-integral, so we obtain the inequality

$$\begin{aligned}v_{\mathfrak{p}}(c_{RG}(M_i, M)) &\geq -v_{\mathfrak{p}}(\chi(inv_G(M_i \otimes M_{tor}))) \\ &\geq -v_{\mathfrak{p}}(\chi(M_i \otimes M_{tor})) = -dv_{\mathfrak{p}}(\chi(M_{tor})),\end{aligned}$$

since $M_i \otimes M_{tor} \simeq M_{tor}^d$ as R-modules. This proves the lower bound.

To prove the upper bound we first note that

$$Coker(inv_G(id_{M_i} \otimes \pi_M)) \simeq Im(c) \subset H^1(G, M_i \otimes M_{tor}), \tag{6.18}$$

where c denotes the connecting homomorphism in the cohomology sequence

$$H^0(G, M_i \otimes M) \to H^0(G, M_i \otimes \overline{M}) \xrightarrow{c} H^1(G, M_i \otimes M_{tor})$$

which results by taking cohomology of the exact sequence

$$0 \to M_i \otimes M_{tor} \to M_i \otimes M \to M_i \otimes \overline{M} \to 0.$$

Furthermore, for any RG-module T which is R-torsion we have that

$$\chi(H^1(G,T)) \supset \chi(T)^n \tag{6.19}$$

because $H^1(G,T)$ is a subquotient of $Hom_R(RG,T) \simeq T^n$ (as R-modules). Applying this to $T = M_i \otimes M_{tor}$ we obtain from equations (6.17), (6.18) and (6.19) that

$$\begin{aligned} v_\mathfrak{p}(c_{RG}(M_i, M)) &\leq 2v_\mathfrak{p}(Coker(inv_G(id_{M_i} \otimes \pi_M))) \\ &\leq 2v_\mathfrak{p}(\chi(H^1(G, M_i \otimes M_{tor}))) \\ &\leq 2n \cdot v_\mathfrak{p}(\chi(M_i \otimes M_{tor})) \\ &= 2nd \cdot v_\mathfrak{p}(\chi(M_{tor})), \end{aligned}$$

which proves the desired upper bound.

Lemma 6.8 *Let $u : L_1 \to L_2$ be an isogeny of RG-lattices of rank s, and let $inv_G(u) : Inv_G(L_1) \to Inv_G(L_2)$ denote the induced map on the invariant spaces. Then we have*

$$n^s\chi(u) \subset \chi(inv_G(u)). \tag{6.20}$$

Proof. Let $L'_i = N'_G L_i$ and $L''_i = L_i^G \oplus L'_i$. Since u is RG-linear, it induces R-homomorphisms $u^G : L_1^G \to L_2^G$, $u' : L'_1 \to L'_2$ and $u'' : L''_1 \to L''_2$. Now since $u'' = u^G \oplus u'$ and $nL_i \subset L''_i \subset L_i$ (cf. the proof of Lemma 5.3b)), we see that all these maps are R-isogenies. Thus, by Corollary 3.3 it follows that

$$\chi(u^G)\chi(u') = \chi(u'') = \chi(u)\chi(L_1/L''_1)\chi(L_2/L'''_2)^{-1}.$$

Since all the χ-ideals are R-integral, we obtain the inclusions

$$\chi(u^G) \supset \chi(u'') \supset \chi(u)\chi(L_1/L''_1).$$

But $\chi(L_1/L''_1) \supset \chi(L_1/nL_1) = n^sR$, and so the assertion follows.

Corollary 6.9 *In the situation of the theorem we have*

$$|v_{\mathfrak{p}}(\chi(inv_G(f\otimes id_{\overline{M}})))| \le drv_{\mathfrak{p}}(n) + rv_{\mathfrak{p}}(\chi(f)).$$

Proof. Apply Lemma 6.8 to $L_i = M_i \otimes \overline{M}$ and $u = f \otimes id_{\overline{M}}$; note that here $s = dr$. Thus the assertion follows from (6.20) because we have $\chi(u) = \chi(f)^r$.

Lemma 6.10 *Let*

$$ch_\alpha(t) = (t-t_1)\cdots(t-t_d)$$

denote the characteristic polynomial of $\alpha \otimes K \in Aut_{KG}(V)$*, where* $V = M_1^* \otimes K$*. Then the characteristic polynomial of* $\alpha' = inv_G(\alpha \otimes id_{M^*\otimes K})$ *is of the form*

$$ch_{\alpha'}(t) = (t-t_1)^{n_1}\cdots(t-t_d)^{n_d},$$

for suitable integers $0 \le n_k \le r$*, where* $1 \le k \le d$*. In particular,* $\det(\alpha') \in R$ *and*

$$\det(\alpha') \mid \det(\alpha)^r.$$

Proof. Write $W = M^* \otimes K$. Since dim(W) = r, the characteristic polynomial of $\alpha \otimes id_W$ on $V \otimes W$ is $ch_\alpha(t)^r$. By Lemma 5.3b) we have $V \otimes W = V' \oplus V''$ where $V' = Inv_G(V \otimes W)$ and $V'' = N'_G(V \otimes W)$. Since α is RG-linear, the map $\alpha \otimes id$ maps each of the spaces V' and V'' into itself. Thus, if $\alpha'' = (\alpha \otimes id)_{|V''}$, then we obtain

$$ch_{\alpha'}(t)ch_{\alpha''}(t) = ch_{\alpha\otimes id}(t) = ch_\alpha(t)^r.$$

Thus $ch_{\alpha'}(t) \mid ch_\alpha(t)^r$, which proves the first statement. Since $\alpha \in End(M_1)$, the coefficients of $ch_\alpha(t)$ are in R, and hence the same is true for $ch_{\alpha'}(t)$. By Gauss's lemma it follows that

$$ch_{\alpha'}(t) \mid ch_\alpha(t)^r \ \text{ in } \ R[t];$$

in particular, the constant coefficient of the former polynomial divides that of the latter, which proves the second statement.

6.3. Frobenius Reciprocity Formulae. We now turn to developing the induction/restriction formalism of the invariant δ. Here we shall first focus on the behaviour of the invariants d_{RG} and Δ, and then later apply this to δ itself. We begin by introducing the following notation.

Notation 6.11 Let $f : G_1 \to G_2$ be a group homomorphism. The restriction map

$$res_f : \mathcal{H}(RG_2, L)^+ \to \mathcal{H}(RG_1, L)^+$$

as defined in section 4 induces a ring homomorphism

$$res_f : \mathcal{H}(RG_2, L) \to \mathcal{H}(RG_1, L)$$

of the associated Grothendieck rings. Furthermore, the induction map

$$ind_f : \mathcal{H}(RG_1, L)^+ \to \mathcal{H}(RG_2, L)^+$$

defined in Definition 4.2 induces a homomorphism

$$ind_f : \mathcal{H}(RG_1, L) \to \mathcal{H}(RG_2, L)$$

of the associated Grothendieck groups. Similarly, the process of coinduction as defined in Definition 4.10 defines an (additive) homomorphism

$$coind_f : \mathcal{H}(RG_1, L) \to \mathcal{H}(RG_2, L).$$

In a similar manner we have a ring homomorphism

$$res_f : Mod(RG_2)_{non-deg} \to Mod(RG_1)_{non-deg}$$

and group homomorphisms

$$ind_f, coind_f : Mod(RG_1)_{non-deg} \to Mod(RG_2)_{non-deg};$$

it is clear that these commute with the forget maps ρ_{G_1} and ρ_{G_2}:

$$\rho_{G_1} \circ res_f = res_f \circ \rho_{G_2}, \quad (co)ind_f \circ \rho_{G_1} = \rho_{G_2} \circ (co)ind_f.$$

Similarly, we have induced maps

$$res_f : K_0(KG_2) \to K_0(KG_1), \quad (co)ind_f : K_0(KG_1) \to K_0(KG_2),$$

which are compatible with the maps κ_{G_1} and κ_{G_2} defined in Notation 5.10:

$$res_f \circ \kappa_{G_2} = \kappa_{G_1} \circ res_f, \quad (co)ind_f \circ \kappa_{G_1} = \kappa_{G_2} \circ (co)ind_f,$$

In particular, we see that

$$res_f(\mathcal{H}(RG_2, L)^0) \subset \mathcal{H}(RG_1, L)^0, \quad ind_f(\mathcal{H}(RG_1, L)^0) \subset \mathcal{H}(RG_2, L)^0, \tag{6.21}$$

and similarly for $coind_f$.

Remark 6.12 Most of the results of section 4 translate into structural properties of the maps res_f, ind_f and $coind_f$. Specifically, we have:

a) If $f : G_1 \to G_2$ is an injective group homomorphism, then by Proposition 4.5 we have the formula

$$ind_f(x \cdot res_f(y)) = ind_f(x) \cdot y, \tag{6.22}$$

which means that ind_f is a homomorphism of $\mathcal{H}(RG_2, L)$-modules, if we view $\mathcal{H}(RG_1, L)$ as an $\mathcal{H}(RG_2, L)$-module via res_f. A similar statement holds if we replace $\mathcal{H}(RG_i, L)$ by $Mod(RG_i)_{non-deg}$.

Similarly, by Proposition 4.18 we have that $coind_f$ is a module homomorphism if $|Ker(f)|$ is invertible in R. Moreover, the formula

$$coind_f(x \cdot res_f(y)) = coind_f(x) \cdot y, \tag{6.23}$$

is also valid if $y \in \mathcal{H}(RG_2, L)_{lat}$ lies in the subring $\mathcal{H}(RG_2, L)_{lat}$ generated by the hermitian RG_2-lattices.

b) If $f_1 : G_1 \to G_2$ and $f_2 : G_2 \to G_3$ are group homomorphisms, then by Propositions 4.6 and 4.13 we have

$$ind_{f_2 \circ f_1} = ind_{f_2} \circ ind_{f_1}, \quad coind_{f_2 \circ f_1} = coind_{f_2} \circ coind_{f_1} \tag{6.24}$$

c) From Corollaries 4.14 and 4.7 we obtain the relations

$$d_{RG_1} = d_{RG_2} \circ coind_f; \quad d'_{RG_1} = d'_{RG_2} \circ ind_f. \tag{6.25}$$

Furthermore, if f is injective, then

$$coind_f = ind_f \tag{6.26}$$

by Proposition 4.12a), so in this case we also have

$$d_{RG_1} = d_{RG_2} \circ ind_f. \tag{6.27}$$

We now formulate the "Frobenius Reciprocity Formulae" for the invariants d_{RG} and Δ.

Proposition 6.13 *Let $f : G_1 \to G_2$ be a group homomorphism with $|Ker(f)|$ invertible in R. Then we have*

$$d_{RG_2}(coind_f(x_1) \cdot x_2) = d_{RG_1}(x_1 \cdot res_f(x_2)), \tag{6.28}$$

if $x_i \in \mathcal{H}(RG_i, L)$, $i = 1, 2$. In particular, for $x_i \in \mathcal{H}(RG_i, L)^0$ and $y_i \in Mod(RG_i)_{non-deg}$, $i = 1, 2$, we have

$$\Delta_{RG_2}(coind_f(x_1), y_2) = \Delta_{RG_1}(x_1, res_f(y_2)), \tag{6.29}$$

$$\Delta_{RG_1}(res_f(x_2), y_1) = \Delta_{RG_2}(x_2, coind_f(y_1)). \tag{6.30}$$

Moreover, if f is injective then these formulae hold with $coind_f$ replaced by ind_f.

Proof. By (6.23) and (6.24) we have

$$\begin{aligned} d_{RG_2}(coind_f(x_1) \cdot x_2) &= d_{RG_2}(coind_f(x_1 \cdot res_f(x_2))) \\ &= d_{RG_1}(x_1 \cdot res_f(x_2)), \end{aligned}$$

which proves equation (6.28) and hence by (5.21) also the other two equations (6.29) and (6.30).

Remark 6.14 If $|Ker(f)|$ is not a unit in R, then formulae (6.28) – (6.30) remain valid if we impose certain additional restrictions on the x_i and y_i. Indeed, the above proof shows that formula (6.28) is still correct as long as $x_2 \in \mathcal{H}(RG_2, L)_{lat}$ (which was defined in Remark 6.12). Moreover, by appealing to the Reciprocity Theorem 4.22 in place of (6.23) and (6.24), we see that (6.28) is true whenever $x_1 = [R[S]]$ is the class of a permutation module. It thus follows that if either $x_1 = [R[S]]$ is of this type or if $y_2 \in Lat(RG_2)_{non-deg}$, the subring of $Mod(RG)_{non-deg}$ which is generated by the classes of (non-degenerate) RG-lattices, then formula (6.29) continues to hold, and similarly, formula (6.30) is true provided that $x_2 \in \mathcal{H}(RG_i, L)^0_{lat} = \mathcal{H}(RG_i, L)^0 \cap \mathcal{H}(RG_i, L)_{lat}$.

