Diagonal Quotient Surfaces

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Introduction

Let X be a smooth projective curve over $\mathbb{C}$ which admits a finite group G of automorphisms. Then $G \times G$ acts on the product surface $Y = X \times X$, and so for each subgroup $H \leq G \times G$, the quotient $Z = H\backslash Y$ is a normal algebraic surface. Here we shall restrict attention to the case that the subgroup H is the graph of a group automorphism $\alpha \in \mathrm{Aut}(G)$ of G, i.e. $H = \Delta_\alpha = \{(g, \alpha(g)) : g \in G\}$; we propose to call the resulting quotient surface a *(twisted) diagonal quotient surface*, and denote it by

$$Z_{X,G,\alpha} = \Delta_\alpha \backslash Y.$$

Such surfaces occur naturally as the (coarse) moduli spaces of certain moduli problems; cf. [17]. In this case $X = X(N)$, the modular curve of level N, and there is a close analogy between these surfaces and the Hilbert modular surfaces studied by Hirzebruch and others (cf. van der Geer[6]).

The main objective of this paper is to analyze the geometry of the diagonal quotient surfaces $Z_{X,G,\alpha}$ and of their desingularizations $\tilde{Z}_\alpha$ by calculating their numerical invariants such as their Betti and Chern numbers, and to determine their place in the Enriques-Kodaira classification table.

One might expect that all these invariants can be expressed in terms of data involving the curve X and the group G, and this indeed turns out to be the case. In fact, they can all be expressed in terms of the genus $g = g(X)$ of X, the genus $\bar{g} = g(\bar{X})$ of the quotient curve $\bar{X} = G\backslash X$, and three fundamental invariants $\mathbb{G}_\alpha$, $\mathbb{S}_\alpha$ and $\mathbb{L}_\alpha$ which are essentially of a local nature in that they depend on the local action of G at the fixed points. For example, we have the following result which is a special case of Theorem 3.7 below:

Theorem 1 *The irregularity, the geometric genus and the square of the first Chern class of the desingularization $\tilde{Z}_\alpha$ of the diagonal quotient surface Z_α are given by:*

$$q = 2\bar{g}, \qquad p_g = \mathbb{G}_\alpha - \mathbb{S}_\alpha, \quad \text{and} \quad c_1^2 = 8(1 - 2\bar{g} + \mathbb{G}_\alpha) - \mathbb{L}_\alpha - 12\mathbb{S}_\alpha.$$

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Of course, the above result is only useful if we succeed in calculating the invariants $\mathbb{G}_\alpha$, $\mathbb{L}_\alpha$ and $\mathbb{S}_\alpha$. Of these, $\mathbb{G}_\alpha$ is perhaps the simplest since it can be expressed in terms of the genera $g_{\bar{x},\alpha}$ of certain subquotients $X_{\bar{x},\alpha}$ of X or, alternatively, as the inner product (h^1_α, h^1) of the character h^1 afforded by the G-module $H^1(X, \mathbb{C})$ (cf. Proposition 1.8). In a similar way, $\mathbb{S}_\alpha$ can be expressed either as a weighted sum of Dedekind sums $\mathbb{S}(q,n)$ or in terms of certain inner products involving the character ω of the G-module $H^0(X, \Omega^1_X)$ of holomorphic differentials (cf. Theorem 1.13). Finally, the invariant $\mathbb{L}_\alpha$, which is closely related to the nature of the singularities of Z_α, is defined in terms of a weighted sum of lengths of certain continued fractions; cf. Proposition 3.5. In both cases, these weights are given by the "local invariants" $s_\nu(\bar{x}, \bar{y})$ which are introduced in section 1.1.

From the knowledge of the Chern numbers it is often possible to determine the Kodaira dimension κ and hence the type of the surface according to the Enriques–Kodaira classification scheme. In particular, we have a complete classification in the case that either $g \leq 1$ or that $\bar{g} \geq 1$, as the following two theorems show (cf. Theorems 4.1 and 4.2):

Theorem 2 a) *If $g = 0$ then $\tilde{Z}_\alpha$ is rational; i.e. $\kappa = -\infty$.*
b) *If $g = \bar{g} = 1$, then $\tilde{Z}_\alpha \simeq Z_\alpha$ is an abelian variety, so $\kappa = 0$.*
c) *If $g = 1$ and $\bar{g} = 0$, then $\tilde{Z}_\alpha$ is an elliptic surface with $\kappa \leq 0$.*
d) *If $g \geq 2$ and $\bar{g} \geq 1$, then $\tilde{Z}_\alpha$ is a minimal surface of general type, i.e. $\kappa = 2$.*

Theorem 3 *If $g = 1$ and $\bar{g} = 0$, then $p_g \leq 1$ and $c_1^2 \leq 0$, and we have the following three possibilities:*
a) *If $p_g = 0$ and $c_1^2 < 0$, then $\tilde{Z}_\alpha$ is rational ($\kappa = -\infty$).*
b) *If $p_g = 0$ and $c_1^2 = 0$, then $\tilde{Z}_\alpha$ is a minimal Enriques surface ($\kappa = 0$).*
c) *If $p_g = 1$, then $\tilde{Z}_\alpha$ is a minimal K3-surface ($\kappa = 0$).*

It should be remarked that all these possibilities actually do occur (cf. section 4.1).

The remaining cases (i.e. $g \geq 2$ and $\bar{g} = 0$) are more subtle in that the Chern numbers themselves are often insufficient in determining the type of the surface, as examples with modular curves show. Nevertheless, one might expect that if $m = |G|$ is small with respect to g, then $\tilde{Z}_\alpha$ is of general type. Theorem 4.8 gives some evidence for this expectation; for example we have:

Theorem 4 *If there are at least $r \geq 8$ points of $\bar{X} = G\backslash X$ which are ramified in X, then $\tilde{Z}_\alpha$ is of general type.*

In order to be able to classify the surfaces $\tilde{Z}_\alpha$ in other cases, additional information seems to be necessary. Since it is difficult to calculate the plurigenera P_n of $\tilde{Z}_\alpha$, we rely instead on the existence of certain curves and/or curve configurations on $\tilde{Z}_\alpha$. To this end we introduce two criteria which are useful for distinguishing between elliptic surfaces and surfaces of general type. The first is based on the invariant $d(S,C) = 2p_g(S) - 2 - (K_S.C)$ of a curve C lying on a regular surface S, and the second studies what we call (-2)-*joins* of two curves C_1 and C_2 on S; the latter are divisors $D = C_1 + C_2 + E_1 + \ldots + E_t$ formed by the (-2)-chains E_i which join C_1 to C_2 (cf. Definition 4.16). More precisely, we have results of the following type (cf. Theorem 4.12 and Proposition 4.17):

Theorem 5 *Let S be a smooth regular surface with $\kappa(S) \geq 1$.*

a) *If S has a curve C with $p_a(C) \geq 2$ and $d(S, C) \geq 1$, then S is of general type.*

b) *If S has a (-2)-join D with $D^2 > (K_S.D)^2$, then S is an elliptic surface.*

Applying Theorem 5 to the curve configurations on $\tilde{Z}_\alpha$ arsing from the resolution of singularities leads to quite explicit criteria. To state these, let $x \in X$ be point with non-trivial stabilzer group $G_x \leq G$ (i.e. x is a ramification point of $\pi : X \to \bar{X} = G\backslash X$), and let $g_{x,\alpha} = g(X_{x,\alpha})$ denote the genus of the quotient curve $X_{x,\alpha} = \alpha^{-1}(G_x)\backslash X$, $r_{1,x,\alpha}$ the number of unramified points above $\bar{x} = \pi(x)$ in the covering $\pi_{x,\alpha} : X_{x,\alpha} \to \bar{X}$, and let

$$c_{x,\alpha} = \sum_{\bar{y} \in \bar{X}} \sum_{\nu=1}^{e_y - 1} s_{\nu,\alpha}(\bar{x}, \bar{y}) \frac{\nu}{e_{\bar{y}}} \quad \text{and} \quad s_{x,\alpha} = \sum_{d | e_x, d \neq e_x} s_{e_x - d, \alpha}(\bar{x}, \bar{x}),$$

where, as before, $s_{\nu,\alpha}(\bar{x}, \bar{y})$ denotes the local invariant defined in section 1. (Note that $g_{\bar{x},\alpha}$ and $r_{1,x,\alpha}$ may also be expressed in terms of these local invariants; cf. Remark 1.5.) Then the aforementioned criteria may be formulated as follows (cf. Proposition 4.19):

Theorem 6 *Suppose that $\bar{g} = 0$.*

a) *If $g_{x,\alpha} = g_{x,\alpha^{-1}} = 0$, $c_{x,\alpha} = c_{x,\alpha^{-1}} = 1$ and $r_{1,x,\alpha} > 0$ for some (ramified) point $x \in X$, then $\tilde{Z}_\alpha$ is rational.*

b) *If we have*

$$s_{x,\alpha} > \tfrac{1}{2}(k_{x,\alpha}^2 - i_{x,\alpha})$$

for some $x \in X$, where

$$k_{x,\alpha} = 2g_{x,\alpha} + 2g_{x,\alpha^{-1}} + c_{x,\alpha} + c_{x,\alpha^{-1}} - 4 \quad \text{and} \quad i_{x,\alpha} = 2r_{1,x,\alpha} - c_{x,\alpha} - c_{x,\alpha^{-1}},$$

then $\tilde{Z}_\alpha$ is not of general type, i.e. $\kappa(\tilde{Z}_\alpha) \leq 1$. If, in addition, $p_g(\tilde{Z}_\varepsilon) = 1$ and $g_{x,\alpha} = g_{x,\alpha^{-1}} = 0$ and $c_{x,\alpha} = c_{x,\alpha^{-1}} = 2$, then $\tilde{Z}_\alpha$ is a (blown-up) K3-surface.

c) *Suppose that $\kappa(\tilde{Z}_\alpha) \geq 1$ and that for some $x \in X$ we have*

$$2p_g(\tilde{Z}_\alpha) > 2g_{x,\alpha} + c_{x,\alpha}.$$

If either $g_{x,\alpha} \geq 2$ or $g_{x,\alpha} = g_{x,\alpha} = 1$ and $r_{1,x,\alpha} > 0$, then $\tilde{Z}_\alpha$ is of general type.

In [17], the above theorem will be used to determine the Enriques-Kodaira classification of the *modular diagonal quotient surfaces* $\tilde{Z}_\alpha = \tilde{Z}_{X,G,\alpha}$ with $X = X(N)$, the modular curve. It is interesting to note that all except two of the cases considered there are actually covered by the above theorem.

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1 Curves with a group action

1.1 The local invariants

Let X be a smooth compact complex curve of genus $g = g(X)$ and let G be a finite group of order m which acts faithfully on X by holomorphic automorphisms. Then the quotient $\bar{X} = G\backslash X$ is again a smooth compact curve, and the quotient map $\pi : X \to \bar{X}$ has degree $\deg(\pi) = m$.

More generally, let $H \le G$ be any subgroup of G, and let X_H denote the quotient of X with respect to H. We then have two associated coverings,

$$\pi_H : X \to X_H \qquad \text{and} \qquad \bar{\pi}_H : X_H \to \bar{X},$$

of degree $\deg(\pi_H) = n_H := |H|$ and $\deg(\bar{\pi}_H) = m/n_H$, respectively, whose composition is $\pi = \bar{\pi}_H \circ \pi_H$. For future reference, we record the following elementary facts.

Remark 1.1 a) If $x \in X$ is any point, then its stabilizer $H_x = \{\tau \in H : \tau x = x\}$ is always a cyclic subgroup of H, and we have $H_{\tau x} = {}^{\tau}H_x = H^{\tau^{-1}} = \tau H_x \tau^{-1}$, for any $\tau \in H$. Its order is the ramification index of π_H at x: $e_x(H) = e_x(\pi_H) = |H_x|$. Since $e_x(H)$ depends only on $\tilde{x} = \pi_H(x) \in X_H$ (see b)), we shall often write $e_{\tilde{x}}(H) = e_x(H)$.

b) Since the group H acts transitively on the points of the fibre $\pi_H^{-1}(\pi_H(x))$, the map $\tau \mapsto \tau x$ induces a bijection

$$f_x = f_{H,x} : H/H_x \xrightarrow{\sim} \pi_H^{-1}(\pi_H(x))$$

between the right coset space H/H_x and the fibre at $\pi_H(x)$.

c) Similarly, the map $\tau \mapsto \pi_H(\tau x)$ induces a bijection

$$\bar{f}_x = \bar{f}_{H,x} : H\backslash G/G_x \xrightarrow{\sim} X_{H,\bar{x}} := \bar{\pi}_H^{-1}(\bar{x})$$

between the double coset space $H\backslash G/G_x$ and the fibre $X_{H,\bar{x}}$ over $\bar{x} = \pi(x)$. In this case the ramification index of $y = \bar{f}_x(HgG_x) \in \bar{\pi}_H^{-1}(\bar{x})$ is given by

$$\bar{e}_y(H) := e_y(\bar{\pi}_H) = \frac{|G_x|}{|H \cap gG_xg^{-1}|} = \frac{|HgG_x|}{n_H}.$$

d) For any $\sigma \in G$ there is a unique isomorphism $\sigma_H : X_H \xrightarrow{\sim} X_{{}^{\sigma}H}$ such that $\pi_{{}^{\sigma}H} \circ \sigma = \sigma_H \circ \pi_H$. Since $({}^{\sigma}H)_{\sigma x} = H_x$ for any $x \in G$, it follows that $e_{\sigma x}({}^{\sigma}H) = e_x(H)$ and that $e_{\sigma_H \tilde{x}}({}^{\sigma}H) = e_{\tilde{x}}(H)$, for $\tilde{x} \in X_H$.

e) The map π_H is ramified at the set $T_H = \{x \in X : e_x(H) \neq 1\}$ of fixed points of H. Since $e_x(H) = e_{\tilde{x}}(H)$ depends only on $\tilde{x} = \pi_H(x) \in X_H$, it follows that $T_H = \pi_H^{-1}(R_H)$, where $R_H = \{\bar{P} \in X_H : e_{\bar{P}}(H) \neq 1\}$ denotes the *ramification locus* of $\bar{\pi}_H$. Let us denote the cardinalities of these sets by $r_H = |R_H|$ and $t_H = |T_H|$. Then by the *Riemann–Hurwitz relation*, the genus $g_H = g(X_H)$ of X_H can expressed in terms of r_H and t_H as follows:

$$(1) \qquad g_H = 1 + \frac{g-1}{n_H} - \frac{1}{2}\sum_{\tilde{x} \in X_H}\left(1 - \frac{1}{e_{\tilde{x}}}\right) = 1 + \frac{g-1}{n_H} - \frac{r_H}{2} + \frac{t_H}{2n_H}.$$

In addition to the ramification indices e_x, there are further invariants attached to the covering π which will be of paramount importance in the study of diagonal quotient surfaces. These arise from the so-called "local action" of the stabilizer G_x at a fixed point $x \in T = T_G$. We briefly recall this notion and thereby also fix some notation.

Notation 1.2 For each (fixed) point $x \in X$, we have a canonical linear action of the stabilizer G_x on the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2 \simeq \mathbb{C}$ at x, which gives rise to a linear character $\lambda_x = \lambda_{G,x} : G_x \to \mathbb{C}$. In terms of a local parameter (or coordinate) $t = t_x$, this character is given by:

$$\tau^* t \equiv \lambda_x(\tau) t \ (\mathrm{mod}\, \mathfrak{m}_x^2), \quad \text{for} \ \ \tau \in G_x. \tag{2}$$

Alternatively, λ_x could also be described as follows. By a more judicious choice of the local coordinate $t = t_x$, we can arrange that the action is locally linear on X, so that we have

$$\tau^* t = \lambda_x(\tau) t, \quad \text{for} \ \ \tau \in G_x. \tag{3}$$

From either description is easy to see that λ_x is a faithful character and hence a generator of the character group of G_x; in particular, $\mathrm{ord}(\lambda_x) = |G_x| = e_x$. For future reference, let us note the formula

$$\lambda_{\sigma x}(\tau) = \lambda_x(\sigma^{-1}\tau\sigma) =: {}^{\sigma}\lambda_x(\tau), \quad \text{for } \tau \in G_{\sigma x} = {}^{\sigma}G_x = \sigma G_x \sigma^{-1}, \tag{4}$$

from which it follows that the induced characters

$$\lambda_{\bar{x},k} = \lambda_{G,\bar{x},k} = \mathrm{ind}_{G_x}^{G}(\lambda_x^k) \tag{5}$$

only depend on $\bar{x} = \pi(x)$. Note that the latter characters satisfy the following *restriction formula*: if $H \leq G$ is any subgroup, then

$$(\lambda_{G,\bar{x},k})_{|H} = \sum_{\tilde{x} \in \bar{\pi}^{-1}(\bar{x})} \lambda_{H,\tilde{x},k}; \tag{6}$$

this follows in view of Remark 1.1c) immediately from Mackey's Subgroup Theorem (cf. [3], (10.13)) together with the fact that ${}^{\tau}(\lambda_x^k)_{|{}^{\tau}G_x \cap H} = \lambda_{H,\tau x,k}$.