We now apply the above formulae to the invariant δ. Since the restriction map defines a ring homomorphism

$$res_f : \mathcal{H}(RG_2)_{uni} \to \mathcal{H}(RG_1)_{uni},$$

we see that the analogue of formula (6.30) is valid:

Theorem 6.15 (Coinduction) *Let $f : G_1 \to G_2$ be a group homomorphism, and let $x_2 \in Mod(RG_2)_{sym}$ and $y_1 \in Mod(RG_1)_{non-deg}$. Then:*

$$\delta_{RG_1}(res_f(x_2), y_1) = \delta_{RG_2}(x_2, coind_f(y_1)). \tag{6.31}$$

Proof. Choose $\tilde{x}_2 \in \mathcal{H}(RG_i)^0_{uni}$ such that $\rho_{sym}(\tilde{x}_2) = x_i$. Then $res(\tilde{x}_2) \in \mathcal{H}(RG_1)^0_{uni}$, and $\eta_{G_1}(res_f(\tilde{x}_2)) = res_f(\eta_{G_2}(\tilde{x}_2))$, so we have by (5.21) that

$$\delta_{RG_1}(res_f(x_2), y_1) = \Delta_{RG_1}(res_f(\eta_{G_2}(\tilde{x}_2)), y_1) = \Delta_{RG_2}(\eta_{G_2}(\tilde{x_2}), coind_f(y_1)),$$

the latter equality resulting from equation (6.30) (which is valid even if $|Ker(f)|$ is not a unit in R because $\eta_{G_2}(\tilde{x}_2) \in \mathcal{H}(RG_2, K)_{lat}$; cf. Remark 6.14). Applying (5.21) once more yields the result.

Corollary 6.16 *Let $H \trianglelefteq G$ be a normal subgroup of G, and put $Q = G/H$. If M_1 and M_2 are symmetric RQ-modules with $M_1 \otimes K \simeq M_2 \otimes K$, then for any non-degenerate RG-module M we have*

$$\delta_{RG}(M_1, M_2; M) = \delta_{RQ}(M_1, M_2; M^H) \tag{6.32}$$

if we view M_1 and M_2 as RG-modules and $M^H = Inv_H(M)$ as an RQ-module.

Proof. Let $f : G_1 = G \to G_2 = G/H$ denote the projection map. Then $Coind_f(M) = M^H$ (cf. Proposition 4.12), so the result follows from the theorem.

On the other hand, in general it is not possible to find an analogue of equation (6.30) for the invariant δ. The basic problem here is that the maps ind_f and $coind_f$ do not necessarily lift to maps between the Grothendieck rings $\mathcal{H}(RG_i)_{uni}$ of *unimodular* hermitian RG_i-modules, as Corollary 4.15 shows. However, we do have:

Proposition 6.17 *If $f : G_1 \to G_2$ is an injective group homomorphism and (M_1, h_1) is a unimodular hermitian RG_1-module, then the hermitian RG_2-module $Coind_f(M_1, h_1)$ is also unimodular.*

Proof. Since f is injective, we have that $Coind_f(M_1, h_1) \simeq Ind_f(M_1, h_1)$ by Proposition 4.12a). Now the definition of induction, particularly (4.10), shows that $h_2 = ind_f(h_1) : M_2 = RG_2 \otimes_{RG_1} M \to M_2^*$ is given by

$$h_2 = \rho_{f,M_1} \circ id_{RG_2} \otimes h_1.$$

Since $\rho_{f,M_1} : Ind_f(M_1^*) \to Ind_f(M_1)^*$ is an isomorphism (cf. Example 4.4a)) and $h_1 : M_1 \to M_1^*$ is an isomorphism by hypothesis, it follows that h_2 is also an isomorphism, as asserted.

Thus, if $f : G_1 \to G_2$ is injective, then we have by the proposition an induced map

$$ind_f : \mathcal{H}(RG_1)_{uni} \to \mathcal{H}(RG_1)_{uni}$$

such that $ind_f([M, h]) = [Ind_f(M, h)]$. Similarly, as is well known, we have a map

$$ind_f : Mod(RG_1)_{sym} \to Mod(RG_2)_{sym}.$$

It is clear from the definitions that these maps are compatible with the maps ρ_{sym}, η and $\overline{\eta}$ (cf. Notation 5.14), and so we obtain as in (6.21) that

$$(6.33)\quad ind_f(\mathcal{H}(RG_1)^0_{uni}) \subset \mathcal{H}(RG_2)^0_{uni}, \;\; ind_f(Mod(RG_1)^0_{sym}) \subset Mod(RG_2)^0_{sym}.$$

We thus obtain the following induction formulae for the invariant δ:

Theorem 6.18 (Induction) *Let $f : G_1 \to G_2$ be an injective group homomorphism. Then for $x_i \in Mod(RG_i)^0_{sym}$ and $y_i \in Mod(RG_i)_{non-deg}$ we have*

$$(6.34)\qquad \delta_{RG_2}(ind_f(x_1), y_2) = \delta_{RG_1}(x_1, res_f(y_2)),$$

$$(6.35)\qquad \delta_{RG_1}(res_f(x_2), y_1) = \delta_{RG_1}(x_2, ind_f(y_1)).$$

Proof. Since f is injective, we have that $ind_f = coind_f$ by (6.26), and so the second equation (6.35) is just a restatement of (6.31). Furthermore, in view of Proposition 6.17, a proof analogous to that of Theorem 6.16 yields equation (6.34).

As was already mentioned, it is not possible to extend Theorem 6.18 to arbitrary maps. However, if one restricts attention to permutation modules, then such an extension is indeed valid:

Theorem 6.19 (Coinduction for Permutation Modules) *Let $f : G_1 \to G_2$ be a group homomorphism. If S_1 and S_2 are G_1-sets such that $K[S_1] \simeq K[S_2]$, then $(Coind_f(R[S_1]), Coind_f(R[S_2]))$ is an admissible pair of RG_2-modules, and for any non-degenerate RG_2-module M_2 we have*

$$(6.36)\qquad \delta_{RG_2}(Coind_f(R[S_1]), Coind_f(R[S_2]); M_2) = \varepsilon_f(S_1, M_2)\varepsilon_f(S_2, M_2)^{-1}\delta_{RG_1}(R[S_1], R[S_2]; Res_f(M_2)),$$

where for $i = 1, 2$

$$\varepsilon_f(S_i, M_2) = \prod_{s \in G_1 \backslash S_i} |Ker(f) \cap (G_1)_s|^{rk(H^0(f((G_1)_s), M_2))}. \tag{6.37}$$

Proof. Since

$$R[S_i] \simeq \bigoplus_{s \in G_1 \backslash S_i} R[G_1/(G_1)_s] = \bigoplus_{X \leq G_1} R[G_1/X]^{n_i(X)}, \tag{6.38}$$

it follows from Corollary 4.15 that

$$Coind_f(R[S_i]) \simeq \bigoplus_{s \in G_1 \backslash S_i} R[G_2/f((G_1)_s)] = \bigoplus_{X \leq G_1} R[G_2/f(X)]^{n_i(X)}. \tag{6.39}$$

Thus $Coind_f(R[S_i])$ is again a permutation module, and hence is symmetric. Moreover, since tensoring with K commutes with the process of coinduction, we see that $Coind_f(R[S_1]) \otimes K \simeq Coind_f(R[S_2]) \otimes K$, which proves the first assertion.

To prove formula (6.36), put $n(X) = n_1(X) - n_2(X)$, where $n_i(X)$ is defined by equation (6.38), and choose a non-degenerate hermitian RG_2-module structure h_2 on M_2. Then by the definition of δ we have

$$\begin{aligned} &\delta_{RG_1}(R[S_1], R[S_2]; Res_f(M_2)) \\ &\quad = \prod_{X \leq G_1} d_{RG_1}((R[G_1/X], h_{G_1/X}) \otimes (Res_f(M), h_2))^{n(X)} \\ &\quad = \prod_{X \leq G_1} d_{RG_2}(Coind_f(R[G_1/X], h_{G_1/X}) \otimes (M_2, h_2))^{n(X)}; \end{aligned} \tag{6.40}$$

here we have used the reciprocity formula (6.28) which is applicable in this case by Remark 6.14.

For a subgroup $X \leq G_1$ put $k(X) = |X \cap Ker(f)|^{-1}$ and

$$r(f(X)) = rk(H^0(G_2, R[G_2/f(X)] \otimes M_2])) = rk(H^0(f(X), M_2)),$$

where the latter equality follows from Corollary 4.17. Now by Corollary 4.15 we have

$$Coind_f(R[G_1/X], h_{G_1/X}) \simeq (R[G_2/f(X)], h_{G_2/f(X)})(k(X)),$$

and so substituting this in the above computation (6.40) we obtain by Proposition 5.4d) that

$$\begin{aligned} &\delta_{RG_1}(R[S_1], R[S_2]; Res_f(M_2)) \\ &\quad = \prod_{X \leq G_1} d_{RG_2}((R[G_2/f(X)], h_{G_2/f(X)})(k(X)) \otimes (M_2, h_2)))^{n(X)} \\ &\quad = \prod_{X \leq G_1} (k(X)^{r(f(X))} d_{RG_2}((R[G_2/f(X)], h_{G_2/f(X)}) \otimes (M_2, h_2))))^{n(X)} \\ &\quad = (\prod_{X \leq G_1} k(X)^{n(X) r(f(X))}) \delta_{RG_2}(Coind_f(R[S_1]), Coind_f(R[S_2]); M_2), \end{aligned}$$

where the last equation follows from the definition of δ and (6.39). This proves (6.36) because the factor in front of δ is by (6.39) equal to $\varepsilon_f(S_1, M_2)\varepsilon_f(S_2, M_2)^{-1}$.

6.4. The exact sequence formula. We next investigate the behaviour of the invariant $\delta(M_1, M_2; M)$ with respect to exact sequences. Although δ is in general not additive on exact sequences, the difference can be understood in terms of group cohomology. To this end we first introduce the following notation.

Notation 6.20 Let $f : M' \to M''$ be a homomorphism of RG-modules. Then for any RG-module M we have the induced R-module homomorphism

$$f_M^G = inv_G(id_M \otimes f) : Inv_G(M \otimes M') \to Inv_G(M \otimes M'')$$

on the invariant spaces. We put

$$\psi_G(M, f) = \chi(Coker(f_M^G)).$$

Moreover, if N is an RG-module with $\psi_G(M, f) \neq 0$, then we put

$$\psi_G(M, N; f) = \psi_G(M, f)\psi_G(N, f)^{-1}.$$

With this notation we can now formulate the exact sequence formula, which may be viewed as a generalization of Proposition 6.3 which relates the δ-invariants of M, M_{tor} and $\overline{M} = M/M_{tor}$.

Theorem 6.21 (Exact Sequence Formula) *Let*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be an exact sequence of RG-modules. If any two of the modules M, M' and M'' are non-degenerate, then so is the third, and for any admissible pair (M_1, M_2) of RG-modules we have the relation

$$\delta(M_1, M_2; M) = \delta(M_1, M_2; M')\delta(M_1, M_2; M'')\psi(M_1, M_2; g)^2, \tag{6.41}$$

where $\psi(M_1, M_2; g) = \psi(M_1, g)\psi(M_2, g)^{-1}$ is as defined above. Furthermore, for $i = 1, 2$ we have

$$\psi(M_i, g) = \chi(Ker((id_{M_i} \otimes f)^1)), \tag{6.42}$$

where $(id_{M_i} \otimes f)^1 : H^1(G, M_i \otimes M') \to H^1(G, M_i \otimes M)$ denotes the induced map on cohomology.

Proof. Since KG is semi-simple we have $M \otimes K \simeq (M' \otimes K) \oplus (M'' \otimes K)$, and so we see that if $M' \otimes K \simeq (M')^* \otimes K$ and $M'' \otimes K \simeq (M'')^* \otimes K$, then also $M \otimes K \simeq M^* \otimes K$. On the other hand, if $M \otimes K \simeq M^* \otimes K$ and $M' \otimes K \simeq (M')^* \otimes K$, then also $M'' \otimes K \simeq (M'')^* \otimes K$ because the cancellation law is valid for KG-modules.

Since the same argument holds when M' and M'' are interchanged, this proves the first assertion.

We next prove equation (6.42). Since M_i is R-torsionfree, hence R-flat, the sequence

$$0 \to M_i \otimes M' \to M_i \otimes M \to M_i \otimes M'' \to 0$$

is exact, so by the long exact sequence of cohomology, viz.

$$(M_i \otimes M)^G \overset{g^G_{M_i}}{\to} (M_i \otimes M'')^G \overset{c}{\to} H^1(G, M_i \otimes M') \overset{(id\otimes f)^1}{\to} H^1(G, M_i \otimes M),$$

we obtain $Coker(g^G_{M_i}) \simeq Ker((id_{M_i} \otimes f)^1)$, and hence (6.42) follows.