Although we are now ready to define the local invariants, it is best to defer the definition for a moment because for the application to the twisted diagonal quotient surfaces we require not only the local invariants themselves but also certain "twisted versions" of these which depend on a group automorphism $\alpha \in \mathrm{Aut}(G)$. The reason for this is that we need to consider not only the "standard" action $\Phi : G \times X \to X$ of G on X given by $\Phi(\sigma, x) = \sigma \cdot x$, but also the *twisted action*

$$\Phi_\alpha : G \times X \to X$$

which is obtained by composing Φ with α; explicitly, $\Phi_\alpha(\sigma, x) := \Phi(\alpha(\sigma), x) = \alpha(\sigma) \cdot x$.

While this is essentially just a renaming of the group elements of G, and hence does not significantly alter the geometry of the action, some care must be taken with the action of subgroups, particularly if we consider several actions at the same time. We observe:

Remark 1.3 a) The quotient $G\backslash X = G\backslash_{\Phi_\alpha} X$ is independent of α, as is the covering $\pi = \pi_{\Phi_\alpha} : X \to \bar{X}$ and hence also the set $T = T_{\Phi_\alpha,G}$ of fixed points of the action Φ_α. The stabilizer $G_{x,\alpha} := G_{\Phi_\alpha,x}$ of a point x with respect to Φ_α, however, is $G_{x,\alpha} = \alpha^{-1}(G_x)$ and hence is cyclic of order $e_{\Phi_\alpha,x} = e_x$. Note that if $\tau \in G$, then $G_{\alpha(\tau)x,\alpha} = {}^\tau G_{x,\alpha} = \tau G_{x,\alpha}\tau^{-1}$. Similarly, the local character of $x \in X$ with respect to the twisted action is given by:

$$\lambda_{x,\alpha} := \lambda_{\Phi_\alpha,x} = \lambda_x \circ \alpha;$$

both are characters on $G_{x,\alpha} = \alpha^{-1}(G_x)$.

In addition, if $H \le G$ is any subgroup, then we have $H\backslash_{\Phi_\alpha} X = H^\alpha\backslash_\Phi X$ and $\pi_{\Phi_\alpha,H} = \pi_{\Phi,H^\alpha}, \bar{\pi}_{\Phi_\alpha,H} = \bar{\pi}_{\Phi,H^\alpha}$, where $H^\alpha = \alpha^{-1}(H)$.

b) If $\alpha = \beta_\sigma$ is the inner automorphism $\beta_\sigma(\tau) = \sigma^{-1}\tau\sigma$, then we had seen above in Remark 1.1d) that $X_{H^\alpha} \simeq X_H$. This, however, need not be true for an outer automorphism α, as the case of the *Fermat curve* X_p of exponent p shows: here $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ acts on X_p, and $\mathrm{Aut}(G) = \mathrm{Gl}(\mathbb{Z}/p\mathbb{Z})$ acts transitively on the subgroups $H \le G$ of order p, but three of the quotients X_H have genus 0, whereas the other $p-2$ have genus $\frac{1}{2}(p-1)$.

We now come to the definition of the (twisted) local invariants which are fundamental for the study of diagonal quotient surfaces. Since these are closely linked to the coverings π_H attached to stabilizer subgroups $H = G_{x,\alpha}$, we first introduce some convenient abbreviations for this map and other related objects.

Notation 1.4 a) If $H = G_{x,\alpha} = \alpha^{-1}(G_x)$ is the stabilizer of a point $x \in X$ with respect to the action Φ_α, then we write $\pi_{x,\alpha} = \pi_H$, $X_{x,\alpha} = X_H$, $g_{x,\alpha} = g_H$, $r_{x,\alpha} = r_H$, etc. for the associated subcover, the genus, the number of ramification points, and so on. Note that the isomorphism class of X_x depends only on $\bar{x} = \pi(x)$, for if $x' = \alpha(\sigma)x \in \pi^{-1}(\bar{x})$, then $G_{x',\alpha} = {}^\sigma G_{x,\alpha} = {}^\sigma H$, and hence $\sigma_{x,\alpha} := \sigma_H : X_{x,\alpha} \to X_{^\sigma H} = X_{x',\alpha}$ defines an isomorphism (cf. Remark 1.1b)). We can thus write $X_{\bar{x},\alpha} = X_{x,\alpha}$ and $g_{\bar{x},\alpha} = g_{x,\alpha}$ in the sequel.

b) To define the local invariants, let $x, y \in X$ be two points, and put $G_{x,y,\alpha} = G_x \cap G_{y,\alpha}$, which is the stabilizer of x with respect to $H = G_{y,\alpha}$. (Dually, $\alpha(G_{x,y,\alpha})$ is that of y with respect to $H = G_{x,\alpha^{-1}}$.) Its order is denoted by

$$n_\alpha(x,y) = |G_{x,y,\alpha}| = e_x(G_{y,\alpha}) = e_y(G_{x,\alpha^{-1}}). \tag{7}$$

We now define the integer $q = q_\alpha(x,y)$ by the condition:

$$(\lambda_x)_{|K} = ((\lambda_{y,\alpha})_{|K})^q \quad \text{and} \quad 0 \le q < n, \tag{8}$$

where $K = G_{x,y,\alpha}$ and $n = n_\alpha = |K|$. Such an integer exists because $(\lambda_x)_{|K}$ and $(\lambda_{y,\alpha})_{|K}$ are both faithful characters of K and hence are generators of the character group; thus, we also have that $(n,q) = 1$. We note that both n and q depend only on the image $\tilde{x} = \pi_{y,\alpha}(x)$ of x in $X_{y,\alpha}$, so we can write $n_\alpha(\tilde{x},y) = n_\alpha(x,y)$ and $q_\alpha(\tilde{x},y) = q_\alpha(x,y)$. For n this clear from its definition (7); for q we observe that if $x' = \tau x$ with $\tau \in H := G_{y,\alpha}$, then we have $G_{x',y,\alpha} = H \cap \tau G_x \tau^{-1} = \tau(H \cap G_x)\tau^{-1} = K$ because H is abelian, and so by (4) we have $(\lambda_{x'})_{|G_{x',y,\alpha}} = {}^\tau((\lambda_x)_{|K}) = (\lambda_x)_{|K}$ (using the commutativity of H again).

Finally, for $\bar{x} \in \bar{X}$ and $1 \le \nu \le e_y$, we define $s_{\nu,\alpha}(\bar{x},y) = |S_{\nu,\alpha}(\bar{x},y)|$ as the cardinality of the set

$$S_{\nu,\alpha}(\bar{x},y) = \left\{ \tilde{x} \in X_{\bar{x},y,\alpha} : q_\alpha(\tilde{x},y) = \frac{\nu}{(\nu,e_y)} \text{ and } n_\alpha(\tilde{x},y) = \frac{e_y}{(\nu,e_y)} \right\}, \tag{9}$$

where $X_{\bar{x},y,\alpha} = X_{H,\bar{x}} = \bar{\pi}_{y,\alpha}^{-1}(\bar{x})$ denotes the fibre of $\bar{\pi}_{y,\alpha}$ at $\bar{x}$. We note that $s_{\nu,\alpha}(\bar{x},y)$ depends only on the image $\bar{y} = \pi(y)$ of y in $\bar{X}$, so that we can write $s_{\nu,\alpha}(\bar{x},\bar{y}) = s_{\nu,\alpha}(\bar{x},y)$. This follows from the fact that if $y' = \alpha(\sigma)y \in \pi^{-1}(\bar{y})$, then $\sigma_{y,\alpha}(S_{\nu,\alpha}(\bar{x},y)) = S_{\nu,\alpha}(\bar{x},y')$, as is easy to check.

Remark 1.5 a) The local invariants $s_{\nu,\alpha}(\bar{x},\bar{y})$ may be viewed as a refinement of the ramification numbers introduced earlier. Indeed, by definition each $\tilde{x} \in S_{\nu,\alpha}$ has ramification index $e_{\tilde{x}}(\pi_{y,\alpha}) = n_\alpha(\tilde{x},y) = \frac{e_y}{(\nu,e_y)}$, so

$$r_{n,\alpha}(\bar{x},\bar{y}) := \sum_{\substack{\nu=1 \\ (\nu,e)=\frac{e}{n}}}^{e} s_{\nu,\alpha}(\bar{x},\bar{y}),$$

is just the number of points $\tilde{x} \in X_{\bar{x},y,\alpha} = \bar{\pi}_{y,\alpha}^{-1}(\bar{x})$ with ramification degree $e_{\tilde{x}}(\pi_{y,\alpha}) = n$. For future reference, let us record here the formula

$$\frac{m}{e_{\bar{x}}} = \sum_{n|e_{\bar{y}}} r_n(\bar{x},\bar{y})\frac{e_{\bar{y}}}{n} \tag{10}$$

which follows from the fact that there are $\frac{e_{\bar{y}}}{n}$ points on X above each $\tilde{x} \in S_\nu(\bar{x},y)$ when $n = (\nu, e_{\bar{y}})$, combined with the fact that there are $\frac{m}{e_x}$ points in the fibre $\pi^{-1}(\bar{x})$.

In a similar vein, if $n > 1$, then

$$r_{n,\alpha}(\bar{y}) := \sum_{\bar{x}\in\bar{X}} r_{n,\alpha}(\bar{x},\bar{y}) = \sum_{\substack{\nu=1 \\ (\nu,e)=\frac{e}{n}}}^{e} s_{\nu,\alpha}(\bar{y}), \text{ where } s_{\nu,\alpha}(\bar{y}) = \sum_{\bar{x}\in\bar{X}} s_{\nu,\alpha}(\bar{x},\bar{y}),$$

is the number of points $\tilde{x} \in X_{y,\alpha}$ which are ramified of degree $e_{\tilde{x}}(\pi_{y,\alpha}) = n$, and hence

$$r_{\bar{y},\alpha} := r_{G_{y,\alpha}} = \sum_{1<n|e} r_{n,\alpha}(\bar{y}) = \sum_{\nu=1}^{e-1} s_{\nu,\alpha}(\bar{y}), \text{ respectively } t_{\bar{y},\alpha} := t_{G_{y,\alpha}} = \sum_{1<n|e} \frac{e}{n} r_{n,\alpha}(\bar{y}),$$

is the total number of ramified points (respectively, the number of fixed points) of $\pi_{y,\alpha} : X \to X_{y,\alpha}$. Thus, by the Riemann–Hurwitz formula (1), the genus $g_{\bar{y},\alpha}$ of $X_{y,\alpha}$ is given by

$$g_{\bar{y},\alpha} - 1 = \frac{g-1}{e_{\bar{y}}} - \tfrac{1}{2}\sum_{1<n|e_{\bar{y}}} r_{n,\alpha}(\bar{y})\left(1-\frac{1}{n}\right) = \frac{g-1}{e_{\bar{y}}} - \frac{r_{\bar{y},\alpha}}{2} + \frac{t_{\bar{y},\alpha}}{2e_{\bar{y}}}. \tag{11}$$

From this it follows that the following quantity $\mathbb{G}_\alpha = \mathbb{G}_\alpha(X) = \mathbb{G}_\alpha(X,G)$, which will occur frequently in many expressions below, can be computed from the numbers $r_{n,\alpha}(\bar{y})$ as follows:

$$\mathbb{G}_\alpha := \bar{g}+(\bar{g}-1)g+\tfrac{1}{2}\sum_{\bar{y}\in\bar{X}}(g-g_{\bar{y},\alpha}) = 2\bar{g}-1+\frac{(g-1)^2}{m}+\tfrac{1}{4}\sum_{\bar{y}\in\bar{X}}\sum_{1<n|e_{\bar{y}}} r_{n,\alpha}(\bar{y})\left(1-\frac{1}{n}\right). \tag{12}$$

b) From their definition it is evident that the local invariants $s_{\nu,\alpha}(\bar{x},\bar{y})$ are closely linked to the local characters $\lambda_{x,\alpha}$. This relation is made more explicit by the following *inner product formula* which is valid for any $\bar{x},\bar{y}\in\bar{X}$ and $k,l\in\mathbb{Z}$ when $e=e_{\bar{y}}$:

$$(13)\qquad (\lambda_{\bar{x},k},\lambda_{\bar{y},\alpha,l})_G \;=\; \sum_{\substack{\nu=1\\ \nu k\equiv(\nu,e)l\ (e)}}^{e} s_{\nu,\alpha}(\bar{x},\bar{y})$$

which follows easily from the restriction formula (6) by using Frobenius reciprocity.

c) Although the local invariants are defined asymmetrically, they satisfy a certain *symmetry formula*. This formula applies when $\nu\frac{e_{\bar{x}}}{e_{\bar{y}}}\in\mathbb{N}$, which is no restriction since otherwise $s_\nu(\bar{x},\bar{y})=0$ (cf. Proposition 1.7c) below). Then there is a unique integer ν^* with $1\le\nu^*\le e_{\bar{x}}$ such that $\frac{e_{\bar{x}}}{(\nu^*,e_{\bar{x}})}=\frac{e_{\bar{y}}}{(\nu,e_{\bar{y}})}$ and $\frac{\nu^*}{(\nu^*,e_{\bar{x}})}\frac{\nu}{(\nu,e_{\bar{y}})}\equiv 1\bmod\left(\frac{e_{\bar{y}}}{(\nu,e_{\bar{y}})}\right)$, and we have

$$(14)\qquad s_{\nu,\alpha}(\bar{x},\bar{y})=s_{\nu^*,\alpha^{-1}}(\bar{y},\bar{x}).$$

To see this, note first that the map $\sigma\mapsto\alpha(\sigma)^{-1}$ induces a bijection $\bar{\alpha}_{x,y}: G_{y,\alpha}\backslash G/G_x\xrightarrow{\sim}G_{x,\alpha^{-1}}\backslash G/G_y$ and hence also via Remark 1.1c) a bijection $\bar{\alpha}_{x,y}: X_{\bar{x},y,\alpha}=\bar{\pi}_{y,\alpha}^{-1}(\bar{x})\xrightarrow{\sim}X_{\bar{y},x,\alpha^{-1}}=\bar{\pi}_{x,\alpha^{-1}}^{-1}(\bar{y})$. Then for $\tilde{x}\in X_{\bar{x},y,\alpha}$ we have $n_{\alpha^{-1}}(\alpha_{x,y}(\tilde{x}),x)=n_\alpha(\tilde{x},y)$ and $q_{\alpha^{-1}}(\alpha_{x,y}(\tilde{x}),x)q_\alpha(\tilde{x},y)\equiv 1 \pmod{n_\alpha(\tilde{x},y)}$, and so it follows that $\bar{\alpha}_{x,y}(S_\nu(\bar{x},y))=S_{\nu^*}(\bar{y},x)$, which proves (14).

d) The local invariants $s_{\nu,\alpha}(\bar{x},\bar{y})$ depend only on the image of α in the outer automorphism group $\mathrm{Out}(G)=\mathrm{Aut}(G)/\mathrm{Inn}(G)$. Indeed, if $\tau\in G$, then we have $n_{\beta_\tau\circ\alpha}(x,y)=n_\alpha(x,\tau y)$ and $q_{\beta_\tau\circ\alpha}(x,y)=q_\alpha(x,\tau y)$, so $\tau_{y,\alpha}(S_{\nu,\beta_\tau\circ\alpha}(\bar{x},y))=S_{\nu,\alpha}(\bar{x},\tau y)$, and hence it follows that $s_{\nu,\beta_\tau\circ\alpha}(\bar{x},\bar{y})=s_{\nu,\alpha}(\bar{x},\bar{y})$.

Since the local invariants $s_{\nu,\alpha}(\bar{x},\bar{y})$ will play such an important role in the sequel, it will be useful to have an efficient method for computing them. It turns out that this can be done by determining the following “normalizing sets”.