Finally, to prove formula (6.41), let $t_0 : M_1 \to M_2$ be an RG-isogeny, let $h_i : M_i \to M_i^*$ be unimodular hermitian structures on M_i for $i = 1, 2$, and put $\alpha_0 = (t_0^{\#} h_2 \circ h_1^{-1}) \otimes K \in Aut_{KG}(V_1)$, where $V_1 = M_1^* \otimes K$. Furthermore, let

$$t = inv_G(t_0 \otimes id_M) : (M_1 \otimes M)^G \to (M_2 \otimes M)^G,$$

and $\alpha = inv_G(\alpha_0 \otimes id_V) \in Aut_{KG}(V_1 \otimes V)$, where $V = M^* \otimes K$, and define similarly t', α', V' (respectively, t'', α'', V'') by replacing M by M' (respectively by M''). Then by formula (5.23) we have

$$\begin{aligned} (6.43) \qquad d &\overset{def}{=} \delta(M_1, M_2; M)\delta(M_1, M_2; M')^{-1}\delta(M_1, M_2; M'')^{-1} \\ &= \chi(t)^2 \det(\alpha)^{-1} \cdot \chi(t')^{-2} \det(\alpha') \cdot \chi(t'')^{-2} \det(\alpha''). \end{aligned}$$

Now since the diagram

$$(6.44) \qquad \begin{array}{ccccccccc} 0 & \to & (V_1 \otimes V'')^G & \overset{(id\otimes g^*)^G}{\to} & (V_1 \otimes V)^G & \overset{(id\otimes f^*)^G}{\to} & (V_1 \otimes V')^G & \to & 0 \\ & & \downarrow \alpha'' & & \downarrow \alpha & & \downarrow \alpha' & & \\ 0 & \to & (V_1 \otimes V'')^G & \overset{(id\otimes g^*)^G}{\to} & (V_1 \otimes V)^G & \overset{(id\otimes f^*)^G}{\to} & (V_1 \otimes V')^G & \to & 0 \end{array}$$

is commutative with exact rows, we see that the determinant terms in (6.43) cancel out, and so we are left with

$$(6.45) \qquad d = \chi(t)^2 \chi(t')^{-2} \chi(t'')^{-2}.$$

On the other hand, since the diagram

$$(6.46) \qquad \begin{array}{ccccccc} 0 & \to & (M_1 \otimes M')^G & \overset{f^G_{M_1}}{\to} & (M_1 \otimes M)^G & \overset{g^G_{M_1}}{\to} & (M_1 \otimes M'')^G \\ & & \downarrow t' & & \downarrow t & & \downarrow t'' \\ 0 & \to & (M_2 \otimes M')^G & \overset{f^G_{M_2}}{\to} & (M_2 \otimes M)^G & \overset{g^G_{M_2}}{\to} & (M_2 \otimes M'')^G \end{array}$$

is also commutative with exact rows, we obtain from Proposition 3.4 the relation

$$\chi(t) = \chi(t')\chi(s''),$$

where $s'' : Im(g^G_{M_1}) \to Im(g^G_{M_2})$ denotes the restriction of t'' to $Im(g^G_{M_1})$. Now by Corollary 3.3 we have

$$\chi(s'') = \chi(t'')\chi(Coker(g^G_{M_1}))\chi(Coker(g^G_{M_2}))^{-1} = \chi(t'')\psi(M_1, M_2, g),$$

which yields

$$\chi(t) = \chi(t')\chi(t'')\psi(M_1, M_2, g).$$

Substituting this in (6.45) gives equation (6.41).

The following corollary will be used and strengthened in the next subsection.

Corollary 6.22 *If, in the situation of Theorem 6.21, M' is cohomologically trivial, then we have*

$$\delta(M_1, M_2; M) = \delta(M_1, M_2; M')\delta(M_1, M_2; M''). \tag{6.47}$$

Similarly, if M is cohomologically trivial then we have

$$\begin{aligned}\delta(M_1, M_2; M) &= \delta(M_1, M_2; M')\delta(M_1, M_2; M'') \\ &\quad \cdot\chi(H^1(G, M_1 \otimes M'))^2\chi(H^1(G, M_2 \otimes M'))^{-2}.\end{aligned} \tag{6.48}$$

Proof. If M' is cohomologically trivial, then so is $M_i \otimes M'$ (cf. Serre[Se], IX.3), and so $Ker((id_{M_i} \otimes f)^1) = H^1(G, M_i \otimes M') = 0$. This proves (6.47), and (6.48) is proved similarly.

6.5. Triviality. The last result which we shall prove here is the following useful formula which justifies in part the unusual definition of the discriminant introduced in section 3:

Theorem 6.23 (Triviality) *Let M be a non-degenerate RG-module which is cohomologically trivial. Then the associated δ-invariant*

$$\delta(M_1, M_2; M) = R$$

is trivial for any pair (M_1, M_2) of symmetric RG-modules with $M_1 \otimes K \simeq M_2 \otimes K$.

We shall prove this theorem first for projective modules and then for arbitrary cohomologically trivial modules by using the exact sequence formula.

The key to proving the theorem for projective modules is the following technical lemma concerning the map

$$\rho_M = \rho_{f,M} : R \otimes M^* \to (R \otimes M)^*$$

which was defined in Remark 4.4; here $f : G \to \{1\}$ denotes the trivial map. More explicitly, ρ_M is defined by the formula

$$\rho_M(r \otimes m^*)(r' \otimes m') = rr'm^*\Big(\sum_{g\in G} gm'\Big) = rr'm^*(N_G m'),$$

where $r, r' \in R, m^* \in M^*$ and $m' \in M$.

Lemma 6.24 *If M is a projective RG-module, then*

$$\rho_M : R \otimes_{RG} M^* \to (R \otimes M)^*$$

is an isomorphism.

Proof. Let us first suppose that $M = RG$. Then $R \otimes_{RG} RG^* = R$, and if $\{g^*\}_{g\in G}$ denotes a dual basis of the standard basis of RG, we have

$$\rho_M(1 \otimes g^*)(1 \otimes g') = g^*\Big(\sum_{g_1\in G} g_1 g'\Big) = 1,$$

for all $g, g' \in G$. Thus $\rho_M(1 \otimes g^*) = 1 \in Hom_R(R \otimes_{RG} RG, R) = R$, for all $g \in G$, which shows that ρ_M is an isomorphism in this case. Since ρ_M is additive in M, it follows that ρ_M is an isomorphism for any direct factor of a free RG-module, which proves the lemma.

Proposition 6.25 *If P is a projective RG-module with a unimodular hermitian RG-module structure $h : P \xrightarrow{\sim} P^*$, then*

$$H_0(G, (P, h)) \simeq H^0(G, (P, h)) \tag{6.49}$$

is a unimodular bilinear R-module. In particular, we have that

$$d_{RG}(P, h) = d'_{RG}(P, h) = R. \tag{6.50}$$

Proof. We first note that since P is projective we have $\hat{H}^{-1}(G, P) = \hat{H}^0(G, P) = 0$, and so by the exact sequence (4.23) and Remark 4.9 we obtain the desired isometry (6.49).

From (4.10) we see that the bilinear structure $ind(h)$ on $H_0(P) = Ind_f(P)$ is given by

$$ind_f(h) = \rho_{f,P} \circ id_R \otimes h,$$

where $f : G \to \{1\}$ denotes the trivial map. Now h is an isomorphism by hypothesis, hence so is $id \otimes h$, and thus it follows from Lemma 6.24 that $ind(h)$ is also an isomorphism, as asserted.

Corollary 6.26 *If (P, h) is as in the proposition, then for any unimodular hermitian RG-module (M, h') we have*

$$d_{RG}((M, h') \otimes (P, h)) = R.$$

In particular, for any admissible pair (M_1, M_2) the associated δ-invariant is trivial:

$$\delta(M_1, M_2; P) = R.$$

Proof. We first note that $P' = M \otimes P$ is again RG-projective. Indeed, it is cohomologically trivial by Serre[Se], IX.3, and hence projective by Serre[Se], IX.5, because P' is clearly R-torsionfree. Thus, since $(P', h'') = (P, h) \otimes (M, h')$ is again unimodular, the first equation follows directly from (6.50).

Moreover, the second equation follows from the first because

$$\delta(M_1, M_2; P) = d_{RG}((M_1, h_1) \otimes (P, h)) \cdot d_{RG}((M_2, h_2) \otimes (P, h))^{-1},$$

for any choice of a unimodular structure h_i on M_i, for $i = 1, 2$.

Remark 6.27 In the case that $(P, h) = (RG, h_G)$ is the regular hermitian RG-module, a more direct proof of this corollary can be given via the Frobenius Reciprocity Theorem. Indeed, since $(RG, h_G) \simeq Ind_H^G(R, id_R)$, where $H = \{1\}$ (cf. Example 4.4b)), we obtain by the Frobenius Reciprocity formula (6.28) (together with (6.26)) that

$$d_{RG}((M, h) \otimes (RG, h_G)) \simeq d_{RH}(Res(M, h) \otimes (R, id_R)) = disc_R(M, h) = R.$$

We have thus proven the Triviality Theorem for *symmetric* projective RG-modules. To go further, we shall use:

Lemma 6.28 *If R is a discrete valuation ring, then every projective non-degenerate RG-module P is symmetric.*

Proof. Since P is non-degenerate, we have an KG-isomorphism $P \otimes K \simeq P^* \otimes K$. But P^* is also projective (cf. Curtis-Reiner[CR1], (10.29)), so it follows from [CR1], (32.1), that $P \simeq P^*$, as asserted.

Lemma 6.29 *If M is a non-degenerate RG-module which is cohomologically trivial, then M has a presentation*

$$0 \to P' \to P \to M \to 0, \tag{6.51}$$

where P and P' are non-degenerate projective RG-modules.

Proof. Let $0 \to M' \to P \to M \to 0$ be an RG-module presentation of M with $P = RG^m$ a free module. Since M and P are non-degenerate, so is M' by Theorem 6.21. Furthermore, since M and P are cohomologically trivial, the long exact sequence of cohomology shows that this is also true for M'. Since $M' \subset P$ is R-torsionfree, it follows that $M' = P'$ is RG-projective (cf. Serre[Se], IX.5).

Proof of the Triviality Theorem. Suppose first that $M = P$ is a (non-degenerate) projective RG-module. If $\mathfrak{p} \in Spec(R)$ is any prime ideal of R, then $M_\mathfrak{p}$ is a symmetric $R_\mathfrak{p}G$-module by Lemma 6.28, and so $\delta_{R_\mathfrak{p}G}((M_1)_\mathfrak{p}, (M_2)_\mathfrak{p}; M_\mathfrak{p}) = R_\mathfrak{p}$ by Corollary 6.26. From the localization formula (6.4) we thus obtain that

$$\delta_{RG}(M_1, M_2 :, M) = \bigcap_{\mathfrak{p}} \delta_{R_\mathfrak{p}G}((M_1)_\mathfrak{p}, (M_2)_\mathfrak{p}; M_\mathfrak{p}) = \bigcap_{\mathfrak{p}} R_\mathfrak{p} = R,$$

which proves the theorem if M is projective.

Now suppose that M is an arbitrary cohomologically trivial module, and let P and P' be as in Lemma 6.29. Applying the exact sequence formula, notably Corollary 6.22, to (6.51), we obtain

$$\delta(M_1, M_2; M) = \delta(M_1, M_2; P)\delta(M_1, M_2; P')^{-1}.$$

Since P and P' are RG-projective, we have $\delta(M_1, M_2; P) = \delta(M_1, M_2; P') = R$ by the case treated above, and so the theorem is proved.

With the help of the Triviality Theorem, Corollary 6.22 can now be strengthened. It is convenient to separate the two cases of the corollary as follows.

Corollary 6.30 *Let*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be an exact sequence of non-degenerate RG-modules, and suppose that M' is cohomologically trivial. Then for any admissible pair (M_1, M_2) of RG-modules we have:

(6.52) $$\delta(M_1, M_2; M) = \delta(M_1, M_2; M'').$$

Proof. Since $\delta(M_1, M_2; M') = R$ by the Triviality Theorem, this follows immediately from (6.47).