Notation 1.6 For $\sigma,\tau\in G$ put $\bar{\sigma}=\sigma^{s/r},\bar{\tau}=\tau^{t/r}$ where $s=|\sigma|,t=|\tau|$ and $r=(s,t)$. Moreover, for any integer $\nu\ge 1$ put $k=(\nu,r)$ and define the *relative normalizing sets*

$$\begin{aligned}
N_\nu(\sigma,\tau) &= N_k(\sigma,\tau) &&:= \{g\in G: g\bar{\sigma}^\nu g^{-1}\in\langle\bar{\tau}^\nu\rangle\} = \{g\in G: g\langle\bar{\sigma}^\nu\rangle g^{-1}=\langle\bar{\tau}^\nu\rangle\}\\
N_\nu^i(\sigma,\tau) &= N_k^i(\sigma,\tau) &&:= \{g\in G: g\bar{\sigma}^k g^{-1}=\bar{\tau}^{ki}\}\\
N_\nu^*(\sigma,\tau) &= N_k^*(\sigma,\tau) &&:= N_k(\sigma,\tau)\setminus\bigcup_{d|k,d\ne k}N_d(\sigma,\tau)\\
N_{\nu,i}^*(\sigma,\tau) &= N_{k,i}^*(\sigma,\tau) &&:= N_\nu^*(\sigma,\tau)\cap N_\nu^i(\sigma,\tau),
\end{aligned}$$

where $i\in\mathbb{Z}$. Clearly $N_\nu^i(\sigma,\tau)=N_\nu^j(\sigma,\tau)$ if $i\equiv j \pmod{\frac{r}{k}}$ and $N_\nu^i(\sigma,\tau)=\emptyset$ if $(i,\frac{r}{k})\ne 1$, and similarly for $N_{\nu,i}^*(\sigma,\tau)$. Moreover, we have the following decompositions:

$$G=\dot{\bigcup_{k|r}}N_\nu^*(\sigma,\tau),\quad N_\nu(\sigma,\tau)=\dot{\bigcup_{1\le i\le r}}N_\nu^i(\sigma,\tau)\quad\text{and}\quad N_\nu^*(\sigma,\tau)=\dot{\bigcup_{1\le i\le r}}N_{\nu,i}^*(\sigma,\tau).$$

Note that $H_1:=\langle\sigma\rangle$ and $H_2:=\langle\tau\rangle)$ act on N_ν^i etc. by multiplication on the right and left, respectively, and hence all these sets are unions of (H_2,H_1)-double cosets of G.

If $\tau = \sigma$ then we shall write $N_\nu(\sigma) = N_\nu(\sigma,\sigma), N^*_\nu(\sigma) = N^*_\nu(\sigma,\sigma)$, and so on. Note that in this case $N_1(\sigma) = N_G(\langle\sigma\rangle)$ and $N^1_1(\sigma) = \mathrm{C}_G(\sigma)$ are just the usual normalizer and the centralizer of $\langle\sigma\rangle$, respectively.

These normalizing sets are connected to the invariants $n_\alpha(x,y), q_\alpha(x,y)$ and $s_{\nu,\alpha}(\bar{x},\bar{y})$ in the following way.

Proposition 1.7 *For $x, y \in X$, let σ_x and σ_y be generators of $G_x = \langle\sigma_x\rangle$ and of $G_{y,\alpha} = \langle\sigma_y\rangle$. Let $\bar\sigma_x, \bar\sigma_y$ and $r = (e_x, e_y)$ be defined as above, and let $c = c_\alpha(\sigma_x,\sigma_y)$ be such that $\lambda_x(\bar\sigma_x) = \lambda_{y,\alpha}(\bar\sigma_y)^c$. Moreover, let $\nu \ge 1$ and put $k = (\nu, r)$. Then for $g \in G$ we have:*

a) $g \in N^*_\nu(\sigma_x,\sigma_y) \iff n_\alpha(gx,y) = \frac{r}{k} \iff G_{gx,y,\alpha} = g\langle\bar\sigma_x^k\rangle g^{-1} = \langle\bar\sigma_y\rangle$.

b) If $g \in N^*_{\nu,i}(\sigma_x,\sigma_y)$, then $iq_\alpha(gx,y) \equiv c \pmod{\frac{r}{k}}$. Thus, if $ij \equiv c \pmod{\frac{r}{k}}$ and $1 \le j \le \frac{r}{k}$, then $g \in N^*_{\nu,i}(\sigma_x,\sigma_y) = N^*_{k,i}(\sigma_x,\sigma_y) \iff \pi_{y,\alpha}(gx) \in S_{jke_y/r,\alpha}(\bar{x},y)$.

c) If $\nu' := \nu\cdot\frac{r}{e_y} \notin \mathbb{N}$ or if $\nu > e_y$, then $s_{\nu,\alpha}(\bar{x},\bar{y}) = 0$. If $\nu' \in \mathbb{N}$ and $\nu \le e_y$, then we have a natural bijection

$$G_{y,\alpha}\backslash N^*_{\nu',c\bar\nu^*}(\sigma_x,\sigma_y)/G_x \xrightarrow{\sim} S_{\nu,\alpha}(\bar{x},y),$$

where $\bar\nu^* \in \mathbb{N}$ is such that $\bar\nu^*\cdot\frac{\nu}{(\nu,e_y)} \equiv 1 \pmod{\frac{e_y}{(\nu,e_y)}}$, and hence

$$s_{\nu,\alpha}(\bar{x},\bar{y}) = \frac{|N^*_{\nu',c\bar\nu^*}(\sigma_x,\sigma_y)|}{(\nu,e_y)e_x}.$$

Proof. a) The first equivalence clearly follows from the following assertion:

$$g \in N_k(\sigma_x,\sigma_y) \iff \tfrac{r}{k} \mid n_\alpha(gx,y).$$

To prove this, note first that if $g \in N_k(\sigma_x,\sigma_y)$ then $g\langle\bar\sigma_x^k\rangle g^{-1} = \langle\bar\sigma_y^k\rangle \le gG_xg^{-1} \cap G_{y,\alpha} = G_{gx,y,\alpha}$, and so $\frac{r}{k} = |\langle\bar\sigma_y^k\rangle|$ divides $|G_{gx,y,\alpha}| = n_\alpha(gx,y)$. Conversely, if $\frac{r}{k}|n_\alpha(gx,y)$, then $g\langle\bar\sigma_x^k\rangle g^{-1}$ and $\langle\bar\sigma_y^k\rangle$ are subgroups of the cyclic group $G_{gx,y,\alpha}$ and hence must be equal, which means that $g \in N_k(\sigma_x,\sigma_y)$. This proves the above assertion and hence the first equivalence. Moreover, the above proof shows that the second equivalence is also valid.

b) By the definition of $q := q_\alpha(gx,y)$ we have $\lambda_{gx}(g\bar\sigma_x^kg^{-1}) = \lambda_{y,\alpha}(g\bar\sigma_x^k)^q = \lambda_{y,\alpha}(\bar\sigma_y^{ki})^q = \lambda_{y,\alpha}(\bar\sigma_y)^{ikq}$. On the other hand, we have $\lambda_{gx}(g\bar\sigma_x^kg^{-1}) = \lambda_x(\bar\sigma_x^k) = \lambda_{y,\alpha}(\sigma_y)^{kc}$, and so it follows that $iq \equiv c \pmod{\frac{r}{k}}$, as asserted. Thus, if $g \in N^*_{\nu,i}$, then $q = j$ and so $\pi_{y,\alpha}(gx) \in S_{\tilde\nu,\alpha}$ by the definition of $S_{\tilde\nu}$, if $\tilde\nu = jke_y/r$. This proves one implication, and the converse follows from this since the sets $N^*_{k,i}$ partition G and since i and j determine each other.

c) Suppose $s_{\nu,\alpha}(\bar{x},y) \ne 0$, so there exists $\tilde{x} = \pi_{y,\alpha}(gx)$ such that $n_\alpha(gx,y) = \frac{e_y}{(\nu,e_y)}$. Since $n_\alpha(gx,y)|r = (e_x,e_y)$, we have $\frac{e_y}{r}|(\nu,e_y)|\nu$, so $\nu' \in \mathbb{N}$. This proves the first statement. Moreover, we then have $\frac{e_y}{(\nu,e_y)} = \frac{r}{(\nu',r)}$ and $\frac{\nu}{(\nu,e_y)} = \frac{\nu'}{(\nu',r)}$, and so by b) we see that the map $\bar{f}_{y,\alpha} : g \mapsto \pi_{y,\alpha}(gx)$ of Remark 1.1c) induces the desired bijection. Thus, $N^*_{\nu',c\bar\nu^*}(\sigma_x,\sigma_y)$ is the disjoint union of precisely $s_{\nu,\alpha}(\bar{x},\bar{y})$ double cosets $G_{y,\alpha}gG_x$ with $g \in N^*_{\nu',c\bar\nu^*}(\sigma_x,\sigma_y)$. But each such double coset has $|G_{y,\alpha}gG_x| = \frac{e_xe_y}{n_\alpha(gx,y)} = e_x(\nu,e_y)$ elements, so $|N^*_{\nu',c\bar\nu^*}(\sigma_x,\sigma_y)| = s_{\nu,\alpha}(\bar{x},\bar{y})e_x(\nu,e_y)$, which proves the desired formula.

1.2 The singular character h^1_X

The action of G on X induces an action on the singular cohomology space $H^1(X,\mathbb{C})$ as well as on the space $H^0(X,\Omega^1_X)$ of global holomorphic differential forms. The corresponding characters will be denoted by $h^1 = h^1_X$ and $\omega = \omega_X$, respectively. If we twist the action by $\alpha \in \mathrm{Aut}(G)$, then the twisted action Φ_α induces a twisted action on $H^1(X,\mathbb{C})$ and $H^0(X,\Omega_X)$, whose corresponding characters are

$$(15) \qquad h^1_\alpha = h^1_{X,\alpha} = h^1_X \circ \alpha \qquad \text{and} \qquad \omega_\alpha = \omega_{X,\alpha} = \omega_X \circ \alpha.$$

We now want to compute the inner product $(h^1, h^1_\alpha)_G$ of the characters h^1 and h^1_α since it will be interpreted below as a Betti number of the twisted diagonal surface Z_α.

Proposition 1.8 *The character h^1_α of $H^1(X,\mathbb{C})$ with respect to the action Φ_α is given by*

$$(16) \qquad h^1_\alpha = 2\cdot 1_G + (2\bar{g}-2)\mathrm{reg}_G + \sum_{\bar{x}\in R}(\mathrm{reg}_G - (1_{G_{\bar{x},\alpha}})^G),$$

where reg_G denotes the regular representation of G, R the ramification locus of $\pi : X \to \bar{X}$, and $(1_{G_{\bar{x},\alpha}})^G = \mathrm{ind}^G_{G_{x,\alpha}}(1_{G_{x,\alpha}})$ the induction to G of the trivial character $1_{G_{x,\alpha}}$ of the stabilizer $G_{x,\alpha} = \alpha^{-1}(G_x) \le G$ of any $x \in \pi^{-1}(\bar{x})$. Thus, the inner product of h^1 with its twist by α is

$$(17) \quad (h^1, h^1_\alpha)_G = (h^1, h^1_{\alpha^{-1}}) = 4\mathbb{G}_\alpha = 4\bar{g} + 4(\bar{g}-1)g + 2\sum_{\bar{x}\in\bar{X}}(g - g_{\bar{x},\alpha})$$
$$= 4\bar{g}(g+1) + 2(r-2)g - 2\sum_{\bar{x}\in R} g_{\bar{x},\alpha}$$

Proof. By Serre [21], p. 106, $h^1_\alpha = 2\cdot 1_G + (2\bar{g}-2)\mathrm{reg}_G + \sum_{\bar{x}\in\bar{X}} a_{\bar{x}}$, where $a_{\bar{x}}$ denotes the Artin character. Since we have tame ramification in characteristic 0, it follows from the definition of $a_{\bar{x}}$ that $a_{\bar{x}} = \mathrm{reg}_G - (1_{G_{\alpha,x}})^G$, which proves (16).

The first equality in (17) is immediate, since for any two characters h, h' of G we have $(h, h'\circ\alpha)_G = (h\circ\alpha^{-1}, h')_G = (h', h\circ\alpha^{-1})_G$. Since the last two equalities are immediate from the definitions, it is enough to verify the second. For this, note first that for any subgroup $H \le G$ we have

$$(18) \qquad (h^1, (1_H)^G)_G = 2g_H$$

because $(h^1,(1_H)^G)_G = (h^1|_H, 1_H)_H = \dim(H^1(X)^H) = \dim(H^1(X_H)) = 2g_H$ by Frobenius reciprocity and the fact that $H^1(X)^H = H^1(X_H)$; cf. Proposition 2.7 below. Thus, from (16) we obtain

$$(h^1, h^1_\alpha) = 2(h^1, 1_G) + (2\bar{g}-2)(h^1, \mathrm{reg}_G) + \sum_{\bar{x}\in\bar{X}}((h^1,\mathrm{reg}_G) - (h^1,(1_{G_{\bar{x},\alpha}})^G)$$
$$= 4\bar{g} + (2\bar{g}-2)2g + \sum_{\bar{x}\in\bar{X}}(2g - 2g_{\bar{x},\alpha})) = 4\mathbb{G}_\alpha,$$

which gives the desired second equation of (17).

1.3 The character ω_X

We now turn to the action of G on $H^0(X, \Omega^1_X)$. Its character ω was first determined by Chevalley and Weil [4] in terms of the local characters λ_x (cf. Notation 1.2). To state this theorem in a convenient form, we first introduce the following *ramification characters* w_x.

Notation 1.9 For $x \in X$ we define the *ramification character* on $G_{x,\alpha}$ and on G by

$$(19)\qquad w_{x,\alpha} = w_{G,x,\alpha} = \sum_{k=1}^{e_x-1} k\lambda_{x,\alpha}^k \quad \text{and} \quad w_{\bar{x},\alpha} = w_{G,\bar{x},\alpha} = \sum_{x\in\pi^{-1}(\bar{x})} (w_{x,\alpha})^G,$$

respectively; here, $\bar{x} = \pi(x)$. We note that the latter character can also be written in the form

$$w_{\bar{x},\alpha} = \frac{m}{e_{\bar{x}}} w'_{\bar{x},\alpha},$$

where $w'_{\bar{x},\alpha} = (w_{x,\alpha})^G$ is independent of the choice of $x \in \pi^{-1}(\bar{x})$ since for $\sigma = \alpha(\tau) \in G$ we have $\lambda_{\sigma x,\alpha} = {}^{\tau}\lambda_{x,\alpha} = \lambda_{x,\alpha} \circ \beta_\tau$, with β_τ as in Remark 1.3b)).

The formula of Chevalley and Weil[4] may now be expressed as follows (cf. Kani[16], Corollary to Theorem 2):

Proposition 1.10 *The character of $H^0(X, \Omega^1_X)$ as a G-module with respect to the action Φ_α is given by*

$$\omega_\alpha = 1_G + (\bar{g} - 1)\mathrm{reg}_G + \overline{w_\alpha},$$

where $w_\alpha = w_{X,\alpha} := \frac{1}{m}\sum_{\bar{x}\in R} w_{\bar{x},\alpha} = \frac{1}{m}\sum_{x\in X}(w_{x,\alpha})^G$.

Remark 1.11 The character ω_α is related to the singular character h^1_α by the formula

$$\omega_\alpha + \overline{\omega_\alpha} = h^1_\alpha;$$

this may be deduced either from Hodge-decomposition or from 1.10 and (16) since it is immediate that $w_{\bar{x},\alpha} + \overline{w_{\bar{x},\alpha}} = ma_{\bar{x}} \ = \ m(\mathrm{reg}_G - (1_{G_{x,\alpha}})^G)$.

Thus, if ω_α is real-valued, i.e. if $\omega_\alpha = \overline{\omega_\alpha}$, then ω_α is exactly one-half of the singular character h^1_α. In general, however, this is not the case; the difference is measured by the character $\mu_\alpha := \omega_\alpha - \overline{\omega_\alpha} = w_\alpha - \overline{w_\alpha}$ as follows:

$$(20)\qquad \omega_\alpha = \tfrac{1}{2}(h^1_\alpha + \mu_\alpha).$$

We observe that the correction term μ_α has a decomposition into local terms, for if we put $\mu_{\bar{y},\alpha} = w_{\bar{y},\alpha} - \overline{w_{\bar{y},\alpha}}$ then clearly

$$(21)\qquad \mu_\alpha = \frac{1}{m}\sum_{\bar{x}\in\bar{X}} \mu_{\bar{y},\alpha}.$$

For future reference, let us note here that these local terms satisfy the following restriction property with respect to a subgroup $H \le G$:

$$(22)\qquad (\mu_{G,\bar{x}})_{|H} = \frac{m}{n_H}\sum_{\tilde{x}\in\bar{\pi}_H^{-1}(\bar{x})} \mu_{H,\tilde{x}};$$

this follows easily from Mackey's subgroup theorem; cf. Kani[16], Proposition 2.

We now want to derive a formula for the inner product $(\omega, \overline{\omega_\alpha})$ which will later be interpreted as a geometric genus (cf. Proposition 3.1). It turns out that this inner product can be conveniently expressed in terms of certain Dedekind sums which we now introduce.