Corollary 6.31 *Let*

$$0 \to M' \to P_r \to \cdots \to P_1 \to M'' \to 0$$

be an exact sequence of non-degenerate RG-modules, where the P_i are cohomologically trivial for $1 \le i \le r$. Fix an admissible pair (M_1, M_2) of RG-modules and write $\delta(M') = \delta(M_1, M_2; M')$ and $\delta(M'') = \delta(M_1, M_2; M'')$. Then we have:

(6.53) $$\begin{aligned} \delta(M') &= \delta(M'')^{(-1)^r} \cdot \chi^r(G, M_1 \otimes M')^2 \chi^r(G, M_2 \otimes M')^{-2}, \\ &= \delta(M'')^{(-1)^r} \cdot \chi_r(G, M_1 \otimes M'')^2 \chi_r(G, M_2 \otimes M'')^{-2}, \end{aligned}$$

where, for an RG-module M, the truncated Euler characteristics $\chi^r(G,M)$ *and* $\chi_r(G,M)$ *are defined by*

$$\chi^r(G,M) = \prod_{i=1}^{r} \chi(H^i(G,M))^{(-1)^{i+1}},$$
$$\chi_r(G,M) = \prod_{i=0}^{r-1} \chi(\hat{H}^{-i}(G,M))^{(-1)^{r-1-i}}.$$

Proof. We first note that the second equation follows from the first since for $i = 1, 2$ we have by the usual dimension-shifting argument that

$$\hat{H}^q(G, M_i \otimes M') \simeq \hat{H}^{q-r}(G, M_i \otimes M''), \quad \text{for all} \quad q \in \mathbb{Z};$$

here we have used the fact that $M_i \otimes P_k$ is cohomologically trivial for $1 \le k \le r$.

To prove the first equality in (6.53), we shall induct on r. If $r = 1$, then the assertion is just equation (6.48) of Corollary 6.22, combined with the fact that $\delta(P_1) = R$ by the Triviality Theorem; thus, assume $r > 1$.

Applying the induction hypothesis to the exact sequence

$$0 \to M' \to P_r \to \cdots P_2 \to M \to 0,$$

where $M = Ker(P_1 \to M'')$, yields

$$\delta(M') = \delta(M)^{(-1)^{r-1}} \cdot \chi^{r-1}(M_1 \otimes M')^2 \chi^{r-1}(M_2 \otimes M')^{-2},$$

and similarly, consideration of the exact sequence

$$0 \to M \to P_1 \to M'' \to 0$$

shows that

$$\delta(M) = \delta(M'')^{-1} \chi(H^1(G, M_1 \otimes M))^2 \chi(H^1(G, M_2 \otimes M))^{-2}.$$

Thus, by combining the last two equations we obtain

$$\delta(M') = \delta(M'')^{(-1)^r} \cdot c_1^2 c_2^{-2}, \tag{6.54}$$

where $c_i = \chi^{r-1}(M_i \otimes M')\chi(H^1(G, M_i \otimes M))^{(-1)^{r-1}}$. But by dimension-shifting as above we have

$$\hat{H}^q(G, M_i \otimes M) \simeq \hat{H}^{q+r-1}(G, M_i \otimes M'),$$

and so it follows that

$$c_i = \chi^{r-1}(M_i \otimes M')\chi(H^r(G, M_i \otimes M'))^{(-1)^{r-1}} = \chi^r(M_i \otimes M').$$

Substituting this into (6.54) yields the first equation of (6.53), as desired.

7. The invariant δ and character relations

We now interpret the results obtained in the previous section in terms of character relations. Since character relations are naturally associated with the *subgroup module* $\mathbb{S}(G)$, we first introduce this space. However, the functorial properties of character relations are best understood in terms of the Burnside ring $\Omega(G)$, and so we study this ring in some detail. Next we observe that the δ-invariant may be viewed as a pairing of a certain ideal $\Omega(G)^0 \subset \Omega(G)$ with the Grothendieck ring $Mod(RG)_{non-deg}$ of non-degenerate RG-modules, and this will then be used to translate the results of the previous section into the form in which they were presented in section 2.

7.1. The space of character relations. Recall from the introduction that a *character relation* is a relation of the form

$$\sum_{H \leq G} n_H 1_H^* = 0, \tag{7.1}$$

where the n_H are integers and $1_H^* = Ind_H^G(1_H)$ denotes the permutation character of the G-set G/H. To study such relations, consider the *subgroup module*

$$\mathbb{S}(G) = \sum_{H \leq G} \mathbb{Z} \cdot H$$

which is the free abelian group generated by the subgroups $H \leq G$. Thus, each element $x \in \mathbb{S}(G)$ may be written as a formal sum $x = \sum n_H H$, or as a tuple $x = \{n_H\}$; the latter was the notation used in the introduction.

We have a natural additive map

$$ch_G : \mathbb{S}(G) \to \mathbf{ch}(\mathbb{Q}G)$$

from the subgroup space $\mathbb{S}(G)$ to the character ring $\mathbf{ch}(\mathbb{Q}G) = K_0(\mathbb{Q}G)$ which associates to each subgroup $H \leq G$ its permutation character:

$$ch_G(H) = 1_H^*.$$

The kernel of the character map ch_G is called the *group of character relations* of G and is denoted by

$$\mathbb{S}(G)^0 = Ker(ch).$$

Clearly an element $x = \sum n_H H \in \mathbb{S}(G)$ lies in $\mathbb{S}(G)^0$ if and only if its coefficients satisfy the character relation (7.1); this justifies the above terminology for $\mathbb{S}(G)^0$.

For our purposes it will be also useful to consider the map

$$ch_{KG} : \mathbb{S}(G) \to \mathbf{ch}(KG) = K_0(KG)$$

defined by $ch_{KG}(X) = Ind_X^G(K)$ for $X \leq G$. However, this map is not really something substantially new for we have

$$ch_{KG} = j_{KG} \circ ch_G,$$

where $j_{KG} : K_0(\mathbb{Q}G) \to K_0(KG)$ denotes the canonical injection. (In the case that $char(K) = 0$, this is just the base-change map induced by the inclusion $\mathbb{Q} \subset K$, whereas when $char(K) = p \neq 0$, this is the map obtained by the composition $K_0(\mathbb{Q}G) \to K_0(\mathbb{Q}_pG) = K_0(\mathbb{F}_pG) \to K_0(KG)$.) It thus follows that

$$\mathbb{S}(G)^0 = Ker(ch_{KG}).$$

We note that the map ch_{KG} factors over the map

$$\overline{\kappa} \circ \overline{\eta} : Mod(RG)_{sym} \to K_0(KG)$$

defined in Notation 5.14; in fact, if we define the map

$$ch_{RG} : \mathbb{S}(G) \to Mod(RG)_{sym}$$

by $ch_{RG}(X) = [Ind_X^G(R)]$ for $X \leq G$, then we clearly have

$$ch_{KG} = \overline{\kappa} \circ \overline{\eta} \circ ch_{RG}. \tag{7.2}$$

In particular, we see that

$$ch_{RG}(\mathbb{S}(G)^0) \subset Mod(RG)^0_{sym}. \tag{7.3}$$

We next want to consider functorial properties of character relations. To this end, let $f : G_1 \to G_2$ be group homomorphism. We then have an induced additive map

$$f_* : \mathbb{S}(G_1) \to \mathbb{S}(G_2)$$

which is defined by the rule $f_*(X) = f(X)$ for $X \leq G_1$. We note that f_* preserves character relations, viz.

$$f_*(\mathbb{S}(G_1)^0) \subset \mathbb{S}(G_2)^0, \tag{7.4}$$

because by Corollary 4.15 we have the formula

$$coind_f \circ ch_{G_1} = ch_{G_2} \circ f_*. \tag{7.5}$$

It is more difficult, however, to construct a map f^* which relates the character relations of G_2 to those of G_1. Here we cannot work integrally, but have to allow denominators. This means that we can define f^* only as a $\mathbb{Q}$-linear map

$$f^* : \mathbb{S}_{\mathbb{Q}}(G_2) \to \mathbb{S}_{\mathbb{Q}}(G_1),$$

where $\mathbb{S}_{\mathbb{Q}}(G_i) = \mathbb{S}(G_i) \otimes \mathbb{Q}$. Explicitly, f^* is given by

$$(7.6) \qquad f^*(Y) = \frac{1}{|G_1|} \sum_{g \in G_2} \frac{|f(gYg^{-1})|}{|Y|} f^{-1}(gYg^{-1}) = \sum_{X \leq G_1} m_f(Y, X) X,$$

where $m_f(Y, X) = \#\{g \in G_2 : f^{-1}(gYg^{-1}) = X\}/([G_1 : X] \cdot |Y|)$. However, the fact that denominators appear here limits the usefulness of f^*, particularly since there is a way to circumvent this difficulty.

7.2. Burnside rings. It is possible to avoid denominators in the above definition of f^* if we replace the subgroup module $\mathbb{S}(G)$ by the *Burnside ring* $\Omega(G)$ of G. By definition, $\Omega(G)$ is the Grothendieck ring associated to the semi-ring $\Omega(G)^+$ consisting of isomorphism classes of G-sets, with addition and multiplication induced by the disjoint sum and direct product of G-sets, respectively; cf. Curtis-Reiner[CR2], ch. 11.

The additive structure of $\Omega(G)$ is easily understood, for $\Omega(G)$ is by [CR2], (80.6) a free abelian group with the canonical basis $\{[G/H]\}_{H \leq_G G}$; in other words,

$$(7.7) \qquad \Omega(G) = \bigoplus_{H \leq_G G} \mathbb{Z} \cdot [G/H],$$

where the notation "$H \leq_G G$" means that the index set runs over a system of representatives of the set of conjugacy classes of all subgroups of G.

We have a natural surjection

$$s_G : \mathbb{S}(G) \to \Omega(G)$$

which associates to each subgroup $H \leq G$ the isomorphism class of the G-set G/H; thus $s_G(H) = [G/H]$. It is clear that the map $ch_G : \mathbb{S}(G) \to \mathbf{ch}(\mathbb{Q}G)$ factors over s_G, so we have an induced map

$$ch_{\Omega(G)} : \Omega(G) \to \mathbf{ch}(\mathbb{Q}G),$$

whose kernel is denoted by

$$\Omega(G)^0 = Ker(ch_{\Omega(G)});$$

thus, by definition we have

$$s_G^{-1}(\Omega(G)^0) = \mathbb{S}(G)^0.$$

More generally, we also have the ring homomorphism

$$\pi_{RG} : \Omega(G) \to Mod(RG)_{sym}$$

which associates to each G-set S its associated permutation module:

$$(7.8) \qquad \pi_{RG}(S) = R[S].$$

From the definitions it is clear that we have

$$(7.9) \qquad \pi_{RG} \circ s_G = ch_{RG}.$$

Remark 7.1 The ranks of the modules $\mathbb{S}(G)^0$ and $\Omega(G)^0$ are easily determined by using Artin's Induction Theorem (cf. [KR1]):

$$\begin{aligned}(7.10)\ rank\,\mathbb{S}(G)^0 &= \#(\text{subgroups of } G)\\ &\quad -\#(\text{conjugacy classes of cyclic subgroups of } G),\\ (7.11)\ rank\,\Omega(G)^0 &= \#(\text{conjugacy classes of non-cyclic subgroups of } G).\end{aligned}$$

In particular, it follows that if G is cyclic, then $\mathbb{S}(G)^0 = \Omega(G)^0 = 0$ (and conversely).

If $f : G_1 \to G_2$ is a group homomorphism, then we have an induced ring homomorphism

$$f^* : \Omega(G_2) \to \Omega(G_1),$$

which is obtained by restriction of operators: each G_2-set S is naturally a G_1-set via the rule $g_1 \cdot s = f(g_1)s$.

To obtain a map in the other direction, let S be a G_1-set, and define the G_2-set $G_2 \times_f S$ as follows. Considering the cartesian product $G_2 \times (Ker(f)\backslash S)$ as a left $f(G_1)$-set via the rule $f(g_1) \cdot (g_2, s) = (g_2 f(g_1)^{-1}, g_1 s)$, we make the orbit space

$$G_2 \times_f S = f(G_1)\backslash(G_2 \times (Ker(f)\backslash S))$$

into a left G_2-set via $g_2 \cdot (f(G_1)(g, s)) = f(G_1)(g_2 g, s)$. We thus obtain an additive map

$$f_* : \Omega(G_1) \to \Omega(G_2)$$

by setting $f_*([S]) = [G_2 \times_f S]$.

In the sequel it will be useful to know the matrix representations of f^* and f_* in terms of the canonical bases of $\Omega(G_1)$ and $\Omega(G_2)$. To this end we prove:

Proposition 7.2 *Let $f : G_1 \to G_2$ be a group homomorphism.*

a) For each subgroup $X \le G_1$ we have

$$(7.12)\qquad f_*([G_1/X]) = [G_2/f(X)].$$

b) For each subgroup $Y \le G_2$ we have

$$(7.13)\ f^*([G_2/Y]) = \sum_{g \in f(G_1)\backslash G_2/Y} [G_1/f^{-1}(gYg^{-1})] = \sum_{X \le_{G_1} G_1} n_f(Y, X)[G_1/X],$$

where $n_f(Y, X) = \#\{g \in G_2 : f^{-1}(gYg^{-1}) \sim X\}/([G_1 : X] \cdot |Y|)$.