Notation 1.12 Let q and n be two relatively prime integers. Then the *Dedekind sum* (cf. [20], [14]) is defined by

$$\mathbb{S}(q,n) := \sum_{k=1}^{n-1} \left(\left(\frac{k}{n}\right)\right)\left(\left(\frac{kq}{n}\right)\right) = \sum_{k=1}^{n-1} \frac{k}{n}\left(\left(\frac{kq}{n}\right)\right) = \sum_{k=1}^{n-1} \frac{k}{n}\left\langle\frac{kq}{n}\right\rangle - \tfrac{1}{4}(n-1), \tag{23}$$

where, as usual, $\langle x\rangle = x - [x]$ denotes the fractional part of x and $((x))$ is the sawtooth function (i.e. $((x)) = \langle x\rangle - \frac{1}{2}$, if $x \notin \mathbb{Z}$ and $((x)) = 0$ if $x \in \mathbb{Z}$). We now put, for $y \in X$ and $\tilde{x} \in X_{y,\alpha}$:

$$\begin{aligned} \mathbb{S}_\alpha(\tilde{x}, y) &= \mathbb{S}(q_\alpha(\tilde{x}, y), n_\alpha(\tilde{x}, y)) \\ \mathbb{S}_\alpha(\bar{x}, y) &= \sum_{\tilde{x} \in X_{\bar{x},y,\alpha}} \mathbb{S}_\alpha(\tilde{x}, y) = \sum_{n=1}^{e_y} s_{n,\alpha}(\bar{x}, y)\mathbb{S}\left(\frac{n}{(n,e_y)}, \frac{e_y}{(n,e_y)}\right), \end{aligned}$$

where $q_\alpha(\tilde{x}, y), n_\alpha(\tilde{x}, y)$ and $s_{n,\alpha}(\bar{x}, y) = s_{n,\alpha}(\bar{x}, \bar{y})$ are defined in Notation 1.4. From the last equality we see that $\mathbb{S}_\alpha(\bar{x}, y)$ only depends on $\bar{y} = \pi(y)$, so we can write $\mathbb{S}_\alpha(\bar{x}, \bar{y}) = \mathbb{S}_\alpha(\bar{x}, y)$. Finally, we define

$$\mathbb{S}_\alpha(X) = \sum_{\bar{x},\bar{y} \in \bar{X}} \mathbb{S}_\alpha(\bar{x}, \bar{y}) = \sum_{n=1}^{e} s_{n,\alpha}\mathbb{S}\left(\frac{n}{(n,e)}, \frac{e}{(n,e)}\right),$$

where $s_{n,\alpha} = \sum_{\bar{x},\bar{y} \in \bar{X}} s_{n,\alpha}(\bar{x}, \bar{y})$ and $e = \mathrm{lcm}_{\bar{y} \in \bar{X}}(e_{\bar{y}})$.

Theorem 1.13 *Let $\bar{x}, \bar{y} \in \bar{X}$. Then the inner product of the local correction terms is*

$$(\mu_{\bar{x}}, \mu_{\bar{y},\alpha}) = 4m^2\mathbb{S}_\alpha(\bar{x}, \bar{y}), \tag{24}$$

and hence the inner product of ω with its twisted contragredient character is

$$(\omega_X, \overline{\omega_{X,\alpha}}) = \tfrac{1}{4}(h^1_X, h^1_{X,\alpha}) - \tfrac{1}{4}(\mu_X, \mu_{X,\alpha}) = \mathbb{G}_\alpha(X) - \mathbb{S}_\alpha(X); \tag{25}$$

in particular, the inner product of the correction terms is $(\mu_X, \mu_{X,\alpha}) = 4\mathbb{S}_\alpha(X)$.

Proof. Before deriving (24), let us see how (25) follows from it. For this, we first note that since clearly $\overline{\mu_\alpha} = -\mu_\alpha$, we have $(h, \mu_\alpha) = 0$ for any character h with $\overline{h} = h$. Thus, from (20) we obtain $(\omega, \overline{\omega_\alpha}) = \frac{1}{4}(h^1 + \mu, h^1_\alpha - \mu_\alpha) = \frac{1}{4}((h^1, h^1_\alpha) - (\mu, \mu_\alpha))$, which proves the first equality of (25). The second follows directly from (24) and the local sum formula (21). Finally, by substituting the formula (17) for (h^1, h^1_α), the second equality of (25) ensues.

It thus remains to prove (24). For this, we shall first derive the following refinement (26). To state it, fix $x, y \in X$, and put $H = G_{y,\alpha}, K = H_x = G_{x,y,\alpha}, e = e_y = |H|, \tilde{x} = \pi_{y,\alpha}(x)$. Moreover, put $\mu_{y,\alpha} = w_{y,\alpha} - \overline{w_{y,\alpha}}$, so $\mu_{\bar{y},\alpha} = \frac{m}{e}\mathrm{ind}_H^G(\mu_{y,\alpha})$. Then we shall prove:

$$(w_{H,\tilde{x}}, \mu_{y,\alpha})_H = 2e^2\mathbb{S}_\alpha(\tilde{x}, y). \tag{26}$$

First note that (24) is an immediate consequence of this formula. Indeed, by Frobenius reciprocity and the restriction formula (22) we have

$$(27)\qquad \begin{aligned}(\mu_{\bar{x}},\mu_{\bar{y},\alpha})_G &= 2(w_{\bar{x}},\mu_{\bar{y},\alpha})_G = \frac{2m}{e}(w_{\bar{x}},\mathrm{ind}_H^G(\mu_{y,\alpha}))_G\\ &= \frac{2m}{e}(w_{\bar{x}},(\mu_{y,\alpha})_{|H})_H = 2\left(\frac{m}{e}\right)^2\sum_{\tilde{x}\in\pi_{y,\alpha}^{-1}(\bar{x})}(w_{H,\tilde{x}},\mu_{y,\alpha}),\end{aligned}$$

and so, in view of the definition of $\mathbb{S}(\bar{x},\bar{y})$, we see that (24) follows directly from (26).

To prove (26), let us first compute $(\lambda_{H,\tilde{x},k},w_{y,\alpha})$ for $k\in\mathbb{Z}$, where $\lambda_{H,\tilde{x},k}$ is as defined in (5). Putting $n=|K|=n_\alpha(\tilde{x},y)$, $q=q_\alpha(\tilde{x},y)$, we claim that

$$(28)\qquad (\lambda_{H,\tilde{x},k},w_{y,\alpha})_H = e\left(\left\langle\frac{qk}{n}\right\rangle+\frac{1}{2}\left(\frac{e}{n}-1\right)\right).$$

To see this, put $\lambda=(\lambda_{y,\alpha})_{|K}$. Then by the definition of q we have $\lambda_{H,x}^k=\lambda^{qk}$, so by Frobenius reciprocity we obtain

$$\begin{aligned}(\lambda_{H,\tilde{x},k},w_{y,\alpha})_H &= (\lambda_{H,x}^k,(w_{y,\alpha})_{|K})_K = \sum_{j=1}^{e-1} j(\lambda^{qk},\lambda^j) = \sum_{\substack{j=1\\ j\equiv qk(n)}}^{e-1} j\\ &= \sum_{j=0}^{\frac{e}{n}-1}\left(n\left\langle\frac{qk}{n}\right\rangle+jn\right) = e\left(\left\langle\frac{qk}{n}\right\rangle+\frac{1}{2}\left(\frac{e}{n}-1\right)\right),\end{aligned}$$

which proves (28). From this we obtain:

$$(29)\qquad (\lambda_{H,\tilde{x},k},\mu_{y,\alpha})_H = e\left(2\left\langle\frac{qk}{n}\right\rangle-1\right),\quad \text{if } qk\not\equiv 0 \pmod n$$

because
$$\begin{aligned}(\lambda_{H,\tilde{x},k},\mu_{y,\alpha})_H &= (\lambda_{H,\tilde{x},k}-\overline{\lambda_{H,\tilde{x},k}},w_{y,\alpha})_H = (\lambda_{H,\tilde{x},k}-\lambda_{H,\tilde{x},-k},w_{y,\alpha})_H\\ &= e\left(\left\langle\tfrac{qk}{n}\right\rangle+\tfrac{1}{2}\left(\tfrac{e}{n}-1\right)\right)-\left(e\left(\left\langle\tfrac{-qk}{n}\right\rangle+\tfrac{1}{2}\left(\tfrac{e}{n}-1\right)\right)\right)\\ &= e\left(\left\langle\tfrac{qk}{n}\right\rangle-\left\langle\tfrac{-qk}{n}\right\rangle\right) = e\left(2\left\langle\tfrac{qk}{n}\right\rangle-1\right),\end{aligned}$$
provided that $qk\not\equiv 0 \pmod n$.

We can now finish the proof of (26) and hence the proof of the theorem. Indeed, using the definitions and formula (29), we obtain

$$\begin{aligned}(w_{H,\tilde{x}},\mu_{y,\alpha}) &= \sum_{k=1}^{n-1}k(\frac{e}{n}\lambda_{H,\tilde{x},k},\mu_{y,\alpha}) = \frac{e}{n}\sum_{k=1}^{n-1}ke\left(2\left\langle\frac{qk}{n}\right\rangle-1\right)\\ &= 2e^2\sum_{k=1}^{n-1}\frac{k}{n}\left(\left(\frac{qk}{n}\right)\right) = 2e^2\mathbb{S}(q,n),\end{aligned}$$

which proves (26).

Before leaving the topic of Dedekind sums, let us note that there is a curious relation between them and the lengths of finite continued fractions which will be useful later on. To be precise, these lengths are defined as follows.

Notation 1.14 Each fraction $x = \frac{n}{q} > 1$ has a unique *continued fraction expansion* of the form

$$\frac{n}{q} = [[c_1, c_2, \ldots, c_r]] := c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_r}}},$$

where the coefficients $c_i \geq 2$ are integers. Its *length* will be denoted by $\mathbb{L}(x) := r \leq q$.

Proposition 1.15 *If* $0 < q < n$ *and* $(n, q) = 1$, *then*

$$12\mathbb{S}(q,n) = \mathbb{L}\left(\frac{n}{n-q}\right) - \mathbb{L}\left(\frac{n}{q}\right) + \frac{q}{n} + \frac{q^*}{n} - 1 \leq n - 1 - \mathbb{L}\left(\frac{n}{q}\right),$$

where q^* *is such that* $qq^* \equiv 1 \pmod{n}$ *and* $0 < q^* < n$.

Proof. If $q = 1$ then $q^* = 1$ and so $\mathbb{L}(\frac{n}{n-q}) + \frac{q}{n} + \frac{q^*}{n} = n - 1 + \frac{2}{n} \leq n$, whereas if $q \geq 2$ then $\mathbb{L}(\frac{n}{n-q}) + \frac{q}{n} + \frac{q^*}{n} \leq n - q + 1 + 1 \leq n$. This proves the asserted inequality.

To prove the identity, let $\frac{n}{q} = [[c_1, \ldots, c_r]]$ denote the continued fraction expansion of $\frac{n}{q}$. Since by Oda[19], Corollary 1.23 (p. 29) we have

$$\sum_{i=1}^{r} c_i = \mathbb{L}\left(\frac{n}{n-q}\right) + 2\mathbb{L}\left(\frac{n}{q}\right) - 1,$$

it is enough to verify the formula (implicit in Hermann[11])

$$12\mathbb{S}(q,n) = \sum_{i=1}^{r} c_i - 3\mathbb{L}\left(\frac{n}{q}\right) + \frac{q}{n} + \frac{q^*}{n}. \tag{30}$$

For this, we shall induct on $r = \mathbb{L}(\frac{n}{q})$. If $r = 1$, then $q = 1$ and then $12\mathbb{S}(1, n) = n - 3 + \frac{2}{n}$ by [20], p. 5. Thus, assume $r > 1$ and write $[[c_2, \ldots, c_r]] = \frac{n_1}{q_1}$ with $(n_1, q_1) = 1$. Then $\frac{n}{q} = c_1 - \frac{q_1}{n_1}$, and so $n = c_1 n_1 - q_1$ and $q = n_1$. Thus $\mathbb{S}(n, q) = \mathbb{S}(c_1 n_1 - q_1, n_1) = \mathbb{S}(-q_1, n_1) = -\mathbb{S}(q_1, n_1)$, and so the Dedekind reciprocity formula yields:

$$12\mathbb{S}(q,n) = 12\mathbb{S}(q_1, n_1) + (-3 + \frac{n}{q} + \frac{1}{nq} + \frac{q}{n}).$$

Applying the induction hypothesis to $\frac{n_1}{q_1}$, we therefore obtain

$$12\mathbb{S}(q,n) = \sum_{i=1}^{r} c_i - 3r + \frac{q}{n} + \frac{nq_1^* + 1}{nq},$$

where $0 < q_1^* < n_1$ and $q_1 q_1^* \equiv 1 \pmod{n_1}$.

It thus remains to show that $nq_1^* + 1 = qq^*$. For this, put $q' := (1 - q_1 q_1^*)/n_1 + c_1 q_1^* \in \mathbb{Z}$. Then $qq' = 1 - q_1 q_1^* + (n + q_1) q_1^* = 1 + nq_1^*$, so in particular $qq' \equiv 1 \pmod{n}$. Moreover, since $0 < qq' = 1 + nq_1^* < nn_1 = nq$, we have $0 < q' < n$. Thus $q' = q^*$ and so $nq_1^* + 1 = qq^*$, as desired.

2 Diagonal quotient surfaces

2.1 Singularities

We now consider the product $Y = X \times X$ of the curve X with itself. Since G acts on X, the product group $G \times G$ acts componentwise on the product surface Y. We shall be particularly interested in the induced action of the "graph subgroup"

$$\Delta_\alpha = \{(\sigma, \alpha(\sigma)) \in G \times G : \sigma \in G\}$$

defined by a group automorphism $\alpha \in \mathrm{Aut}(G)$ of G. Since the map $\delta_\alpha : \sigma \mapsto (\sigma, \alpha(\sigma))$ defines a group isomorphism $\delta_\alpha : G \xrightarrow{\sim} \Delta_\alpha$, this action can also be considered as the action of G on Y given by $(\sigma, (x, x')) \mapsto (\sigma \cdot x, \alpha(\sigma) \cdot x')$. The quotient

$$Z_\alpha \;=\; G \setminus Y \;=\; \Delta_\alpha \setminus Y$$

of Y by this "twisted diagonal action" will be called a *(twisted) diagonal quotient surface.*

Let $\varphi = \varphi_\alpha : Y \to Z_\alpha = \Delta_\alpha \backslash Y$ denote the quotient map. Since $\Delta_\alpha \le G \times G$, we also have an induced quotient map $\psi = \psi_\alpha$ from Z_α to the product $\bar{Y} = (G \times G)\backslash Y = \bar{X} \times \bar{X}$ of $\bar{X}$ with itself:

$$Y = X \times X \xrightarrow{\varphi} Z_\alpha = G \setminus Y \xrightarrow{\psi} \bar{Y} = \bar{X} \times \bar{X}.$$

Moreover, for $i = 1, 2$, we shall denote the composition of ψ with the i-th projection of $\bar{Y}$ to $\bar{X}$ by $\psi_i = \psi_{\alpha,i} = \mathrm{pr}_i \circ \psi_\alpha : Z_\alpha \to \bar{X}$.

We begin our study of these surfaces by analyzing their singularities. For this, it is useful to first examine the fibres of the morphisms φ, ψ, and ψ_i.

Proposition 2.1 *a) Let $y = (x_1, x_2) \in Y$ be a point on the product surface Y. Then $G_{y,\alpha} := G_{x_1,x_2,\alpha} = G_{x_1} \cap \alpha^{-1}(G_{x_2})$ is the stabilizer of y with respect to the G-action on Y, so φ is ramified at y of order $e_y(\varphi) = |G_{y,\alpha}| = n_\alpha(x_1, x_2)$, and hence the map $f_y : \tau \mapsto \tau y = (\tau x_1, \alpha(\tau) x_2)$ induces a bijection*

$$f_y : G_{y,\alpha}\backslash G \;\xrightarrow{\sim}\; \varphi^{-1}(\varphi(y)).$$

b) For $i = 1, 2$ and for each $\bar{x} \in \bar{X}$, the reduced fibre $C_{\bar{x},i} := \psi_i^{-1}(\bar{x})_{red}$ is a smooth irreducible curve on Z_α. More precisely, if we put $\alpha_i = \alpha^{(-1)^i}$, then for each $x \in \pi^{-1}(\bar{x})$ we have isomorphisms

$$\varphi_i = \varphi_{x,i} : X_{x,\alpha_i} = \alpha_i^{-1}(G_x)\backslash X \;\xrightarrow{\sim}\; C_{\bar{x},i} = \psi_i^{-1}(\bar{x})_{red} \subset Z_\alpha$$

which are characterized by the properties that $\varphi_1(\pi_{x,\alpha^{-1}}(x')) = \varphi(x, x')$ and $\varphi_2(\pi_{x,\alpha}(x')) = \varphi(x', x)$, if $x' \in X$. Moreover, they satisfy the relations $\psi(\varphi_1(\tilde{x}_1)) = (\bar{x}, \bar{\pi}_{x,\alpha^{-1}}(\tilde{x}_1))$ and $\psi(\varphi_2(\tilde{x}_2)) = (\bar{\pi}_{x,\alpha}(\tilde{x}_2), \pi(x))$, for all $\tilde{x}_i \in X_{x,\alpha_i}$.

c) The fibre of ψ at $\bar{y} = \psi\varphi(y) = (\bar{x}_1, \bar{x}_2) \in \bar{Y}$ is the intersection of the two curves $C_{\bar{x}_i,i}$ and corresponds bijectively under $\varphi_{x_1,1}$ to the fibre $X_{\bar{x}_2,x_1,\alpha^{-1}}$ of $\bar{\pi}_{x_1,\alpha^{-1}}$ at $\bar{x}_2$, as well as under $\varphi_{x_2,2}$ to the fibre $X_{\bar{x}_1,x_2,\alpha}$ of $\bar{\pi}_{x_1,\alpha}$ at $\bar{x}_1$:

$$\psi^{-1}(\bar{y}) = C_{\bar{x}_1,1} \cap C_{\bar{x}_2,2} = \varphi_{x_1,1}(X_{\bar{x}_2,x_1,\alpha^{-1}}) = \varphi_{x_2,2}(X_{\bar{x}_1,x_2,\alpha}).$$

In particular, the map $\tau \mapsto \varphi(\tau x_1, x_2)$ induces a bijection

$$\bar{f}_y : G_{x_2,\alpha}\backslash G/G_{x_1} \xrightarrow{\sim} \psi^{-1}(\bar{y})$$

between the double coset space $G_{x_2,\alpha}\backslash G/G_{x_1}$ and the fibre of ψ over $\bar{y}$.

Proof. a) The first assertion is clear, and the second follows in view of Remark 1.1b) since G acts transitively on the fibres of φ.

b) We first prove that $C_{\bar{x},i}$ is smooth. For this, consider the curve $C' = \varphi^{-1}(C_{\bar{x},2}) = \cup_{x\in\pi^{-1}(\bar{x})} X \times \{x\}$ on which G acts. Since the components of C' are smooth and disjoint, C' is locally normal and hence so is the quotient $G\backslash C'$ (cf. Mumford[18], p. 5). Thus $G\backslash C'$ is a normal, hence smooth curve. Since the natural morphism $G\backslash C' \to G\backslash Y = Z_\alpha$ is a closed immersion with image $C_{\bar{x},2}$, it follows that $C_{\bar{x},2}$ is smooth. Similarly, one shows that $C_{\bar{x},1}$ is smooth.

Next we construct the morphism φ_2. The map $x' \mapsto \varphi(x', x)$ is $\alpha^{-1}(G_x)$-equivariant because $\varphi(\tau x', x) = \varphi(x', \alpha(\tau^{-1})x) = \varphi(x', x)$ since $\alpha(\tau^{-1}) \in G_x$. Thus, the map factors over $\pi_{x,\alpha}$, and so there a unique morphism $\varphi_2 : X_{x,\alpha} \to Z_\alpha$ such that $\varphi_2(\pi_{x,\alpha}(x')) = \varphi(x', x)$. Moreover, if $\tilde{x}_2 = \pi_{x,\alpha}(x_2) \in X_{x,\alpha}$, then $\psi(\varphi_2(\tilde{x}_2)) = \psi\varphi(x_2, x) = (\pi(x_2), \bar{x}) = (\bar{\pi}_{x_2,\alpha}(\tilde{x}_2), \bar{x}))$, which yields the second formula.

In particular, we see that $\mathrm{Im}(\varphi_2) \subset \psi_2^{-1}(\bar{x}))$. To prove equality, let $y' = \varphi(x_1', x_2') \in \psi_2^{-1}(\bar{x})$. Then $x_2' \in \pi^{-1}(x_2)$, so there is a $\tau \in G$ such that $\alpha(\tau)x = x_2'$. But then $\varphi_2(\tau^{-1}x_1') = \varphi(\tau^{-1}x_1', x) = \varphi(x_1', \alpha(\tau)x) = \varphi(x_1', x_2') = y'$, so φ_2 is surjective.

This shows that $C_{\bar{x},2} = \varphi_2(X_{x,\alpha})$ is irreducible. Since we already know that $C_{\bar{x},2}$ is smooth, it will therefore follow that φ_2 is an isomorphism once we have verified that φ_2 is injective.

To see this, let $\tilde{x}_1' = \pi_{x,\alpha}(x_1')$, $\tilde{x}_2' = \pi_{x,\alpha}(x_2') \in X_{x,\alpha}$ be such that $\varphi_2(\tilde{x}_1') = \varphi_2(\tilde{x}_2')$. Then we have $\varphi(x_1', x) = \varphi(x_2', x)$, so there exits a $\tau \in G$ such that $\tau x_1' = x_2'$ and $\alpha(\tau)x = x$. Thus $\tau \in \alpha^{-1}(G_x) = G_{x,\alpha}$, so $\tilde{x}_1' = \pi_{x,\alpha}(x_1') = \pi_{x,\alpha}(\tau^{-1}x_2') = \tilde{x}_2'$, and hence φ_2 is injective.

The proof for φ_1 is entirely analogous.

c) The first three assertions are obvious; in particular, we see that the bijection φ_2 constructed above restricts to a bijection $\bar{f}_{2,y} : X_{\bar{x}_1,x_2,\alpha} \to \psi^{-1}(\bar{y})$. Next, consider the bijection $\bar{f}_2 := \bar{f}_{G_{x_2,\alpha}} : G_{x_2,\alpha}\backslash G/G_{x_1} \to X_{\bar{x}_1,x_2,\alpha}$ which is defined as in Remark 1.1c) by $\tau \mapsto \pi_{x_2,\alpha}(\tau x_2)$. Then $\bar{f}_{2,y}(\bar{f}_2(\tau)) = \bar{f}_{2,y}(\pi_{x_2,\alpha}(\tau x_2)) = \varphi(\tau x, x_2) = \bar{f}_y$, which shows that our desired map $\bar{f}_y$ is well-defined and bijective.

We now investigate the singularities of Z_α. It turns out that they are all cyclic quotient singularities and hence are of the form $A_{n,q} = G_n\backslash\mathbb{C}^2$; cf. BPV[1], p. 84. Explicitly, this means that the cyclic group G_n acts on $\mathbb{C}^2$ via $\sigma(z_1, z_2) = (\chi_1(\sigma)z_1, \chi_2(\sigma)z_2)$, where the "weights" $\chi_i : G_n \to \mathbb{C}^\times$ have order n and satisfy $\chi_1 = \chi_2^q$.

Remark 2.2 There is a slight ambiguity concerning the *type* (n, q) of a quotient singularity $A_{n,q} = G_n\backslash\mathbb{C}^2$ in that it depends not only on the analytic isomorphism class of $A_{n,q}$ but also on an ordering of the two weights χ_i. If we interchange the weights, then we obtain an (isomorphic) singularity of type (n, q^*), with $qq^* \equiv 1 \pmod n$. However, this choice of an ordering is the only ambiguity, for two singularities of type $A_{n,q}$ and $A_{n',q'}$ are analytically isomorphic if and only if $n = n'$ and $q = q'$ or $qq' \equiv 1 \pmod n$; cf. Hirzebruch [12], 3.4(18).

In our case, each quotient singularity $z = \varphi(y)$ of Z_α is induced by the action of the stabilizer groups $G_{y,\alpha}$ on the tangent space $T_{Y,y} = T_{X,x_1} \oplus T_{X,x_2} \simeq \mathbb{C}^2$ which comes equipped with a natural *ordered* basis (of eigenvectors), the ordering being induced by the ordering of the factors of $Y = X \times X$. We thus have a natural ordering of the weights attached to this action, and hence (in this sense) a uniquely determined type (n, q).

Theorem 2.3 *a) A point $z = \varphi(y) = \varphi(x_1, x_2) \in Z_\alpha$ is a singularity of Z_α if and only if $e_z(\varphi) > 1$ (cf. Proposition 2.1a)). If this is the case, then z is a cyclic quotient singularity of type $A_{n,q}$, where $n = n(z) = n_\alpha(x_1, x_2) = e_z(\varphi)$ and $q = q(z) = q_\alpha(x_1, x_2)$, where $q_\alpha(x_1, x_2)$ is as defined in 1.4.*

b) For each $\nu \geq 1$, the map $\varphi_{x_2,2}$ of Proposition 2.1b) induces a bijection

$$\bar{f}_{2,y,\nu} : S_{\nu,\alpha}(\pi(x_1), x_2) \to S_{\nu,\alpha}(\psi(z))$$

between the set $S_{\nu,\alpha}(\pi(x_1), x_2)$ defined in 1.4 and the set $S_{\nu,\alpha}(\psi(z))$ of quotient singularities of fixed type $(n, q) = (\frac{e}{(e,\nu)}, \frac{\nu}{(e,\nu)})$ in the fibre $\psi^{-1}(\psi(z))$, where $e = e_{x_2}$. In particular, the number of such singularities is $s_{\nu,\alpha}(\pi(x_1), \pi(x_2))$.

c) In each fibre $\psi_2^{-1}(\bar{x})$ there are precisely $r_{\bar{x},\alpha}$ singularities, where $r_{\bar{x},\alpha}$ is as defined in Remark 1.5. Similarly, in $\psi_1^{-1}(\bar{x})$ there are $r_{\bar{x},\alpha^{-1}}$ singularities, and hence the total number of singularities of Z_α is

$$|S_\alpha| = \sum_{\bar{x} \in \bar{X}} r_{\bar{x},\alpha} = \sum_{\bar{x} \in \bar{X}} r_{\bar{x},\alpha^{-1}}.$$

In particular, Z_α is smooth if and only if $\pi_{\bar{x},\alpha} : X \to X_{\bar{x},\alpha}$ is unramified for all $\bar{x} \in \bar{X}$.

Proof. a) Let us factor φ as $\varphi : Y \overset{\varphi_y}{\to} \tilde{Y} := G_{y,\alpha} \backslash Y \overset{\tilde{\varphi}_y}{\to} Z_\alpha$, where $G_{y,\alpha}$ denotes the stabilizer of y with respect to G, and put $\tilde{y} = \varphi_y(y)$. Then $\tilde{\varphi}_y^* : \mathcal{O}^{an}_{Z_\alpha,z} \to \mathcal{O}^{an}_{\tilde{Y},\tilde{y}}$ is unramified (cf. Bourbaki, Commutative Algebra, ch. V, §2.2, Prop. 4), hence local-etale (since Z_α is normal; cf. [5], Lemma I.1.5) and thus an isomorphism (cf. [5], Prop. 1.7). It follows that $\mathcal{O}^{an}_{Z_\alpha,z}$ is a quotient singularity with respect to the group $G_{y,\alpha}$; in particular, if $e_z(\varphi) = e_y(\varphi) = 1$, then $\mathcal{O}^{an}_{Z_\alpha,z} \simeq \mathcal{O}^{an}_{Y,y}$, and so z is a smooth point of Z_α.

Now suppose that $e_z(\varphi) > 1$. Choose local coordinates t_i at x_i which linearize the action of G_{x_i} as in (3). Then (t_1, t_2) are local coordinates at $y = (x_1, x_2) \in Y$ and we have

$$g^* \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \lambda_1(g) t_1 \\ \lambda_2(g) t_2 \end{pmatrix}, \quad \text{for } g \in G_{y,\alpha} = G_{x_1} \cap \alpha^{-1}(G_{x_2}),$$

where $\lambda_1 = \lambda_{x_1}$ and $\lambda_2 = \lambda_{x_2,\alpha}$. Thus, the weights of the action are $(\lambda_1)_{|K}$ and $(\lambda_2)_{|K}$, where $K = G_{y,\alpha}$. Fix a generator τ of K, and write $\lambda_k(\tau) = e^{2\pi i p_k/n}$, $k = 1, 2$, where $n = |K| = n_\alpha(x_1, x_2)$ by Proposition 2.1a). We note that $(p_k, n) = 1$ since $(\lambda_k)_{|K}$ are both faithful characters. By definition (8) of $q = q_\alpha(x_1, x_2)$, we have $\lambda_1(\tau) = \lambda_2(\tau)^q$, so it follows that $qp_2 \equiv p_1 \pmod{n}$, and hence by Proposition 5.3 of BPV[1], p. 84, we have that z is a quotient singularity of type $A_{n,q}$.

b), c) These assertions follow immediately from Proposition 2.1b) and Remark 1.5a).

Remark 2.4 It is possible for G to have fixed points, yet that Z_α is smooth for a suitable twist α, for it can happen that all $\pi_{\bar{x},\alpha} : X \to X_{\bar{x},\alpha}$ are unramified, as the example of the Fermat curves shows (cf. Remark 1.3b)).

For computational purposes, it is useful to restate the above theorem in terms of the "normalizing sets" $N^*_{k,i}(\sigma_1,\sigma_2)$ introduced in Notation 1.6.

Corollary 2.5 *Let $x_1, x_2 \in X$, and let σ_1 and σ_2 be generators of $G_{x_1} = \langle\sigma_1\rangle$ and $G_{x_2,\alpha} = \langle\sigma_2\rangle$, respectively. Put $r = (e_1, e_2)$ where $e_i = e_{x_i}$, and let $c = c_\alpha(\sigma_1,\sigma_2)$ be such that*

$$\lambda_{x_1}(\sigma_1^{e_1/r}) = \lambda_{x_2,\alpha}(\sigma_2^{e_2/r})^c. \tag{31}$$

a) Let $z \in \psi^{-1}(\psi\varphi(x_1,x_2))$. Then $z = \varphi(gx_1,x_2)$ for some $g \in N^*_{k,i}(\sigma_1,\sigma_2)$, where k and i satisfy $1 \le k|r$, $1 \le i \le r$ and $(i, \frac{r}{k}) = 1$. Furthermore, k and i are uniquely determined by z.

b) The point $z = \varphi(gx_1,x_2)$ is a singularity of Z_α if and only if $k \neq r$. If this is the case, then its type (n,q) is uniquely determined by the relations

$$n = \frac{r}{k} \qquad \text{and} \qquad iq \equiv c \pmod{\frac{r}{k}}. \tag{32}$$

c) Let (k,i) and (n,q) be related by (32), and put $\nu = qe_yk/r$. Then the map $g \mapsto \varphi(gx_1,x_2)$ induces a bijection

$$G_{x_2,\alpha}\backslash N^*_{k,i}(\sigma_1,\sigma_2)/G_{x_1} \xrightarrow{\sim} S_{\nu,\alpha}(\psi\varphi(x_1,x_1)) \tag{33}$$

between the indicated double coset space of normalizing elements and the set of singularities of type (n,q) which lie in the fibre $\psi^{-1}(\psi\varphi(x_1,x_1))$.

Proof. a) The first assertion is clear since the sets $\{N^*_{k,i}(\sigma_1,\sigma_2)\}_{(k,i)\in S}$, where $S = \{(k,i) : 1 \le k|r, 1 \le i \le r, (i,\frac{r}{k}) = 1\}$ form a partition of G (cf. Notation 1.6). The uniqueness assertion follows immediately from part b).

b) By Theorem 2.3 we have $n = n_\alpha(gx_1,x_2)$ and $q = q_\alpha(gx_1,x_2)$. On the other hand, by Proposition 1.7a) we have $n_\alpha(gx_1,x_2) = \frac{r}{k}$ and by Proposition 1.7b) we see that q satisfies the congruence (32). This determines q uniquely since $(i,\frac{r}{k}) = 1$ and since $1 \le q \le n_\alpha = \frac{r}{k}$ by the definition of q.

c) By composing the bijection of Proposition 1.7b), c) with that of Theorem 2.3b) we obtain a bijection between the two sets of (33). Now by (the proof of) Proposition 2.1c) we see that this composite bijection is the map induced by the map $g \mapsto \varphi(gx_1,x_2)$, and so the assertion follows.