Proof. a) Consider the map

$$t : G_2 \times (Ker(f)\backslash G_1/X) \to G_2/f(X)$$

defined by $t(g_2, Ker(f)g_1X) = g_2 f(g_1)f(X)$. It is clear that t is constant on the $f(G_1)$-orbits of the cartesian product and hence induces a map

$$\bar{t} : G_2 \times_f (G_1/X) \to G_2/f(X).$$

Clearly, $\bar{t}$ is a G_2-equivariant surjection, and is easily checked that $\bar{t}$ is also injective. Thus $G_2 \times_f G_1/X \simeq G_2/f(X)$ as G_2-sets, and the assertion follows.

b) To prove the first equality, we shall use the decomposition

$$G_2/Y = \bigcup_{g \in f(G_1)\backslash G_2/Y} f(G_1)gY/Y$$

as G_1-sets. Now for each $g \in G_2$, the assignment $t_g(g_1) = f(g_1)gY$ induces a map

$$t_g : G_1/f^{-1}(gYg^{-1}) \to f(G_1)gY/Y,$$

which is clearly a surjection of G_1-sets. Moreover, t_g is injective because if $t_g(g_1) = t_g(g_1')$, then $f(g_1')g = f(g_1)gy$ with $y \in Y$, so $g_1^{-1}g_1' \in f^{-1}(gYg^{-1})$ and hence $g_1' f^{-1}(gYg^{-1}) = g_1 f^{-1}(gYg^{-1})$. Thus t_g is an isomorphism of G_1-sets, which proves the first equality in (7.13).

By what has just been proved we have

$$\begin{aligned}\sum_{g \in f(G_1)\backslash G_2/Y} [G_1/f^{-1}(gYg^{-1})] &= \sum_{g \in G_2} \frac{1}{|f(G_1)gY|}[G_1/f^{-1}(gYg^{-1})] \\ &= \sum_{X \le_{G_1} G_1} \sum_{\substack{g \in G_2 \\ f^{-1}(gYg^{-1}) \sim X}} \frac{1}{|f(G_1)gY|}[G_1/X].\end{aligned}$$

Now if $f^{-1}(gYg^{-1}) \sim X$, then $|X| = |Ker(f)| \cdot |f(G_1) \cap gYg^{-1}| = |Ker(f)| \cdot |Y| \cdot |f(G_1)|/|f(G_1)gY| = |G_1| \cdot |Y|/|f(G_1)gY|$, and so the coefficient of $[G_1/X]$ is precisely $n_f(Y, X)$, which proves the second equality of (7.13).

Corollary 7.3 *If S_i is a G_i-set, then*

(7.14) $$\pi_{RG_1}(f^*([S_2])) = Res_f(R[S_2]),$$

(7.15) $$\pi_{RG_2}(f_*([S_1])) = Coind_f(R[S_2]).$$

In particular, the maps f_* and f^* preserve character relations:

(7.16) $$f_*(\Omega(G_1)^0) \subset \Omega(G_2)^0; \quad f^*(\Omega(G_2)^0) \subset \Omega(G_1)^0.$$

Proof. The first equation is clear from the definitions. To prove the second, we may assume $S_1 = G_1/X$, where $X \le G_1$. Then by (7.12) and Corollary 4.15 we have

$$\pi_{RG_2}(f_*([G_1/X])) = R[G_2/f(X)] = Coind_f(R[G_1/X]),$$

which proves (7.15). The last assertion follows from (7.14) and (7.15) by taking $R = K$ and noting that $\pi_{KG} = ch_{KG}$.

Remark 7.4 It also follows from the proposition that the maps f_* and f^* on the Burnside rings $\Omega(G_i)$ agree with those defined on the subgroup modules $\mathbb{S}(G_i)$:

$$s_{G_2} \circ f_* = f_* \circ s_{G_1}, \tag{7.17}$$

$$s_{G_1} \circ f^* = f^* \circ s_{G_2}. \tag{7.18}$$

Here the second equation is understood to hold only after tensoring all spaces and maps with $\mathbb{Q}$.

7.3. The pairing δ on $\mathbb{S}(G)^0 \times Mod(RG)_{non-deg}$. Recall that in the previous two sections we had constructed and studied the pairing

$$\delta : Mod(RG)^0_{sym} \times Mod(RG)_{non-deg} \to Id(R,K) = K^\times/R^\times.$$

Since the homomorphism

$$ch_{RG} : \mathbb{S}(G) \to Mod(RG)_{sym}$$

maps $\mathbb{S}(G)^0$ into $Mod(RG)^0_{sym}$ (cf. (7.3)), we can pull δ back to a pairing

$$\tilde{\delta} = \tilde{\delta}_{RG} : \mathbb{S}(G)^0 \times RG \to Id_R(K)$$

via the rule

$$\tilde{\delta}(\textstyle\sum n_H H, M) = \delta(ch_{RG}(\sum n_H H), M). \tag{7.19}$$

Whenever convenient, we shall denote this pairing just by $\delta = \tilde{\delta}$.

Since $ch_{RG} = \pi_{RG} \circ s_G$, we see that the pairing factors over $\Omega(G)^0$ to yield a pairing which we shall also denote by δ:

$$\delta : \Omega(G)^0 \times RG \to Id_R(K).$$

We first show that this pairing $\delta = \tilde{\delta}$ is precisely the pairing which was introduced in section 2.

Proposition 7.5 *If (M,h) is a non-degenerate, L-valued hermitian RG-module, then for any character relation $\{n_H\} \in \mathbb{S}(G)^0$ we have*

$$\delta(\{n_H\}, M) = \prod_{H \le G} disc(Inv_H(M,h))^{n_H}. \tag{7.20}$$

Proof. By Example 4.4b) and the reciprocity formula (6.28) we have

$$\begin{aligned}
\delta(\{n_H\}, M) &= \prod_{H\le G} d_{RG}((R[G/H], h_{G/H}) \otimes (M,h))^{n_H} \\
&= \prod d_{RG}(Ind_H^G(R, id) \otimes (M,h))^{n_H} \\
&= \prod d_{RH}(Res_H^G(M,h))^{n_H},
\end{aligned}$$

which proves (7.20).

Next we shall use the maps f^* and f_* which were defined in the previous subsection to formulate the coinduction formulae. For this, recall the following notation from section 2:

Notation 7.6 For an element $x = \sum n_X X \in \mathbb{S}(G)^0$ and a RG-module M, let

$$\delta^*_{RG}(x, M) = \varepsilon_{RG}(x, M)\delta_{RG}(x, M), \tag{7.21}$$

where the element $\varepsilon_{RG}(x, M) \in K^\times$ is defined by

$$\varepsilon_{RG}(x, M) = \prod_X |X|^{n_X rk(M^X)}. \tag{7.22}$$

Since ε is additive in x and is constant on conjugacy classes of subgroups, it follows that ε_{RG} and hence δ^* factor over $\Omega(G)$. Explicitly we have for a G-set S:

$$\varepsilon_{RG}([S], M) = \prod_{s \in G \backslash S} |G_s|^{rk(M^{G_s})}.$$

Note that if x is fixed, then $\varepsilon_{RG}(x, M)$ depends only on the KG-module structure of $M \otimes K$, and hence is additive on exact sequences.

Remark 7.7 If (M, h) is a non-degenerate hermitian RG-module, then it follows from Proposition 7.5 and Proposition 5.4d) that

$$\delta^*_{RG}(\{n_H\}, M) = \prod_{H \leq G} disc(M^H, h_{|M^H}))^{n_H}. \tag{7.23}$$

Thus, δ^* is the same invariant as that of Remark 2.6.

Theorem 7.8 (Coinduction) *Let $f : G_1 \to G_2$ be a group homomorphism, and let M_i be a non-degenerate RG_i-module. Then we have for any $x_i \in \Omega(G_i)^0$ the formulae:*

$$\delta_{RG_1}(f^*(x_2), M_1) = \delta_{RG_2}(x_2, Coind_f(M_1)), \tag{7.24}$$

$$\delta^*_{RG_2}(f_*(x_1), M_2) = \delta^*_{RG_1}(x_1, Res_f(M_2)). \tag{7.25}$$

In other words, we have

$$\delta_{RG_2}(f_*(x_1), M_2) = \varepsilon_f(x_1, M_2)\delta_{RG_1}(x_1, Res_f(M_2)) \tag{7.26}$$

where for $x_1 = \sum n_X X \in \mathbb{S}(G_1)$ the correction term $\varepsilon_f(x_1, M_2)$ is given by

$$\varepsilon_f(x_1, M_2) = \varepsilon(x_1, Res_f(M_2))\varepsilon(f_*(x_1), M_2)^{-1} = \prod_{X \leq G_1} |Ker(f) \cap X|^{n_X rk(M_2^{f(X)})}. \tag{7.27}$$

Proof. In view of (7.14), the first equation follows directly from the Coinduction Theorem 6.15. To prove the second one, write $x_i = [S_1] - [S_2]$. Then by (7.15) we obtain from the Coinduction Theorem for Permutation Modules (Theorem 6.19) that

$$\begin{aligned} \delta(f_*(x_1), M_2) &= \delta_{RG_2}(Coind_f(R[S_1]), Coind_f(R[S_2]); M_2) \\ &= \varepsilon_f(S_1, M)\varepsilon_f(S_2, M)^{-1}\delta_{RG_1}(R[S_1], R[S_2]; Res_f(M_2)), \end{aligned}$$

where, as in Theorem 6.19,

$$\begin{aligned} \varepsilon_f(S_i, M) &= \prod_{s \in G_1 \backslash S_i} |Ker(f) \cap (G_1)_s|^{rk(H^0(f((G_1)_s), M_2))} \\ &= \prod_{s \in G_1 \backslash S_i} \left(\frac{|(G_1)_s|}{|f((G_1)_s)|} \right)^{rk(H^0(f((G_1)_s), M_2))} \\ &= \varepsilon([S_i], Res_f(M_2))\varepsilon(f_*([S_i]), M_2)^{-1}; \end{aligned}$$

here we have used (7.12) and the fact that $H^0(f(X), M_2) = H^0(X, Res_f(M_2))$ for every $X \leq G_1$. This proves (7.26) and (7.27), from which (7.25) follows readily.

7.4. Proofs of the main results. We can now proceed to prove the main results stated in section 2, with the exception of Theorem 2.7, which will be considered separately in the next section.

For the most part, the proofs just consist of translating the results of section 6 into the language of character relations, which is done as follows. Let $\sum n_H 1_H^* = 0$ be a character relation, and put

$$M_1 = \bigoplus_{n_H > 0} Ind_H^G(R)^{n_H}, \quad M_2 = \bigoplus_{n_H < 0} Ind_H^G(R)^{-n_H}.$$

Then $M_1 \otimes K \simeq M_2 \otimes K$ and we have in the notation of sections 5 and 6:

$$\delta(\{n_H\}, M) = \delta(M_1, M_2; M);$$

this follows immediately from the definitions (cf. (7.19) and (5.22)).

Proof of Theorem 2.5: Since by Proposition 7.5 we have

$$\delta(M_1, M_2; M) = \prod_H disc_R(Inv_H(M, h))^{n_H},$$

the assertions of Theorem 2.5 follow immediately from those of Theorem 6.1.

Proof of Theorem 2.8: This follows immediately from the more precise Uniform Boundedness Theorem 6.6.

Proof of Theorem 2.9: This is a special case of the Triviality Theorem 6.23.

Proof of Theorem 2.10 and of Corollary 2.11: These follow from the Exact Sequence Formula (Theorem 6.21) and Corollary 6.31, combined with the fact that by Shapiro's lemma (cf. [CE], ch. XII, Ex. 7) we have

$$H^q(G, M_i \otimes M) = \bigoplus_{X \le G} H^q(X, Ind_X^G(M))^{\oplus n_X} \simeq \bigoplus_{X \le G} H^q(X, M)^{\oplus n_X}, \text{ for } q \ge 0.$$

Proof of Theorem 2.12: Let $f : X \to G$ denote the inclusion map. Then by the definition of f_* in subsection 7.1 we have

$$ind_X^G\Big(\sum m_Y Y\Big) = f_*\Big(\sum m_Y Y\Big),$$

and so (2.14) follows directly from (7.26) (and (7.17)). (Note that (7.27) shows that $\varepsilon_f(x, M) = 1$ if f is injective.) Moreover, by (7.6) we have

$$res_X^G\Big(\sum n_H H\Big) = \sum_{Y \le X} \sum_{H \le G} n_H m(Y, H) Y = \sum_{H \le G} n_H f^*(H) = f^*\Big(\sum n_H H\Big),$$

so (2.15) follows from (7.24) and (7.18). This proves Theorem 2.12.