As an application of (the proof of) Theorem 2.3, we can interpret the *signature defect* def_y (cf. Hirzebruch[13]) of a fixed point $y \in Y$ of G in terms of an inner product of characters, and hence give a similar interpretation for the *signature* $\mathrm{sign}(Z_\alpha)$ of Z_α:

Corollary 2.6 *The signature defect of a point $y = (x_1,x_2) \in Y$ with respect to G is*

$$\mathrm{def}_y = -4e_y(\varphi)\mathbb{S}_\alpha(\tilde{x}_1,x_2) = -\frac{e_y(\varphi)}{e_{x_2}^2}(\mu_{H,\tilde{x}_1},\mu_{x_2,\alpha})_H, \tag{34}$$

where $H = G_{x_2,\alpha}$, $\tilde{x}_1 = \pi_H(x_1)$ and $\mu_{H,\tilde{x}_1}, \mu_{x_2,\alpha}$ are as defined in 1.11. Thus we have

$$\sum_{y'\in\varphi^{-1}\psi^{-1}(\bar{y})} \mathrm{def}_{y'} = -\frac{1}{m}(\mu_{\bar{x}_1},\mu_{\bar{x}_2,\alpha})_G = 4m\mathbb{S}_\alpha(\bar{x}_1,\bar{x}_2), \tag{35}$$

where $\bar{y} = \psi(\varphi(y))$, and hence the signature of Z_α is

$$\mathrm{sign}(Z_\alpha) = -(\mu_X,\mu_{X,\alpha}) = -4\mathbb{S}_\alpha(X). \tag{36}$$

Proof. Comparing the above computation (in the proof of 2.3) to that of Hirzebruch/Zagier [14], p. 179ff, we see from their formula (17) that their $q = q_\alpha(x_1, x_2)$ and $p = n_\alpha(x_1, x_2)$. (Moreover, their $n = 1$ and their $m = 1$.) Thus, by [14], formula (24) on p. 180, we obtain

$$\mathrm{def}_y = -4p \sum_{k=1}^{p} \left(\left(\frac{k}{p}\right)\right) \left(\left(\frac{kq}{p}\right)\right) = -4p\mathbb{S}_\alpha(\tilde{x}_1, x_2),$$

which yields the first equality of (34). On the other hand, by formula (26) we have

$$(\mu_{H,\tilde{x}_1}, \mu_{x_2,\alpha})_H = 2(w_{H,\tilde{x}_1}, \mu_{x_2,\alpha})_H = 4e^2\mathbb{S}(\tilde{x}_1, x_2),$$

where $e = e_{x_2}$, from which the second equality of (34) follows.

To prove (35), first note that the second equality is just a restatement of (24). Next, since def_y depends only on $z = \varphi(y)$, we obtain from (34) (and Prop. 2.1b)) and formula (27) that

$$\sum_{y\in\varphi^{-1}\psi^{-1}(\bar{y})} \mathrm{def}_y = \sum_{z\in\psi^{-1}(\bar{y})} \frac{m}{e_z(\varphi)}\mathrm{def}_z = -\sum_{\tilde{x}\in X_{\bar{x}_1,x_2,\alpha}} \frac{m}{e^2}(\mu_{H,\tilde{x}}, \mu_{x_2,\alpha})_H = -\frac{1}{m}(\mu_{\bar{x}_1}, \mu_{\bar{x}_2,\alpha})_G,$$

which proves the first equality of (35).

Finally, to prove (36), we shall apply the equivariant signature formula (cf. [13], formula (24) or [14], p. 181):

$$m \cdot \mathrm{sign}(Z_\alpha) = \mathrm{sign}(Y) + \sum_{y\in Y} \mathrm{def}_y.$$

Since $\mathrm{sign}(Y) = 0$ by [14], Th. 2.1.2, p. 26, we obtain from (35):

$$\mathrm{sign}(Z_\alpha) = \frac{1}{m} \sum_{\bar{y}\in\bar{Y}} \sum_{y\in(\psi\varphi)^{-1}(\bar{y})} \mathrm{def}_y = -\frac{1}{m^2} \sum_{\bar{x}_1,\bar{x}_2\in\bar{X}} (\mu_{\bar{x}_1}, \mu_{\bar{x}_2,\alpha}) = -(\mu_X, \mu_{X,\alpha}).$$

This proves the first equality of (36), and the second follows from (25).

2.2 The Betti and Hodge numbers

We now turn to calculate the Betti numbers $b_i(Z_\alpha) = h^i(Z_\alpha, \mathbb{C})$ as well as the Hodge numbers $h^{0,i}(Z_\alpha) = h^i(Z_\alpha, \mathcal{O}_{Z_\alpha})$ of Z_α. They can be computed from those of Y via

Proposition 2.7 *For all $q \geq 0$ we have*

$$H^q(Z_\alpha, \mathbb{C}) = H^q(Y, \mathbb{C})^{\Delta_\alpha} \quad \textit{and} \quad H^q(Z_\alpha, \mathcal{O}_{Z_\alpha}) = H^q(Y, \mathcal{O}_Y)^{\Delta_\alpha}$$

where $(\ )^G$ denotes the G-invariant subspace.

Proof. Grothendieck [9], Corollary to Proposition 5.2.3 and Griffiths [7], formula (2.8).

Using this relation we find

Proposition 2.8 *The Betti numbers of Z_α are*

$$b_0(Z_\alpha) = b_4(Z_\alpha) = 1, \qquad b_1(Z_\alpha) = b_3(Z_\alpha) = 4\bar{g}, \qquad b_2(Z_\alpha) = 2 + (h^1, h^1_\alpha) = 2 + 4\mathbb{G}_\alpha.$$

In particular, the topological Euler characteristic of Z_α is

$$\chi_{top}(Z_\alpha) \;=\; 4(1 - 2\bar{g} + \mathbb{G}_\alpha) \;=\; 4(\bar{g}-1)(g-1) + 2\sum_{\bar{x}\in R}(g - g_{\bar{x},\alpha}).$$

Proof. The first equation is obvious. For the other equations we consider $H^i(Z_\alpha, \mathbb{C})$ as a G-module with corresponding character $h^i(Z_\alpha)$. Using the Künneth formula (and the fact that $H^i(X, \mathbb{C}) = 0$ for $i > 2 = \dim_{\mathbb{R}}(X)$), we get

$$H^1(Y, \mathbb{C}) \cong \Big(H^0(X, \mathbb{C}) \otimes H^1(X, \mathbb{C})\Big) \oplus \Big(H^1(X, \mathbb{C}) \otimes H^0(X, \mathbb{C})\Big) \cong H^3(Y, \mathbb{C}).$$

Observing the twisted action on the second factor of each summand, this implies $h^1(Z_\alpha) = h^3(Z_\alpha) = h^1 + h^1_\alpha$. Thus the preceding proposition gives $b_1(Z_\alpha) = b_3(Z_\alpha) = (h^1(Z_\alpha), 1_G) = (h^1, 1_G) + (h^1_\alpha, 1_G) = 4\bar{g}$ by formula (18).

Similarly we get $b_2(Z_\alpha) = (h^2(Z_\alpha), 1_G) = (1_G + h^1 \otimes h^1_\alpha + 1_G, 1_G) = 2 + (h^1 \otimes h^1_\alpha, 1_G) = 2 + (h^1, h^1_\alpha)$ since h^1_α is real-valued (cf. equation (16)). Moreover, the inner product (h^1, h^1_α) was computed in Proposition 1.8, from which the last formula follows.

Proposition 2.9 *The Hodge numbers $h^{0,q}$ of Z_α are*

$$h^{0,0} = h^0(\mathcal{O}_{Z_\alpha}) = 1, \qquad h^{0,1} = h^1(\mathcal{O}_{Z_\alpha}) = 2\bar{g}, \qquad h^{0,2} = h^2(\mathcal{O}_{Z_\alpha}) = (\omega, \overline{\omega_\alpha}) = \mathbb{G}_\alpha - \mathbb{S}_\alpha.$$

In particular, the arithmetic genus of Z_α is given by

$$p_a(Z_\alpha) \;=\; (\omega, \overline{\omega_\alpha}) - 2\bar{g} \;=\; -1 + (\bar{g}-1)(g-1) + \tfrac{1}{2}\sum_{\bar{x}\in R}(g - g_{\bar{x},\alpha}) - \mathbb{S}_\alpha(X).$$

Proof. Let χ^i be the character of the G-module $H^i(Y, \mathcal{O}_Y)$ for $i = 1, 2$. From the Künneth formula and Serre duality on X we obtain

$$\chi^1 \;=\; \overline{\omega} + \overline{\omega_\alpha} \quad \text{and} \quad \chi^2 \;=\; \overline{\omega} \cdot \overline{\omega_\alpha}.$$

Hence, applying the second part of Proposition 2.7, we obtain $h^1(\mathcal{O}_{Z_\alpha}) = (\chi^1, 1_G) = 2\bar{g}$ by formula (18), and $h^2(\mathcal{O}_{Z_\alpha}) = (\chi^2, 1_G) = (\overline{\omega} \cdot \overline{\omega_\alpha}, 1_G) = (\omega, \overline{\omega_\alpha})$. Finally, by substituting the expression (25) for $(\omega, \overline{\omega_\alpha})$, the formula for p_a follows.

3 The desingularization of diagonal quotient surfaces

3.1 The geometric genus of $\tilde{Z}_\alpha$

Since the diagonal quotient surface Z_α is almost always a singular surface (cf. Theorem 2.3c)), we shall mainly focus our attention on its (minimal) desingularization

$$\sigma : \tilde{Z}_\alpha \to Z_\alpha.$$

In this subsection we shall to compute some of the numerical invariants of $\tilde{Z}_\alpha$ with the help of the group action on Y, whereas in the subsequent sections we shall take a closer look at the desingularization map σ to compute further invariants.

Proposition 3.1 *The cohomology of the structure sheaf of $\tilde{Z}_\alpha$ is the same as that of Z_α. Therefore, the irregularity q, the geometric genus p_g and the arithmetic genus p_a of $\tilde{Z}_\alpha$ are given by*

$$\begin{array}{rclcrclcl} q(\tilde{Z}_\alpha) & := & h^1(\tilde{Z}_\alpha, \mathcal{O}_{\tilde{Z}_\alpha}) & = & q(Z_\alpha) & = & (h^1, 1_G) & = & 2\overline{g}, \\ p_g(\tilde{Z}_\alpha) & := & h^2(\tilde{Z}_\alpha, \mathcal{O}_{\tilde{Z}_\alpha}) & = & p_g(Z_\alpha) & = & (\omega, \overline{\omega_\alpha}) & = & \mathbb{G}_\alpha - \mathbb{S}_\alpha, \\ p_a(\tilde{Z}_\alpha) & := & p_g(\tilde{Z}_\alpha) - q(\tilde{Z}_\alpha) & = & p_a(Z_\alpha) & = & (\omega, \overline{\omega_\alpha}) - 2\overline{g} & = & \mathbb{G}_\alpha - \mathbb{S}_\alpha - 2\overline{g}. \end{array}$$

Proof. Since σ is birational and Z_α is normal, we have $\sigma_*\mathcal{O}_{\tilde{Z}_\alpha} = \mathcal{O}_{Z_\alpha}$. Moreover, we have $R^1\sigma_*\mathcal{O}_{\tilde{Z}_\alpha} = 0$ because by Proposition 2.3 all singularities of Z_α are cyclic quotient singularities and hence are rational; cf. Barth et al. [1] III Prop. (3.1). It thus follows from Hartshorne [10] Ex. III.8.1 that

$$H^i(\tilde{Z}_\alpha, \mathcal{O}_{\tilde{Z}_\alpha}) \simeq H^i(Z_\alpha, \sigma_*\mathcal{O}_{\tilde{Z}_\alpha}) \simeq H^i(Z_\alpha, \mathcal{O}_{Z_\alpha}) \text{ for all } i \geq 0.$$

This proves the first statement, and the others follow immediately from this in view of Proposition 2.9.

Corollary 3.2 *If $h^1_\alpha = h^1$ but $\omega_\alpha \neq \omega$, then $p_g(\tilde{Z}_\alpha) > 0$ if and only if $g(X) > 0$.*

Proof. If $(\omega, \overline{\omega_\alpha}) = 0$, then these characters have no common irreducible components, so $h^1 = \omega + \overline{\omega} = \omega_\alpha + \overline{\omega_\alpha}$ implies that $\omega_\alpha = \omega$, which contradicts the assumption.

Corollary 3.3 *If there is an automorphism $\overline{\alpha}$ of G such that $\omega_{\overline{\alpha}} = \overline{\omega_\alpha}$ then*

$$\begin{array}{rcl} p_g(\tilde{Z}_\alpha) + p_g(\tilde{Z}_{\overline{\alpha}}) & = & \frac{1}{2}(h^1, h^1_\alpha) = 2\mathbb{G}_\alpha, \\ p_g(\tilde{Z}_\alpha) - p_g(\tilde{Z}_{\overline{\alpha}}) & = & -\frac{1}{2}(\mu_X, \mu_{X,\alpha}) \;=\; \frac{1}{2}\mathrm{sign}(Z_\alpha) = -2\mathbb{S}_\alpha. \end{array}$$

Proof. By Theorem 3.1 we have

$$p_g(\tilde{Z}_\alpha) \pm p_g(\tilde{Z}_{\overline{\alpha}}) = (\omega, \overline{\omega_\alpha}) \pm (\omega, \overline{\omega_{\overline{\alpha}}}) = (\omega, \overline{\omega_\alpha}) \pm (\omega, \omega_\alpha) = \tfrac{1}{2}(\omega \pm \overline{\omega}, \overline{\omega_\alpha} \pm \omega_\alpha),$$

where the last equality follows because $(\overline{\omega}, \overline{\omega_\alpha} \pm \omega_\alpha) = (\omega, \omega_\alpha \pm \overline{\omega_\alpha}) = \pm(\omega, \overline{\omega_\alpha} \pm \omega_\alpha)$. Since $\overline{\omega_\alpha} + \omega_\alpha = h^1_\alpha$ and $\overline{\omega_\alpha} - \omega_\alpha = -\mu_\alpha$, the first two assertions follow. The last equality is just a restatement of formula (36).

Remark 3.4 We shall see in [17] that the hypothesis of Corollary 3.3 is satisfied when $X = X(N)$ is a modular curve. Thus, the above formula

$$p_g(\tilde{Z}_\alpha) - p_g(\tilde{Z}_{\overline{\alpha}}) = \tfrac{1}{2}\mathrm{sign}(Z_\alpha)$$

may be viewed as a generalization of Satz 3a) of Hermann[11], where this formula is proven in the special case that $X = X(N)$.

3.2 The Euler characteristic of $\tilde{Z}_\alpha$

To compute the Euler characteristic of $\tilde{Z}_\alpha$ we need to know that of the exceptional divisor E. We recall the following description of E given in BPV [1] III.5:

Proposition 3.5 *If $s \in Z_\alpha$ is a singularity of type $A_{n,q}$, then*

$$\sigma^{-1}(s) = \bigcup_{j=1}^{r_s} C_{s,j} \text{ with } C_j = C_{s,j} \cong \mathbb{P}^1, \quad C_j^2 = -c_{s,j}, \quad (C_i.C_j) = \begin{cases} 1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases},$$

where $\frac{n}{q} = [[c_{s,1}, \ldots, c_{s,r_s}]]$ is the continued fraction expansion of $\frac{n}{q}$ of length $\mathbb{L}(\frac{n}{q}) = r_s$ as in Notation 1.14. Thus, if S_α denotes the set of singular points of Z_α, then the exceptional divisor of $\sigma : \tilde{Z}_\alpha \to Z_\alpha$ is

$$E = \sum_{s \in S_\alpha} E_s \qquad \text{where} \qquad E_s = \sum_{j=1}^{r_s} C_{s,j},$$

and hence the number of irreducible components of E is

$$(37) \qquad \mathbb{L}_\alpha(X) := \sum_{\bar{x},\bar{y} \in \bar{X}} \mathbb{L}_\alpha(\bar{x},\bar{y}) \qquad \text{where} \qquad \mathbb{L}_\alpha(\bar{x},\bar{y}) = \sum_{\nu=1}^{e_{\bar{y}}-1} s_\nu(\bar{x},\bar{y}) \mathbb{L}(\frac{e_{\bar{y}}}{\nu}).$$

Proposition 3.6 *The topological Euler characteristic of $\tilde{Z}_\alpha$ is*

$$\chi_{top}(\tilde{Z}_\alpha) \;=\; \chi_{top}(Z_\alpha) + \#\textit{irreducible components of } E \;=\; 4(1 - 2\bar{g} + \mathbb{G}_\alpha) + \mathbb{L}_\alpha.$$

Proof. For every closed subset A of a compact topological space T we have a long exact sequence

$$\cdots \to H_c^i(T \setminus A, \mathbb{C}) \to H^i(T, \mathbb{C}) \to H^i(A, \mathbb{C}) \to H_c^{i+1}(T \setminus A, \mathbb{C}) \to \cdots.$$

Applying this to $(\tilde{Z}_\alpha, E)$ and (Z_α, S_α) we get:

$$\begin{aligned} \chi_{top}(\tilde{Z}_\alpha) &= \chi_{top}(E) + \chi_{top}^c(\tilde{Z}_\alpha \setminus E), \\ \chi_{top}(Z_\alpha) &= \chi_{top}(S_\alpha) + \chi_{top}^c(Z_\alpha \setminus S_\alpha). \end{aligned}$$

Since $\tilde{Z}_\alpha \setminus E \cong Z_\alpha \setminus S_\alpha$ and S_α consists of isolated points this implies

$$\chi_{top}(\tilde{Z}_\alpha) = \chi_{top}(Z_\alpha) + \chi_{top}(E) - |S_\alpha| = \chi_{top}(Z_\alpha) + \sum_{s \in S_\alpha} (\chi_{top}(E_s) - 1).$$

Since

$$\chi_{top}(E_s) = r_s + 1,$$

as is easily verified by using the Mayer-Vietoris sequence, the formula follows in view of Propositions 2.8 and 3.5.

3.3 The Betti, Hodge and Chern numbers of $\tilde{Z}_\alpha$

By using results of the previous two subsections, we can now reap the fruits of our labour and compute the basic invariants of the surface $\tilde{Z}_\alpha$.

Theorem 3.7 *The Betti, Hodge and Chern numbers of the surface $\tilde{Z}_\alpha$ may all be expressed in terms of the three fundamental invariants of X, which were defined in 1.5, 1.12, and 3.5:*

$$\begin{aligned}
\mathbb{G}_\alpha &= \bar{g} + (\bar{g}-1)g + \tfrac{1}{2}\sum_{\bar{x}\in\bar{X}}(g - g_{\bar{x},\alpha}) &&= \tfrac{1}{4}(h^1, h^1_\alpha) \\
\mathbb{S}_\alpha &= \sum_{\bar{x}\in\bar{X}}\sum_{n=1}^{e_{\bar{x}}-1} s_{n,\alpha}(\bar{x})\mathbb{S}\left(\frac{n}{(n,e_{\bar{x}})}, \frac{e_{\bar{x}}}{(n,e_{\bar{x}})}\right) &&= \tfrac{1}{4}(h^1, h^1_\alpha) - (\omega, \overline{\omega_\alpha}) \\
\mathbb{L}_\alpha &= \sum_{\bar{x}\in\bar{X}}\sum_{n=1}^{e_{\bar{x}}-1} s_{n,\alpha}(\bar{x})\mathbb{L}\left(\frac{e_{\bar{x}}}{n}\right),
\end{aligned}$$

and hence also in terms of the numbers $s_{n,\alpha}(\bar{x})$. Explicitly, we have the following formulae:

$$\begin{aligned}
h^{0,1}(\tilde{Z}_\alpha) &= q \;=\; \tfrac{1}{2}b_1(\tilde{Z}_\alpha) \;=\; \tfrac{1}{2}b_1(Z_\alpha) &&= 2\bar{g} \\
h^{0,2}(\tilde{Z}_\alpha) &= p_g \;=\; h^{0,2}(Z_\alpha) &&= \mathbb{G}_\alpha - \mathbb{S}_\alpha \\
h^{1,1}(\tilde{Z}_\alpha) &= b_2(\tilde{Z}_\alpha) - 2p_g(\tilde{Z}_\alpha) &&= 2 + 2\mathbb{G}_\alpha + 2\mathbb{S}_\alpha + \mathbb{L}_\alpha \\
b_2(\tilde{Z}_\alpha) &= b_2(Z_\alpha) + \mathbb{L}_\alpha &&= 2 + 4\mathbb{G}_\alpha + \mathbb{L}_\alpha. \\
\mathrm{sign}(\tilde{Z}_\alpha) &= \mathrm{sign}(Z_\alpha) - \mathbb{L}_\alpha &&= -4\mathbb{S}_\alpha - \mathbb{L}_\alpha \\
c_2(\tilde{Z}_\alpha) &= \chi_{top}(\tilde{Z}_\alpha) = \chi_{top}(Z_\alpha) + \mathbb{L}_\alpha &&= 4(1 - 2\bar{g} + \mathbb{G}_\alpha) + \mathbb{L}_\alpha \\
c_1^2(\tilde{Z}_\alpha) &= 2c_2(\tilde{Z}_\alpha) + 3\mathrm{sign}(\tilde{Z}_\alpha) &&= 8(1 - 2\bar{g} + \mathbb{G}_\alpha) - \mathbb{L}_\alpha - 12\mathbb{S}_\alpha.
\end{aligned}$$

Proof. The first equality for $\mathbb{G}_\alpha$ is just its definition (12), and the second is derived in Proposition 1.8. Similarly, the first equality for $\mathbb{S}_\alpha$ is its definition (cf. Notation 1.12), whereas the second is formula (25), combined with (17). Moreover, the formula for $\mathbb{L}_\alpha$ is just its definition (37).

The formulae for $h^{0,1}$ and $h^{0,2}$ are merely a restatement of Proposition 3.1, from which it also follows that $b_2(\tilde{Z}_\alpha) - b_2(Z_\alpha) = \chi_{top}(\tilde{Z}_\alpha) - \chi_{top}(Z_\alpha) = \mathbb{L}_\alpha$, the latter by Proposition 3.6. By substituting the values of $b_2(Z_\alpha)$ and $\chi_{top}(Z_\alpha)$ which were obtained in Proposition 2.8, the indicated formulae for $b_2(\tilde{Z}_\alpha)$ and $\chi_{top}(\tilde{Z}_\alpha)$ follow. (Recall that $c_2 = \chi_{top}$ by the Gauss–Bonnet formula; cf. [8], p. 416.) From this the second identity for $h^{1,1}$ follows immediately, whereas the first is just a consequence of the Hodge decomposition for $H^2(\tilde{Z}_\alpha, \mathbb{C})$.

We can now compute the signature of $\tilde{Z}_\alpha$ via the formula

$$\mathrm{sign}(\tilde{Z}_\alpha) = b^+ - b^- = (2p_g + 1) - (h^{1,1} - 1) = 2p_g - h^{1,1} + 2,$$

which follows from BPV[1], Theorem (IV.2.6), together with the fact that $b^+ + b^- = b_2 = 2p_g + h^{1,1}$. Substituting the above values for p_g and $h^{1,1}$ yields $\mathrm{sign}(\tilde{Z}_\alpha) = -4\mathbb{S}_\alpha - \mathbb{L}_\alpha = \mathrm{sign}(Z_\alpha) - \mathbb{L}_\alpha$, the latter by Corollary 2.6.

From this, the second formula for c_1^2 follows immediately, whereas the first formula is the Index Theorem of Thom–Hirzebruch for $\tilde{Z}_\alpha$ (cf. [1], p. 18).

Corollary 3.8 *For each $n \in \mathbb{N}$ and $\bar{x} \in \bar{X}$, let $n^* = n^*_{\bar{x}}$ be defined by the conditions $0 < n^* \le e_{\bar{x}}$, $(n^*, e_{\bar{x}}) = (n, e_{\bar{x}})$ and $\frac{nn^*}{(n,e_{\bar{x}})^2} \equiv 1 \pmod{\frac{e_{\bar{x}}}{(n,e_{\bar{x}})}}$. Then we have*

$$\begin{aligned}
c_1^2(\tilde{Z}_\alpha) &= 8(1 - 2\bar{g} + \mathbb{G}_\alpha) - \sum_{\bar{x}\in\bar{X}}\sum_{n=1}^{e_{\bar{x}}-1} s_{n,\alpha}(\bar{x})\left(\mathbb{L}\left(\frac{e_{\bar{x}}}{e_{\bar{x}} - n}\right) + \frac{n}{e_{\bar{x}}} + \frac{n^*}{e_{\bar{x}}} - 1\right) \\
&\ge 8(1 - 2\bar{g} + \mathbb{G}_\alpha) - \sum_{\bar{x}\in\bar{X}}\sum_{1<k|e_{\bar{x}}} r_{k,\alpha}(\bar{x})(k-1).
\end{aligned} \tag{38}$$

In particular, if (as before) $r = |R|$ denotes the number of ramified points in $\bar{X}$, then

$$(39)\quad 2c_2(\tilde{Z}_\alpha) \ \geq\ c_1^2(\tilde{Z}_\alpha) \ \geq\ (2g-2)(r+4(\bar{g}-1)) + 2\sum_{\bar{x}\in\bar{R}}(e_{\bar{x}}-2)(g_{\bar{x},\alpha}-1).$$

Moreover, this lower bound is an equality if and only if $e_{\bar{x}} = 2$ for all $\bar{x} \in R$ for which $\pi_{\bar{x},\alpha} : X \to X_{\bar{x},\alpha}$ is ramified.

Proof. From Proposition 1.15 and the definitions it follows that

$$(40)\qquad 12\mathbb{S}_\alpha + \mathbb{L}_\alpha = \sum_{x\in\bar{X}}\sum_{n=1}^{e_{\bar{x}}-1} s_{n,\alpha}(\bar{x})\left(\mathbb{L}\left(\frac{e_{\bar{x}}}{e_{\bar{x}}-n}\right) + \frac{n}{e_{\bar{x}}} + \frac{n^*}{e_{\bar{x}}} - 1\right),$$

from which the formula follows in view of Theorem 3.7. Moreover, the stated inequality follows immediately from this and the inequality of Proposition 1.15, together with Remark 1.5.

To prove (39), note first that from (40) we see that $12\mathbb{S}_\alpha + \mathbb{L}_\alpha \geq 0$. Thus, by Theorem 3.7 we have $c_1^2 \leq 8(1-2\bar{g}+\mathbb{G}_\alpha) + 2\mathbb{L}_\alpha = 2c_2$, which proves the upper bound on c_1^2. To prove the lower bound, we use the estimate $(k-1) \leq \frac{e_{\bar{x}}}{k}(k-1) = e_{\bar{x}}(1-\frac{1}{k})$ and (11) to obtain

$$\sum_{1<k|e_{\bar{x}}} r_{k,\alpha}(\bar{x})(k-1) \leq e_{\bar{x}} \sum_{1<k|e_{\bar{x}}} r_{k,\alpha}(\bar{x})(1-\frac{1}{k}) = (2g-2) - e_{\bar{x}}(2g_{\bar{x},\alpha}-2).$$

Thus, since $1-2\bar{g}+\mathbb{G}_\alpha = (\bar{g}-1)(g-1) + \frac{1}{2}\sum_{\bar{x}\in\bar{X}}(g-g_{\bar{x},\alpha})$, we obtain

$$c_1^2 \geq 8(\bar{g}-1)(g-1) + \sum_{\bar{x}\in\bar{X}}(4(g-g_{\bar{x},\alpha}) - (2g-2) + e_{\bar{x}}(2g_{\bar{x},\alpha}-2)),$$

which yields the desired lower bound. Note that if this lower bound is an equality, then so is that of (38), which happens precisely when $r_{n,\alpha}(\bar{x}) = 0$ for $n \geq 3$ (cf. the proof of Proposition 1.15). Moreover, from the above we see that we also require that $(k-1) = \frac{e_{\bar{x}}}{k}(k-1)$ whenever $r_{k,\alpha}(\bar{x}) \neq 0$, which means that $e_{\bar{x}} = 2$ if $\pi_{\bar{x},\alpha}$ is ramified.

Remark 3.9 It follows from Remark 1.5d) that all the above invariants $b_i(\tilde{Z}_\alpha)$ etc. depend only on the class of α in the outer automorphism group $\mathrm{Out}(G)$. However, this may also be deduced more directly from the fact that if $\alpha' = \beta_\tau \circ \alpha$, then the map $\mathrm{id} \times \tau : Y \to Y$ induces an isomorphism $Z_\alpha \xrightarrow{\sim} Z_{\alpha'}$ which lifts to an isomorphism $\tilde{Z}_\alpha \xrightarrow{\sim} \tilde{Z}_{\alpha'}$ of the desingularizations.

3.4 The canonical divisor

Although we have already computed the self-intersection number $c_1^2(\tilde{Z}_\alpha) = K_\alpha^2$ of the canonical divisor class $c_1(Z_\alpha) = K_\alpha$, it will be often useful to have an explicit representative of the canonical divisor class at our disposal. Here we describe such a representative in two steps. First we give a conceptual description in Theorem 3.10, and then we work out explicitly the terms involved in Proposition 3.13.

Theorem 3.10 *Let $K_{\bar{Y}}$ be a canonical divisor on $\bar{Y}$ and let*

$$(41)\qquad D(\tilde{\psi}) := \sum_{\bar{x}\in R}\sum_{i=1}^{2}(\tilde{\psi}^*(\bar{C}_{\bar{x},i}) - \tilde{\psi}^{-1}(\bar{C}_{\bar{x},i})) + \sum_{s\in S_\alpha}\sum_{j=1}^{r_s} C_{s,j},$$

where $\bar{C}_{\bar{x},1} = \bar{x} \times \bar{Y}$, $\bar{C}_{\bar{x},2} = \bar{Y} \times \bar{x}$ *and* $\tilde{\psi}^{-1}(\bar{C}) = (\tilde{\psi}^*(C))_{red}$ *(viewed as a divisor on* $\tilde{Z}_\alpha$*). Then*

$$K_\alpha = \tilde{\psi}^* K_{\bar{Y}} + D(\tilde{\psi})$$

is a canonical divisor of $\tilde{Z}_\alpha$*.*

Remark 3.11 a) The above theorem may be deduced from the general ramification formula (Iitaka [15], §5.6), viz.

$$K_{Z_\alpha} \sim \tilde{\psi}^*(K_{\bar{Y}}) + R_{\tilde{\psi}},$$

by showing that $R_{\tilde{\psi}} = D(\tilde{\psi})$. However, it is just as easy to prove Theorem 3.10 directly as it is to verify this identity.

b) Let $\tilde{C}_{\bar{x},i}$ denote the proper transform of the curve $C_{\bar{x},i} = \psi_i^{-1}(\bar{x})_{red}$ with respect to σ, which by Proposition 2.1b) is a smooth curve of genus $g_{\bar{x},\alpha_i}$. Then by definition and 2.1, the divisor $\tilde{\psi}^{-1}(\bar{C}_{\bar{x},i})$ consists of $\tilde{C}_{\bar{x},i}$ together with all the exceptional divisors of σ which map to $\bar{C}_{\bar{x},i}$, all with multiplicity one. However, the pullback $\tilde{\psi}^*(\bar{C}_{\bar{x},i})$ of $\bar{C}_{\bar{x},i}$ is more complicated to describe and will be determined in Proposition 3.13 below.

Proof of 3.10. Since the validity of the assertion does not depend on the choice of the canonical divisor $K_{\bar{Y}}$, it is enough verify it for $K_{\bar{Y}} = \mathrm{div}(\eta)$, where η is a two-form on $\bar{Y}$ such that $K_{\bar{Y}}$ has no common component with $\sum_{\bar{x}\in R}(\bar{C}_{\bar{x},1} + \bar{C}_{\bar{x},2})$. Put $\tilde{\eta} = \tilde{\psi}^*(\eta)$ and

$$K_\alpha := \mathrm{div}(\tilde{\eta}) = \sum n_C C.$$

The coefficients n_C can be computed locally as follows. For each $z \in \tilde{Z}_\alpha$ choose local coordinates (z_1, z_2) around $\tilde{\psi}(z)$ of $\bar{Y}$. Thus $\eta = f dz_1 \wedge dz_2$ (locally). If there is a point $z \in C$ at which $\tilde{\psi}$ is unramified, then (z_1, z_2) (or more precisely $z_i \circ \tilde{\psi}$) are local coordinates around z and thus n_C is equal to the coefficient of $\tilde{\psi}(C)$ in $K_{\bar{x}}$.

Thus, suppose that all points of C are ramified with respect to $\tilde{\psi}$. Then either $C = \tilde{C}_{\bar{x},i}$ with $\bar{x} \in R$ or $C = C_{s,j}$ is an exceptional curve.

For $C = \tilde{C}_{\bar{x},1}$ with $\bar{x} \in R$, choose a point $z \in C$ not lying over any $(\bar{x}_1, \bar{x}_2)$ where $\bar{x}_i \in R$. Then $\tilde{\psi}$ is ramified at z of index $e_{\bar{x}}$ and we can find local coordinates $(\tilde{z}_1, \tilde{z}_2)$ around z such that $z_1 = \tilde{z}_1^{e_{\bar{x}}}$ and $z_2 = \tilde{z}_2$. Therefore $\tilde{\eta} = f d\tilde{z}_1^{e_{\bar{x}}} \wedge d\tilde{z}_2 = f(e_{\bar{x}} - 1)\tilde{z}_1^{e_{\bar{x}}-1} d\tilde{z}_1 \wedge d\tilde{z}_2$. This implies $n_C = e_{\bar{x}} - 1$ because $C_{\bar{x},1}$ is not a component of $K_{\bar{Y}}$.

The same argument also gives the coefficients for $\tilde{C}_{\bar{x},2}$. Thus, any further components of K_α can only be exceptional ones, and hence

$$K_\alpha = \tilde{\psi}^*(K_{\bar{Y}}) + \sum_{\bar{x}\in\bar{X}} (e_{\bar{x}} - 1)(\tilde{C}_{\bar{x},1} + \tilde{C}_{\bar{x},2}) + D_0,$$

where $D_0 \in D(S_\alpha) := \bigoplus_{s,j} \mathbb{Z}C_{s,j} \leq \mathrm{Div}(\tilde{Z}_\alpha)$ lies in the subgroup $D(S_\alpha)$ generated by the exceptional curves. On the other hand, from Proposition 2.1b) one easily deduces that

$$\tilde{\psi}^*(\bar{C}_{\bar{x},i}) = e_{\bar{x}}\tilde{C}_{\bar{x},i} + D_i,$$

with $D_i \in D(S_\alpha)$, and so it follows that $D := K_\alpha - \tilde{\psi}^*(K_{\bar{Y}}) - D(\tilde{\psi}) \in D(S_\alpha)$. Thus, to prove the assertion of the theorem (i.e. that $D = 0$), it is enough to verify that

$$(D.C_{s,j}) = 0, \quad \text{for all } s \in S_\alpha,\ 1 \le j \le r_s,$$

because the intersection matrix $(C_{s,j}.C_{s',j'})_{s,s',j,j'}$ is negative definite.