Proof of Example 2.13: a) This follows immediately from Theorem 2.12 by noting that if X is cyclic, then $res_X^G(\{n_H\}) \in \Omega(X)^0 = 0$ by (7.16) and Remark 7.1.

b) The hypothesis means that $M = R[S]$ is a direct sum of modules induced from cyclic subgroups (cf. Example 4.4b)), and so the assertion follows from a).

c) By decomposing S into its orbits, we may reduce to the case that G acts transitively on S. Then $R[S] = Ind_X^G(R)$ by Example 4.4b), where $X = G_s$, and so we obtain by applying Theorem 2.12 with $N = R$ that

$$\delta_R(\{n_H\}, R[S]) = \delta(\{n_Y^*\}, R) = \prod_Y |Y|^{-n_Y^*}$$

by (2.7). Now by (2.16) we have

$$\begin{aligned}
\prod_Y |Y|^{-n_Y^*} &= \prod_H \prod_Y |Y|^{-n_H m(H,Y)} \\
&= \prod_H \prod_{g \in G} |gHg^{-1} \cap X|^{-n_H/(|H|[X:gHg^{-1}\cap X])} \\
&= \prod_H \prod_{g \in H\backslash G/X} |H \cap gXg^{-1}|^{-n_H},
\end{aligned}$$

which proves (2.19).

Proof of Theorem 2.14: a) We shall apply Theorem 7.8 to the quotient map $f : G \to Q = G/X$. By definition of f^* (cf. (7.6)) we have

$$f^*\Big(\sum_{Y \le Q} n_Y Y\Big) = \sum_{H \le G} \sum_{Y \le Q} n_Y m_f(Y, H) H.$$

Now $m_f(Y,H)=0$ unless $H\sim f^{-1}(Y)$, and in this case

$$\begin{aligned} m_f(Y,H) &= \#\{g\in Q : f^{-1}(gYg^{-1})=H\}/([G:H]|Y|) \\ &= \#\{g\in Q : gYg^{-1}=f(H)\}/([G:H]|Y|) \\ &= |N_Q(f(H))|/([G:H]|Y|) \\ &= |N_G(H)|/|G| = [G:N_G(H)]^{-1} \end{aligned}$$

since $H\geq X$. Thus

$$f^*\Big(\sum_{Y\leq Q} n_Y Y\Big) = \sum_{X\leq H\leq G}\sum_{Y\sim f(X)} n_Y/[G:N_G(H)]H = inf_G^Q(\{n_Y\}).$$

Since $Coind_f(M)=M^X$ by Proposition 4.12b), the assertion (2.20) follows immediately from (7.24).

b) Here we have

$$f_*\Big(\sum_H n_H H\Big) = \sum_H n_H f(H) = \sum_Y \sum_{\substack{H \\ f(H)=Y}} n_H Y = ind_G^Q(\{n_H\}),$$

so equations (2.21) – (2.23) follow directly from equations (7.25) – (7.27).

Proof of Example 2.15: a) Apply Theorem 2.14b) with $X=G$.

b) By hypothesis, M may be viewed as an RQ-module, so the assertion follows from (2.22) because $ind_G^Q(\{n_H\})\in\mathbb{S}(Q)^0=0$ by (7.16) and Remark 7.1.

Proof of Corollary 2.16: Here we have $rank(M^H)=0$ for all subgroups $H\leq G$, so $\varepsilon_X(\{n_H\},M)=1$. This proves the first assertion, and the second follows from Example 2.15b).

Proof of Proposition 2.18: Let $H_0,\ldots,H_p$ denote the $p+1$ subgroups of G of order p. We then have the relation

$$\sum_{i=0}^{p} N_{H_i} = N_G + p\cdot 1,$$

where $N_H=\sum_{g\in H} g\in RG$. This yields $\sum n_H\varepsilon_H=0$, where $\varepsilon_H=\frac{1}{|H|}N_H\in KG$, and so we obtain $\sum n_H 1_H^*=0$ because for any irreducible character χ we have

$$(\textstyle\sum n_H 1_H^*,\chi)_G = \chi(\sum n_H\varepsilon_H)=0.$$

Thus $x=\sum n_H H\in\mathbb{S}(G)^0$ is a character relation. Moreover, since $rk(\mathbb{S}(G)^0)=1$ by Remark 7.1, it follows that every character relation of G is a rational and hence an integral multiple of x because some $n_H=1$.

a) We first note that since $M_{\mathfrak{p}} = (M^{(p)})_{\mathfrak{p}}$ for $\mathfrak{p} = (p)$, it follows from the localization formula (2.2) that $\delta(M) = \delta(M^{(p)})$, and so we may assume henceforth that $M = M^{(p)} = \mathbb{Z}/p^r\mathbb{Z}$. Let $X = Ker(\alpha : G \to Aut(M))$, where α denotes the map defining the action of G on M. If $X \neq 1$, then G/X is cyclic and $M^X = M$, so $\delta(M) = \mathbb{Z}$ by Corollary 2.16.

Thus, assume that $X = 1$. Here G is a subgroup of $Aut(M) = (\mathbb{Z}/p^r\mathbb{Z})^\times$, so $Aut(M)$ cannot be cyclic, which forces the conditions $p = 2$ and $r \geq 3$. Moreover, in that case we have that α induces an isomorphism $\alpha : G \xrightarrow{\sim} Aut(M)_2 = \{\pm id_M, \pm(1 + 2^{r-1})id_M\}$, so if we choose $\sigma, \tau \in G$ such that $\alpha(\sigma) = -id, \alpha(\tau) = (1 + 2^{r-1})id$, then we have that

$$M^{\langle\sigma\rangle} = M^{\langle\sigma\tau\rangle} = M^G = M_2 \text{ and } M^{\langle\tau\rangle} = 2M. \tag{7.28}$$

We thus obtain $\delta(M) = (|M^{\langle\sigma\rangle}| \cdot |M^{\langle\tau\rangle}| \cdot |M^{\langle\sigma\tau\rangle}| \cdot |M|^{-1} \cdot |M^G|^{-2})^{-1}\mathbb{Z} = (2 \cdot 2^{r-1} \cdot 2 \cdot 2^{-r} \cdot 2^{-2})^{-1}\mathbb{Z} = 2\mathbb{Z}$, which proves a).

b) By the exact sequence formula (Theorem 2.10) we have

$$\delta(M) = \delta(M_{tor})\delta(\overline{M})\psi(\pi_M)^2, \tag{7.29}$$

where $\psi(\pi_M) = \prod_H \chi(Coker(\pi_M^H : M^H \to \overline{M}^H))^{n_H}$. Since $\overline{M}^H = \overline{M}$, we have the exact sequence

$$0 \to M_{tor}/M_{tor}^H \to M/M^H \to Coker(\pi_M^H) \to 0,$$

from which we obtain the relation

$$\chi(Coker(\pi_M^H)) = \chi(M/M^H)\chi(M_{tor}/M_{tor}^H)^{-1} = \chi(M/M^H)\chi(M_{tor})^{-1}\chi(M_{tor}^H)$$

and hence

$$\begin{aligned}\psi(\pi_M) &= \chi(M_{tor})^{-\Sigma n_H} \prod \chi(M_{tor}^H)^{n_H} \prod \chi(M/M^H)^{n_H} \\ &= \delta(M_{tor})^{-1} \prod \chi(M/M^H)^{n_H},\end{aligned}$$

where we have used the fact that $\sum n_H = 0$. Substituting this in (7.29) yields

$$\delta(M) = \delta(M_{tor})^{-1}\delta(R)^r \prod_H \chi(M/M^H)^{2n_H},$$

because $\overline{M} = R^r$, where $r = rk(M)$. Moreover, by (2.7) we have

$$\delta(R) = \prod_H |H|^{-n_H} R = \left(\prod_{i=0}^{p} |H_i|^{-1}\right) |1|^{-(-1)}|G|^{-(-p)}R = p^{-(p+1)+2p}R$$

and so b) follows.

c) The non-degeneracy hypothesis means that $V \simeq V^*$, where $V = \overline{M} \otimes K$. Since $V^* \simeq V^{\otimes(p-1)}$, we see that this is only possible for $p = 2$. This proves the first assertion.

To prove the next one, put $X = Ker(G \to Gl(\overline{M})) = Ker(G \to Gl(V))$; by hypothesis we have that $X \neq G$. Moreover, since $\dim(V) = 1$, it follows by elementary representation theory that G/X is cyclic, so $|X| = p = 2$. This proves the second statement.

To prove the first equality in equation (2.29), we shall use the exact sequence formula (7.29) again. Here we have

$$\psi(\pi_M) = \chi(Coker(M^X \to \overline{M})) = \chi(\overline{M}/\overline{M^X}) = \chi(M/(M_{tor} + M^X)),$$

because $V^H = \overline{M}^H = 0$ if $H \neq X, \{1\}$ and $Coker(M \to \overline{M}) = 0$, and so by (7.29) we obtain

(7.30) $$\delta(M) = \delta(M_{tor})\delta(\overline{M})\chi(M/(M_{tor} + M^X))^2.$$

Next we compute $\delta(\overline{M})$ via the formula (7.20). In order to apply this, we shall view $\overline{M} = R$ as a hermitian RG-module via the standard pairing $h(x, y) = xy$. Since $\overline{M}^H = 0$ for $H \neq X, 1$ and $\overline{M}^X = \overline{M}$ we see that formula (7.20) reduces to

$$\delta(\overline{M}) = disc(\overline{M}^X, \tfrac{1}{2}h) \cdot disc(\overline{M}, h)^{-1} = \tfrac{1}{2}R$$

Substituting this in (7.30) proves the first equality of (2.29).

To prove the last equation, consider the exact sequence

$$M^X \to \overline{M} \to H^1(X, M_{tor}) \to H^1(X, M) \to H^1(X, \overline{M}).$$

Since X acts trivially on $\overline{M}$ we have $H^1(X, M) = Hom(X, \overline{M}) = 0$ because $\overline{M}$ is R-torsionfree. We thus obtain the exact sequence

(7.31) $$0 \to \overline{M}/\overline{M^X} \to H^1(X, M_{tor}) \to H^1(X, M) \to 0$$

which yields the relation

(7.32) $$\chi(\overline{M}/\overline{M^X}) = \chi(H^1(X, M_{tor})) \cdot \chi(H^1(X, M))^{-1}.$$

Now since X is a cyclic group, the theory of Herbrand quotients (cf. Serre[Se], VIII.4) is applicable, and so by Serre[Se], VIII.4, Proposition 8 and its corollary we have:

$$\begin{aligned} \chi(H^1(X, M_{tor})) &= \chi(\hat{H}^0(X, M_{tor})), \\ \chi(\hat{H}^0(X, M))\chi(H^1(X, M))^{-1} &= \chi(\hat{H}^0(X, \overline{M}))\chi(H^1(X, \overline{M}))^{-1} = 2R. \end{aligned}$$

Here the last equality is valid because $\overline{M} = R$ is a trivial RX-module and so

$$\hat{H}^0(X, \overline{M}) = R/2R \quad \text{and} \quad H^1(X, \overline{M}) = 0.$$

Substituting these relations in (7.32) yields

$$\chi(\overline{M}/\overline{M^X}) = \chi(\hat{H}^0(X, M_{tor})) \cdot 2\chi(\hat{H}^0(X, M))^{-1},$$

and so the second equality of (2.29) follows.

d) Put $Y = Ker(G \to Aut(M_4))$. Then $Y \neq G$ because X acts non-trivially on M_4, and $Y \neq 1$ because $Aut(M_4) = \mathbb{Z}/2\mathbb{Z}$ is cyclic. Thus $Y = \langle\tau\rangle$ is cyclic of order 2, which proves the first assertion.

To prove formula (2.30), put $d = [M : (M_{tor} + M^X)]$. We shall show that the formulae

$$d \cdot |\hat{H}^0(Y, M)| = |\hat{H}^0(Y, M_{tor})| \qquad (7.33)$$

$$\delta(M_{tor}) \cdot |\hat{H}^0(Y, M_{tor})| = 2\mathbb{Z} \qquad (7.34)$$

are valid, from which (2.30) follows immediately because by (2.29) we have

$$\delta(M_{tor})\delta(M) = \tfrac{1}{2}[\delta(M_{tor})d]^2 = \tfrac{1}{2}[2|\hat{H}^0(Y, M)|^{-1}]^2.$$

Moreover, it follows from (7.33) and (7.34) that $|\hat{H}^0(Y, M)|$ divides 2 because $\delta(M_{tor})$ is integral by a), and so the last assertion of d) is also evident.

It thus suffices to prove formulae (7.33) and (7.34). For the first, it is clearly enough to construct the exact sequence

$$0 \to M/(M_{tor} + M^X) \overset{\bar{N}_Y}{\to} \hat{H}^0(Y, M_{tor}) \to \hat{H}^0(Y, M) \to 0. \qquad (7.35)$$

To construct this sequence we shall start with the obvious sequence

$$0 \to N_Y(M)/N_Y(M_{tor}) \to \hat{H}^0(Y, M_{tor}) \to \hat{H}^0(Y, M) \to 0 \qquad (7.36)$$

which is exact because $M^Y = M^Y_{tor}$. Next we observe that

$$N_Y(M^X) \subset M^Y \cap 2M_{tor} \subset N_Y(M_{tor}), \qquad (7.37)$$

and hence N_Y induces a surjective homomorphism

$$\bar{N}_Y : M/(M_{tor} + M^X) \to N_Y(M)/N_Y(M_{tor}),$$

which will be seen to be an isomorphism.

To prove (7.37), let us first verify the following fact:

$$(\sigma - 1)x + (1 + \tau)x \in 2M_{tor}, \forall x \in M. \qquad (7.38)$$

To see this, let a be a generator of $M_{tor} = \mathbb{Z}/m\mathbb{Z}$. Then $\sigma(a) = -a$ and $\tau(a) = ka$ with $(k, m) = 1$; we note that $k \equiv 1 \bmod 4$ because by definition of τ

we have $\tau(\frac{m}{4}a) = \frac{m}{4}a$, so $k\frac{m}{4} \equiv \frac{m}{4} \bmod m$. Now since $(\sigma - 1)x, (1 + \tau)x \in M_{tor}$, we can write $(\sigma - 1)x = k_1 a, (1 + \tau)x = k_2 a$ for some integers k_1, k_2. Then

$$\begin{aligned} \sigma\tau(x) &= \sigma(-x + k_2 a) &= -(x + k_1 a) - k_2 a, \\ \tau\sigma(x) &= \tau(x + k_1 a) &= -x + k_2 a - kk_2 a, \end{aligned}$$

which yields the relation $(kk_1 + k_2)a = -(k_1 + k_2)a$ or $k_1(1+k) + 2k_2 \equiv 0 \bmod m$. Since $k \equiv 1 \bmod 4$ and $4|m$ this gives $k_1 + k_2 \equiv 0 \bmod 2$, which proves (7.38).

From (7.38) it follows immediately that $N_Y(M^X) \subset M^Y \cap 2M_{tor}$, which is the first inclusion of (7.37). Next, suppose $x \in M^Y \cap 2M_{tor}$. Then $x = 2y$ with $y \in M_{tor}$ and we have $\tau(y) - y = z \in M_2$. Thus $x = 2y = N_Y(y) - x \in N_Y(M_{tor})$ because $N_Y(M_4) = M_2$, and so (7.37) follows.

We thus obtain the desired map $\bar{N}_Y$. To prove that $\bar{N}_Y$ is injective, it is enough to show that

(7.39) $$Ker(N_Y : M \to M) \subset M_{tor} + M^X.$$

This again follows from (7.38). Indeed, if $x \in Ker(N_Y)$, then by (7.38) we have $(\sigma - 1)x = 2y$ with $y \in M_{tor}$, so $\sigma(x + y) = x + 2y + \sigma(y) = x + y$, which means that $x + y \in M^X$. Thus $x = -y + (x + y) \in M_{tor} + M^X$, which proves (7.39). This shows that $\bar{N}_Y$ is an isomorphism, and so we obtain the desired exact sequence (7.35).

It remains to prove equation (7.34), for which we shall distinguish two cases:

Case 1: G does not act faithfully on $M^{(2)}$.
Since $\tilde{Y} := Ker(G \to Aut(M^{(2)})) \subset Ker(G \to Aut(M_4)) = Y$, and $\tilde{Y} \neq 1$ by hypothesis, it follows that $\tilde{Y} = Y$. Thus Y acts trivially on $M^{(2)}$ and hence

$$\hat{H}^0(Y, M_{tor}) = \hat{H}^0(Y, M^{(2)}) = M^{(2)}/2M^{(2)} \simeq \mathbb{Z}/2\mathbb{Z}.$$

Since we have $\delta(M_{tor}) = \mathbb{Z}$ by a), this proves (7.34).

Case 2: G acts faithfully on $M^{(2)}$.
Here we shall show that

(7.40) $$\hat{H}^0(Y, M_{tor}) = 0,$$

from which (7.34) follows immediately because by a) we have in this case that $\delta(M_{tor}) = 2\mathbb{Z}$.

To prove (7.40), first note that since τ acts trivially on M_4, it follows from (7.28) that $(M^{(2)})^Y = 2M^{(2)}$. Moreover, since $a' = \frac{m}{2^r}a$ is a generator of $M^{(2)}$, we see that $N_Y(a') = a' + ka' = 2(\frac{1+k}{2})a'$ is a generator of $2M^{(2)}$ because $k \equiv 1 \bmod 4$, and so $N_Y(M^{(2)}) = 2M^{(2)}$. Thus $\hat{H}^0(Y, M_{tor}) = \hat{H}^0(Y, M^{(2)}) = 0$, as desired.

This concludes the proof of equation (7.34) and hence, as was explained above, also of Proposition 2.18.

8. Applications to class number relations

In this final section we apply the theory of discriminants of hermitian $\mathbb{Z}G$-modules to study relations among S-regulators of number fields. For this, we shall first introduce a hermitian structure on the group of S-units and relate its discriminant to the S-regulator. This will then be used to compute the δ-invariant of $U_S(K)$, from which Theorem 2.7 (and hence Brauer's Theorem 1.2) follows readily. Finally, we shall prove Dirichlet's Theorem 1.1.

8.1. The S-unit group. Let K be a number field and let $S \supset S_\infty$ be a finite set of places of K containing the archimedean places. As usual, the S-unit group of K is defined by

$$U_S(K) = \{x \in K : |x|_v = 1, \forall v \notin S\}.$$

Here we shall view it as a bilinear $\mathbb{Z}$-module via the pairing ρ_S defined by

$$\rho_S(u, u') = \sum_{v \in S} d_v \log |u|_v \log |u'|_v, \text{ for } u, u' \in U_S(K), \tag{8.1}$$

where $d_v = [\hat{K}_v : \mathbb{Q}_v]$ denotes the absolute degree of the completion $\hat{K}_v$ and $|\cdot|_v$ is the absolute value associated to v which is normalized in such a way that its restriction to $\mathbb{Q}$ coincides with the usual p-adic or archimedean absolute values of $\mathbb{Q}$. Thus, with this normalization the product formula for $u \in U_S(K)$ becomes:

$$\prod_{v \in S} |u|_v^{d_v} = 1. \tag{8.2}$$

We now want to compute the discriminant of the bilinear module $(U_S(K), \rho_S)$. As one might expect, its essential ingredient is the *S-regulator* of K, which, as in Tate[Ta], p. 22, is defined by

$$reg_S(K) = |\det((d_v \log |u_i|_v)_{\substack{1 \le i \le r \\ v \in S \setminus v_0}})|, \tag{8.3}$$

where $v_0 \in S$ is an arbitrarily chosen place and $u_1, \ldots, u_r$ denotes a basis of $\overline{U}_S(K) = U_S(K)/U_S(K)_{tor}$; recall that by the S-unit theorem its rank is $r = rk(U_S(K)) = |S| - 1$.

Proposition 8.1 *The discriminant of the bilinear $\mathbb{Z}$-module $(U_S(K), \rho_S)$ is given by the formula*

$$disc(U_S(K), \rho_S) = reg_S(K)^2 d(S)/(w(K)p(S))\mathbb{Z}, \tag{8.4}$$

where $w(K) = |U_S(K)_{tor}| = |U_{S_\infty}(K)_{tor}|$ denotes the number of roots of unity in the field K,

$$d(S) = \sum_{v \in S} d_v \quad \text{and} \quad p(S) = \prod_{v \in S} d_v.$$

Proof. Let $u_1, \ldots, u_r$ be a basis of $\overline{U}_S(K) = U_S(K)/U_S(K)_{tor}$, and let $G = (g_{ij})$ denote the associated Gram matrix; thus

$$g_{ij} = \sum_{v \in S} d_v \log |u_i|_v \log |u_j|_v, \text{ for } 1 \leq i, j \leq r.$$

Then by (3.38) and (3.37) we have

$$disc(U_S(K), \rho_S) = |U_S(K)_{tor}|^{-1} \det(G)\mathbb{Z} = w(K)^{-1} \det(G)\mathbb{Z}. \tag{8.5}$$

In order to relate $\det(G)$ to the regulator, write $S = \{v_1, \ldots, v_{r+1}\}$ and consider the matrix

$$A = \begin{pmatrix} d_{v_1} \log |u_1|_{v_1} & \cdots & d_{v_1} \log |u_r|_{v_1} & d_{v_1} \\ \vdots & & \vdots & \vdots \\ d_{v_{r+1}} \log |u_1|_{v_{r+1}} & \cdots & d_{v_{r+1}} \log |u_r|_{v_{r+1}} & d_{v_{r+1}} \end{pmatrix},$$

which is related to G by the formula

$$A^t diag(\frac{1}{d_{v_1}}, \ldots, \frac{1}{d_{v_{r+1}}})A = \begin{pmatrix} G & 0 \\ 0 & d(S) \end{pmatrix};$$

this follows by multiplying out the left hand side and applying (8.2). We thus obtain:

$$\det(G) = \det(A)^2/(d(S)p(S)). \tag{8.6}$$

Finally we want to compute $\det(A)$. By adding the first r rows to the last one and applying (8.2) again, we obtain

$$\det(A) = \det \begin{pmatrix} d_{v_1} \log |u_1|_{v_1} & \cdots & d_{v_1} \log |u_r|_{v_1} & d_{v_1} \\ \vdots & & \vdots & \vdots \\ d_{v_r} \log |u_1|_{v_r} & \cdots & d_{v_r} \log |u_r|_{v_r} & d_{v_r} \\ 0 & \cdots & 0 & d(S) \end{pmatrix} = \pm reg_S(K) \cdot d(S).$$

Substituting this in (8.6) and (8.5), the assertion follows.

From now on we shall consider the case that S is invariant under a group $G \leq Aut(K)$ of automorphisms of K, so $U_S(K)$ is a naturally a $\mathbb{Z}G$-module. It is immediate that ρ_S is G-equivariant, and thus $(U_S(K), \rho_S)$ is in fact a hermitian $\mathbb{Z}G$-module.

To compute its δ-invariant, we shall determine the invariant modules of $\tilde{U}_S(K)$ with respect to subgroups $H \leq G$.

Proposition 8.2 *For each subgroup $H \leq G$ there is a natural isometry*

$$i_H : (U_{H\backslash S}(K^H), \rho_{H\backslash S}) \xrightarrow{\sim} Inv_H(U_S(K), \rho_S), \tag{8.7}$$

where we view $H\backslash S$ as the set of places of the fixed field K^H which lie below S.

Proof. Clearly $U_S(K)^H = U_S(K) \cap K^H = U_{H\backslash S}(K^H)$, and hence the inclusion map $i_H : U_{H\backslash S}(K^H) \to U_S(K)$ induces an isomorphism

$$i_H : U_{H\backslash S}(K^H) \xrightarrow{\sim} U_S(K)^H.$$

To compare the hermitian structures, let $u, u' \in U_{\bar{S}}(K^H)$, where $\bar{S} = H\backslash S$. Then

$$\rho_S(i_H(u), i_H(u')) = \sum_{\bar{v}\in\bar{S}} \sum_{v|\bar{v}} d_v \log|u|_{\bar{v}} \log|u'|_{\bar{v}} = |H|\rho_{\bar{S}}(u, u'),$$

because

(8.8) $$\sum_{v|\bar{v}} d_v = d_{\bar{v}}|H|.$$

Thus $inv_H(\rho_S) = \rho_{\bar{S}}$, which proves the assertion.

Corollary 8.3 *Let $x = \sum n_H H \in \mathbb{S}(G)^0$ be a character relation. Then*

(8.9) $$\delta(x, U_S(K)) = \prod_H \left(\frac{d(H\backslash S)reg_{H\backslash S}(K^H)^2}{p(H\backslash S)w(K^H)} \right)^{n_H}.$$

Proof. By Proposition 7.5 and Proposition 8.2 we have

$$\delta(x, U_S(K)) = \prod_H disc(Inv_H(U_S(K), \rho_S))^{n_H} = \prod_H disc(U_{H\backslash S}(K^H), \rho_{H\backslash S})^{n_H},$$

and so (8.9) follows from Proposition 8.1.

The above formula (8.9) can be simplified by means of the following lemma.

Lemma 8.4 *Let S be a G-invariant set of places of K, and let $x = \sum n_H H \in \mathbb{S}(G)^0$ be a character relation. Then:*

(8.10) $$\prod_H d(H\backslash S)^{n_H} = \delta(x, \mathbb{Z}),$$

(8.11) $$\prod_H p(H\backslash S)^{n_H} = \delta(x, \mathbb{Z}[S]).$$

Proof. a) If $H \leq G$ is a subgroup, then by (8.8) we have

$$d(S) = |H| d(H\backslash S),$$

and so we obtain

$$\prod_H d(H\backslash S)^{n_H} = \prod_H (|H|^{-1} d(S))^{n_H} = \left(\prod_H |H|^{-n_H}\right) \cdot \left(\prod_H d(S)^{n_H}\right) = \delta(x, \mathbb{Z});$$

here we have used (2.7) and the fact that

$$\sum_H n_H = \sum_H n_H (1_G, 1_H^*)_G = 0. \tag{8.12}$$

b) By decomposing S into its G-orbits and treating each orbit separately, we may assume without loss of generality that G acts transitively on S. Thus, if we fix $v_0 \in S$ and let $Z = G_{v_0}$ denote the stabilizer (or decomposition group) of v_0, then the map $g \to gv_0$ induces a bijection $G/Z \xrightarrow{\sim} S$. Moreover, since the decomposition group of $v = gv_0$ is gZg^{-1}, we obtain the relation

$$d_v = |gZg^{-1}| d_w = |Z| d_w, \tag{8.13}$$

where w is the place of K^G lying below any $v \in S$.

If $H \le G$ is any subgroup, then the H-orbit space $\bar{S} = H\backslash S$ may be identified with the double coset space $H\backslash G/Z$, and for each $\bar{v} \in \bar{S}$ we have the analogous formula

$$d_v = |H \cap gZg^{-1}| d_{\bar{v}}, \quad \text{if } v = gv_0|\bar{v}, \tag{8.14}$$

because $Z_H(v) = H \cap gZg^{-1}$ is the decomposition group of $v = gv_0$ with respect to H. Comparing (8.14) with (8.13) gives

$$d_{\bar{v}} = \frac{|Z| d_w}{|H \cap gZg^{-1}|}, \tag{8.15}$$

and so we obtain

$$p(H\backslash S) = \prod_{\bar{v} \in H\backslash S} d_{\bar{v}} = \prod_{g \in H\backslash G/Z} \frac{|Z| d_w}{|H \cap gZg^{-1}|}. \tag{8.16}$$

This yields

$$\prod_H p(H\backslash S)^{n_H} = \prod_H \prod_{g \in H\backslash G/Z} \left(\frac{|Z| d_w}{|H \cap gZg^{-1}|} \right)^{n_H} = \prod_H \prod_{g \in H\backslash G/Z} |H \cap gZg^{-1}|^{-n_H},$$

where the latter equality follows from the fact that

$$\sum_H n_H |H\backslash G/Z| = \sum_H n_H (1_H^*, 1_Z^*)_G = 0. \tag{8.17}$$

(In the last formula we have used the fact that $|H\backslash G/Z| = (1_H^*, 1_Z^*)$ by Mackey's Theorem and Frobenius Reciprocity.) In view of Example 2.13c), this proves (8.11).

8.2. Proofs of the main results. We can now proceed to prove the main theorem on class number relations:

Proof of Theorem 2.7: Write $x = \sum n_H H$. Then by Corollary 8.3 and Lemma 8.4 we obtain

$$\delta(x, U_S(K)) = \prod_H \left(\frac{d(H\backslash S) reg_{H\backslash S}(K^H)^2}{p(H\backslash S) w(K^H)} \right)^{n_H} = \prod_H \left(\frac{reg_{H\backslash S}(K^H)^2}{w(K^H)} \right)^{n_H} \delta(x, \mathbb{Z}) \delta(x, \mathbb{Z}[S])^{-1}, \tag{8.18}$$

which proves the first assertion of the theorem. To deduce equation (2.8) from this, we first note that Brauer's formula (1.6) may be generalized to S-regulators and S-class numbers in the following manner:

$$\prod_H \left(reg_{H\backslash S}(K^H) h_{H\backslash S}(K^H) w(K^H)^{-1} \right)^{n_H} = 1. \tag{8.19}$$

This follows from the usual Artin formalism applied to the Artin L-functions $L_S(s, \chi)$ (as defined in Tate[Ta], p. 23), using the fact that

$$L_S(s, 1_H^*) = \zeta_{K^H, H\backslash S}(s) \sim -\frac{h_{H\backslash S}(K^H) reg_{H\backslash S}}{w(K^H)} s^{|H\backslash S|-1}$$

in a neighbourhood of $s = 0$ (cf. [Ta], p. 23).

Thus, by (8.19) and (8.18) we obtain

$$\prod_H h_{H\backslash S}(K^H)^{2n_H} = \prod_H w(K^H)^{n_H} \prod_H \left(\frac{reg_{H\backslash S}(K^H)^2}{w(K^H)} \right)^{-n_H} = \delta(x, U(K)_{tor})^{-1} \left(\delta(x, U_S(K)) \delta(x, \mathbb{Z})^{-1} \delta(x, \mathbb{Z}[S]) \right)^{-1},$$

which proves equation (2.8). The last assertion follows immediately from Example 2.13b).

As an illustration of the above class number formula, let us now prove Dirichlet's Theorem:

Proof of Theorem 1.1: Since $G = Gal(K/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we obtain from the first assertion of Proposition 2.18 and Theorem 2.7 the class number relation

$$[h(\mathbb{Q}(\sqrt{d})) h(\mathbb{Q}(\sqrt{-d})) h(\mathbb{Q}(i)) h(K)^{-1} h(\mathbb{Q})^{-2}]^2 = 2[\delta(U(K)) \delta(U(K)_{tor})]^{-1}, \tag{8.20}$$

where we have also used the fact that $\delta(\mathbb{Z}) = 2\mathbb{Z}$; cf. Proposition 2.18b).

In order to compute the right hand side of 8.20, we shall apply Proposition 2.18d) to $M = U(K)$, and thus need to check that M satisfies the hypotheses there. For this, note first that $\overline{M}$ is as in Proposition 2.18c) because by Dirichlet's Unit Theorem we have $rk(U(K)) = r = r_1 + r_2 - 1 = 1$ since $r_1 = 2$ and $r_2 = 1$, and hence G acts non-trivially on $\overline{M}$ since $rk(M^G) = rk(U(\mathbb{Q})) = 0$. Moreover, if $K^+ = \mathbb{Q}(\sqrt{d})$ denotes the real subfield of the CM-field K, then $rk(U(K^+)) = 1$, so the σ of 2.18c) is just the complex conjugation automorphism. Next we note that M_{tor} is cyclic, being generated by a suitable root of unity ζ of K, and so σ acts by multiplication by -1 since $\sigma(\zeta) = \bar{\zeta} = \zeta^{-1}$. In addition we see that 4 divides $|M_{tor}|$ because $i = \sqrt{-1} \in K$. Thus M satifies all the hypotheses of Proposition 2.18d), and so substituting (2.30) in the right hand side of (8.20) and using the well-known fact that $h(\mathbb{Q}(i)) = h(\mathbb{Q}) = 1$ yields the relation

$$[h(d)h(-d)h(K)^{-1}]^2 = |\hat{H}^0(Y, M)|^2.$$

We thus see that (1.1) holds with

$$Q = 2|\hat{H}^0(Y, M)|^{-1}. \tag{8.21}$$

It remains to verify that this definition of Q coincides with the one given in (1.2). For this, we first note that $Y = Gal(K/\mathbb{Q}(i))$ because Y fixes $M_4 = \langle i \rangle$. Thus, if $K \neq \mathbb{Q}(\zeta_8)$, then $U(K)^Y = U(\mathbb{Q}(i)) = \langle i \rangle$ and $N_Y(U(K)) = \langle \pm 1, N_Y(\varepsilon) \rangle$, and so the desired coincidence is immediate in view of the last assertion of Proposition 2.18d). On the other hand, if $K = \mathbb{Q}(\zeta_8)$, then $M_{tor} = M_8 = \langle \zeta_8 \rangle$, and so by the proof of Proposition 2.18d) (or otherwise) it follows that $\hat{H}^0(Y, M) = \hat{H}^0(Y, M_{tor}) = 0$. We thus obtain from (8.21) that $Q = 2$ in this case, which proves the assertion.

Remark 8.5 The above proof also shows that the invariant Q can be interpreted as an index of unit groups. Indeed, if $K \neq \mathbb{Q}(\zeta_8)$, then it follows from (2.30) and (2.29) that

$$Q = [U(K) : (U(K)_{tor} + U(K^+))],$$

which is the usual index considered in the theory of abelian fields and CM-fields (cf. Washington[Wash], p. 39).

In order to include also the case $K = \mathbb{Q}(\zeta_8)$, we note with Herglotz[He] (cf. also Kubota[Kb]) that the above index is equal to the index

$$Q = [U(K) : \tilde{U}(K)],$$

where $\tilde{U}(K)$ denotes the subgroup generated by the unit groups of the *proper* subfields of K, and that this formula is also true for $K = \mathbb{Q}(\zeta_8)$.

We have thus also proven Herglotz's version of Dirichlet's Theorem. Note that this version is actually considerably easier to obtain by our methods than Dirichlet's version because it uses essentially only Proposition 2.18c) and not part d) which required a much more detailed analysis.

References

[BA] N. Bourbaki, *Algebra I, ch. 1-3.* Addison-Wesley, Reading, 1974.

[BCA] N. Bourbaki, *Commutative Algebra.* Addison-Wesley, Reading, 1972.

[Br] R. Brauer, Beziehungen zwischen Klassenzahlen von Teilkörpern eines galoisschen Körpers. *Math. Nachr.* **4** (1951), 158 - 174.

[CE] H. Cartan, S. Eilenberg, *Homological Algebra.* Princeton U Press, Princeton, 1956.

[CR1] C. Curtis, I. Reiner, *Methods of Representation Theory I.* John Wiley & Sons, New York, 1981.

[CR2] C. Curtis, I. Reiner, *Methods of Representation Theory II.* John Wiley & Sons, New York, 1987.

[Di1] G. L. Dirichlet, Untersuchungen über die Theorie der komplexen Zahlen. *Königl. Preuss. Akad. Wiss.* 1841, 190 - 194 = Werke I, 504 - 508.

[Di2] G. L. Dirichlet, Recherches sur les formes quadratiques a coefficients et a indéterminées complexes. *J. reine angew. Math.* **24** (1842), 291 - 371 = Werke I, 535 - 618.

[Fu] W. Fulton, *Intersection Theory.* Springer-Verlag, Berlin, 1984.

[He] G. Herglotz, Über einen Dirichletschen Satz. *Math. Z.* **12** (1922), 225 - 261.

[Hi] D. Hilbert, Über den Dirichletschen biquadratischen Zahlkörper. *Math. Ann.* **45** (1894), 309 - 340 = Ges. Abh. I, 24 - 52.

[Hu] B. Huppert, N. Blachburn, *Finite Groups II.* Springer-Verlag, Berlin, 1982.

[KR1] E. Kani, M. Rosen, Idempotent relations and factors of Jacobians.*Math. Ann.* **284** (1989), 307 - 327.

[KR2] E. Kani, M. Rosen, Idempotent relations among arithmetic invariants attached to number fields and algebraic varieties. *J. Number Th.*, to appear.

[Kb] T. Kubota, Über die Beziehung der Klassenzahlen der Ünterkörper des bizyklischen biquadratischen Zahlkörpers. *Nagoya Math. J.* **6** (1953), 119-127.

[Ku] S. Kuroda, Über die Klassenzahlen algebraischer Zahlkörper. *Nagoya Math. J.* **1** (1950), 1 - 10.

[Neh] H. Nehrkorn, Über absolute Idealklassengruppen und Einheiten in algebraischen Zahlkörpern. *Abh. Math. Sem. Univ. Hamburg* **9** (1933), 318 - 334.

[Neu] J. Neukirch, *Algebraische Zahlentheorie.* Springer Verlag, Berlin, 1992.

[Sch] W. Scharlau, *Quadratic and Hermitian Forms.* Springer Verlag, Berlin, 1985.

[Se] J.-P. Serre, *Local Fields.* Springer Verlag, New York, 1979.

[Sz] L. Szpiro, Présentation de la théorie d'Arakélov. In: *Current Trends in Arithmetical Algebraic Geometry*, K. Ribet, ed., Contemp. Math. **67**, AMS, Providence, 1987, pp. 279 – 293.

[Ta] J. Tate, *Les Conjectures de Stark sur les Fonctions L d'Artin.* Birkhäuser, Boston, 1984.

[Wa1] C. Walter, Brauer's class number relation. *Acta Arith.* **35** (1979), 33 - 40.

[Wa2] C. Walter, Kuroda's class number relation. *Acta Arith.* **35** (1979), 41 - 51.

[Wash] L. Washington, *Introduction to Cyclotomic Fields.* Springer Verlag, New York, 1982.