For this, fix s and j and write $\psi(s) = (\bar{x}_1, \bar{x}_2)$. Then we have

$$\left(\left(\sum_{\bar{x},i} \tilde{\psi}^{-1}(\bar{C}_{\bar{x},i}) - \sum_{s',j'} C_{s',j'}\right).C_{s,j}\right) = \left(\left(\tilde{C}_{\bar{x}_1,1} + C_{s,1} + \ldots + C_{s,r_s} + \tilde{C}_{\bar{x}_2,2}\right).C_{s,j}\right) = C_{s,j}^2 + 2,$$

where we have used Proposition 3.5 and also Lemma 3.12 below. On the other hand, by the adjunction formula we have

$$(K_\alpha.C_{s,j}) = p_a(C_{s,j}) - C_{s,j}^2 = -2 - C_{s,j}^2,$$

whereas the projection formula yields

$$(\tilde{\psi}^*(K_{\bar{Y}} + \sum_{\bar{x},i} \bar{C}_{\bar{x},i}).C_{s,j}) = ((K_{\bar{Y}} + \sum_{\bar{x},i} \bar{C}_{\bar{x},i}).\tilde{\psi}(C_{s,j})) = 0.$$

Combining these equations gives $(D.C_{s,j}) = (-2 - C_{s,j}^2) - 0 + (2 + C_{s,j}^2) = 0$, which proves the assertion of the theorem.

In the above proof we made use of the following crucial fact.

Lemma 3.12 *Let $s \in S_\alpha$ and write $\psi(s) = (\bar{x}_1, \bar{x}_2)$. Then for each $\bar{x} \in \bar{X}$ and $1 \le j \le r_s$ we have*

$$(\tilde{C}_{\bar{x},1}.C_{s,j}) = \delta_{\bar{x}\bar{x}_1}\delta_{j1} \quad \text{and} \quad (\tilde{C}_{\bar{x},2}.C_{s,j}) = \delta_{\bar{x}\bar{x}_2}\delta_{jr_s}.$$

Proof. Choose $y = (x_1, x_2) \in Y$ such that $\varphi(y) = s$. Then by Proposition 2.1, $\varphi(x_1 \times X) = C_{\bar{x}_1,1}$ and $\varphi(X \times x_2) = C_{\bar{x}_2,2}$; recall that by definition $\tilde{C}_{\bar{x}_i,i}$ is the proper transform of $C_{\bar{x},i}$.

As in the proof of Proposition 2.3b), let U be a small neighbourhood of y such that $G_{y,\alpha}$ acts linearly on U. Then, via the isomorphism $U \simeq \mathbb{C}^2$ constructed there, the curves $(x_1 \times X) \cap U$ and $X \cap x_2 \cap U$ correspond to the coordinate axes $z_1 = 0$ and $z_2 = 0$, respectively. Thus, it follows from the explicit construction of the desingularization of the quotient singularity that the proper transform of the image of $z_1 = 0$ (respectively, of $z_2 = 0$) in $G_{y,\alpha}\backslash\mathbb{C}^2$ meets the first component $C_{s,1}$ (respectively, the last component C_{s,r_s}) transversally and none of the others, as is explained in detail in van der Geer [6], p. 42-3. Note, however, that the numbering used there is opposite of that used here. This proves the assertion if $\bar{x} = \bar{x}_1$ or $\bar{x} = \bar{x}_2$. In the other cases we have $s \notin C_{\bar{x},1}$ or $s \notin C_{\bar{x},1}$, so the curve $\tilde{C}_{\bar{x},i}$ does not meet any $C_{s,j}$, and hence the assertion follows.

Although Theorem 3.10 is often already sufficient for many applications, it is useful to complement it by describing the the pullback of the divisors $\tilde{\psi}^*(\bar{C}_{\bar{x},i})$ explicitly.

Proposition 3.13 *For each $\bar{x} \in \bar{X}$, the pullback of the divisors $\bar{C}_{\bar{x},1} = \bar{x} \times \bar{X}$ and $\bar{C}_{\bar{x},1} = \bar{X} \times \bar{x}$ on $\bar{Y}$ via $\tilde{\psi} = \psi \circ \sigma$ is*

$$\tilde{\psi}^*(\bar{C}_{\bar{x},i}) = e_{\bar{x}}\tilde{C}_{\bar{x},i} + \sum_{s \in S_\alpha \cap \psi_i^{-1}(\bar{x})} \sum_{j=1}^{r_s} a_{s,i,j} C_{s,j}, \tag{42}$$

where $i = 1, 2$ and the coefficients $a_{s,i,j}$ are determined from the continued fraction expansion of $\frac{n_s}{q_s} = [[c_{s,1}, \ldots, s_{s,r_s}]]$ (cf. Proposition 3.5) by the recursion relations

$$a_{s,i,j+1} = c_{s,j} a_{s,i,j} - a_{s,i,j-1}, \quad 1 \le j \le r_s, \tag{43}$$

together with the boundary conditions $a_{s,1,0} = a_{s,2,r_s+1} = e_{\bar{x}}$, $a_{s,1,r_s+1} = a_{s,2,0} = 0$. Thus we have

$$\begin{array}{rcl} a_{s,1,1} &=& \frac{e_{\bar{x}}}{n_s} q_s > a_{s,1,2} > \ldots > a_{s,1,r_s} = \frac{e_{\bar{x}}}{n_s}, \\ a_{s,2,1} &=& \frac{e_{\bar{x}}}{n_s} < a_{s,2,2} < \ldots < a_{s,2,r_s} = \frac{e_{\bar{x}}}{n_s} q_s^*, \end{array}$$

where $q_s^* q_s \equiv 1 \pmod{n_s}$ and $0 < q_s^* < n_s$. Furthermore, the divisor $D(\tilde{\psi})$ is given by

$$D(\tilde{\psi}) = \sum_{\bar{x} \in \bar{X}} (e_{\bar{x}} - 1)(\tilde{C}_{\bar{x},1} + \tilde{C}_{\bar{x},2}) + \sum_{s \in S_\alpha} \sum_{j=1}^{r_s} (a_{s,1,j} + a_{s,2,j} - 1) C_{s,j} \le (e_{\bar{x}} - 1) D(\tilde{\psi})_{red}. \tag{44}$$

Proof. Clearly, $\tilde{\psi}^*(\bar{C}_{\bar{x},i})$ has a representation (42) with $a_{s,i,j} \in \mathbb{N}$. To determine the coefficients, fix $i = 1, 2$ and $s \in S_\alpha \cap \psi_i^{-1}(\bar{x})$. The projection formula gives $(\tilde{\psi}^*(\bar{C}_{\bar{x},i}).C_{s,j}) = (\bar{C}_{\bar{x},i}.\tilde{\psi}_*(C_{s,j}) = 0$ for $1 \le j \le r_s$. In view of Proposition 3.5 and Lemma 3.12, this leads to the system of recursion relations (43), together with the indicated boundary conditions, and these determine the $a_{s,i,j}$ uniquely.

On the other hand, since $\frac{n_s}{q_s} = [[c_{s,1}, \ldots, c_{s,r_s}]]$, the sequence $\{\mu_j\}$ defined by $\mu_0 = 0, \mu_1 = 1$ and the recursion relation $\mu_{j+1} = c_{s,j}\mu_k - \mu_{j-1}$, $1 \le j \le r_s$, also satisfies $\mu_{r_s} = q_s^*$ and $\mu_{r_s+1} = n_s$, and similarly, the sequence $\{\lambda_j\}$ defined by $\lambda_0 = n_s, \lambda_1 = q_s$ and the same recursion relation also satisfies $\lambda_{r_s} = 1, \lambda_{r_s+1} = 0$; cf. BPV [1], p. 81. Thus, $a_{s,1,j} = \frac{e_{\bar{x}}}{n_s}\lambda_j, a_{s,2,j} = \frac{e_{\bar{x}}}{n_s}\mu_j$, satisfy the recursion relation (43) and the boundary conditions. Furthermore, since by induction $\mu_{j+1} - \mu_j \ge \ldots \ge \mu_1 - \mu_0 = 1$ and $\lambda_j - \lambda_{j+1} \ge \ldots \ge \lambda_0 - \lambda_1 = n_s - q_s \ge 1$, we see that the $a_{s,i,j}$ satisfy the indicated values and inequalities.

The last formula/inequality follows immediately from the definition of $D(\tilde{\psi})$ and from the inequalities $\mu_j + \lambda_j \le n_s$.

Remark 3.14 It is also possible to give determinantal expressions for the $a_{s,i,j}$'s. Indeed, if for integers $d_1, \ldots, d_n$ we put

$$[d_1, \ldots, d_n] := \det \begin{pmatrix} d_1 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & d_n \end{pmatrix} \text{ and } [\,] := 1,$$

then we have $[-d_1, \ldots, -d_n] = (-1)^n [d_1, \ldots, d_n]$ and the Laplace expansion gives

$$[d_1, \ldots, d_n] = d_1[d_2, \ldots, d_n] - [d_3, \ldots, d_n].$$

Comparing this recursion relation with (43) shows that we have

$$a_{s,1,j} = \frac{e_{\bar{x}}}{n_s}[c_{s,j+1}, \ldots, c_{s,r_s}] \text{ and } a_{s,2,j} = \frac{e_{\bar{x}}}{n_s}[c_{s,1}, \ldots, c_{s,j-1}]. \tag{45}$$

Corollary 3.15 *For $\bar{x} \in \bar{X}$ we have*

$$(\tilde{C}_{\bar{x},1})^2 = -\sum_{s\in S_\alpha\cap\psi_1^{-1}(\bar{x})} \frac{q_s}{n_s} = -\sum_{\bar{y}\in\bar{X}}\sum_{\nu=1}^{e_{\bar{y}}-1} s_{\nu,\alpha}(\bar{x},\bar{y})\frac{\nu}{e_{\bar{y}}},$$
$$(\tilde{C}_{\bar{x},2})^2 = -\sum_{s\in S_\alpha\cap\psi_2^{-1}(\bar{x})} \frac{q_s^*}{n_s} = -\sum_{\bar{y}\in\bar{X}}\sum_{\nu=1}^{e_{\bar{x}}-1} s_{\nu,\alpha}(\bar{y},\bar{x})\frac{\nu^*}{e_{\bar{x}}},$$

where q_s^* and $\nu^* = \nu_{\bar{x}}^*$ are as in 3.8. In particular, if α is such that $\alpha^2 \in \mathrm{Inn}(G)$, then $\tilde{C}_{\bar{x},1}^2 = \tilde{C}_{\bar{x},2}^2$. Moreover, for any $\bar{x}, \bar{y} \in \bar{X}$ we have

$$(\tilde{C}_{\bar{x},1}.\tilde{C}_{\bar{y},2}) = r_{1,\alpha}(\bar{x},\bar{y}).$$

Proof. We first note that $\tilde{\psi}_*(\tilde{C}_{\bar{x},i}) = \frac{m}{e_{\bar{x}}}\bar{C}_{\bar{x},i}$. Thus, since $(\bar{C}_{\bar{x},i})^2 = 0$, the projection formula combined with (42) yields

$$0 = (\tilde{\psi}^*(\bar{C}_{\bar{x},i}).\tilde{C}_{\bar{x},i}) = e_{\bar{x}}(\tilde{C}_{\bar{x},i})^2 + \sum_{s\in S_\alpha\cap\psi_i^{-1}(\bar{x})}\sum_{j=1}^{r_s} a_{s,i,j}(C_{s,j}.\tilde{C}_{\bar{x},i}),$$

so $(\tilde{C}_{\bar{x},i})^2 = -\sum_{s\in S_\alpha\cap\psi_i^{-1}(\bar{x})}\sum_{j=1}^{r_s}(a_{s,i,j}/e_{\bar{x}})(C_{s,j}.\tilde{C}_{\bar{x},i})$. Now for $s \in \psi_1^{-1}(\bar{x})$ we have that $\sum_{j=1}^{r_s}(a_{s,1,j}/e_{\bar{x}})(C_{s,j}.\tilde{C}_{\bar{x},i}) = a_{s,1,1}/e_{\bar{x}} = \frac{q_s}{n_s}$ by Lemma 3.12 and Proposition 3.13. This proves the first equality, and the second follows in view of 2.3b). The formulae for $(\tilde{C}_{\bar{x},2})^2$ is proved analogously, and the identity $\tilde{C}_{\bar{x},1}^2 = \tilde{C}_{\bar{x},2}^2$ follows from the symmetry formula (14) and Remark 1.5d). A similar computation for $(\tilde{C}_{\bar{x},1}.\tilde{C}_{\bar{y},2})$, using the fact that $(\bar{C}_{\bar{x},1}.\bar{C}_{\bar{y},2}) = 1$, leads to

$$(\tilde{C}_{\bar{x},1}.\tilde{C}_{\bar{y},2}) = \frac{m}{e_{\bar{x}}e_{\bar{y}}} - \sum_{s\in\psi^{-1}(\bar{x},\bar{y})}\frac{1}{n_s} = \frac{m}{e_{\bar{x}}e_{\bar{y}}} - \sum_{1<d|e_{\bar{y}}} r_{d,\alpha}(\bar{x},\bar{y})\frac{1}{d},$$

which, by formula (10), yields the desired formula.

Remark 3.16 Using the above formulae it is possible to compute the the self-intersection number $c_1^2 = K_\alpha^2$ once more; this leads to a second proof of Corollary 3.8, one that does not use Proposition 1.15.

4 Classification theorems

4.1 General results

In this section we want to examine how the twisted diagonal surfaces $\tilde{Z}_\alpha$ fit into the classification scheme of Enriques–Kodaira. Since it is rather difficult to determine the Kodaira dimension $\kappa = \kappa(\tilde{Z}_\alpha)$ in some cases, a complete classification does not seem to be possible; nevertheless, it is possible to obtain a partial classification which will be presented in this section. In the next sections we study criteria which are useful in handling some of the more subtle cases.

We begin by examining the effect of the genera $g = g(X)$ and $\bar{g} = g(\bar{X})$ on the Kodaira dimension.

Theorem 4.1 *a) If $g = 0$ then $\tilde{Z}_\alpha$ is rational; i.e. $\kappa = -\infty$.*
b) If $g = \bar{g} = 1$, then $\tilde{Z}_\alpha \simeq Z_\alpha$ is an abelian variety, so $\kappa = 0$.
c) If $g = 1$ and $\bar{g} = 0$, then $\tilde{Z}_\alpha$ is an elliptic surface with $\kappa \leq 0$.
d) If $g \geq 2$ and $\bar{g} \geq 1$, then $\tilde{Z}_\alpha$ is a minimal surface of general type, i.e. $\kappa = 2$.

Proof. a) Since $\mathbb{C}(\tilde{Z}_\alpha) \subset \mathbb{C}(Y)$ and $Y = \mathbb{P}^1 \times \mathbb{P}^1$ in this case, we see that $\tilde{Z}_\alpha$ is unirational and hence is rational by Castelnuovo's theorem.

b) In this case $Y = X \times X$ is the product of two elliptic curves and hence is an abelian surface. Moreover, since $g = \bar{g}$, G acts without fixed points and hence consists of translations of X. Thus, $\Delta_\alpha \le G \times G$ consists of translations of Y and so $Z_\alpha = \Delta_\alpha \backslash Y$ is again an abelian surface. In particular, Z_α is smooth and hence $\tilde{Z}_\alpha = Z_\alpha$.

c) Here Y is again an abelian surface, so $\kappa(\tilde{Z}_\alpha) \le \kappa(Y) = 0$ (cf. [1], (I.7.4)). Moreover, almost all of the fibres of the fibration $\tilde{\psi}_i : \tilde{Z}_\alpha \to \bar{X} \simeq \mathbb{P}^1$ are isomorphic to X (cf. Proposition 2.1b)), and hence are elliptic curves, so $\tilde{Z}_\alpha$ is an elliptic surface.

d) First note that $p_g \ge 1$ because the canonical class formula 3.10 shows that we can find $D \sim K_\alpha$, $D \ge 0$. (Here we use the fact that $p_g(\bar{Y}) \ge 1$ because $\bar{g} \ge 1$.) Thus, $\tilde{Z}_\alpha$ is not rational, so it is enough to show that $K_\alpha^2 > 0$, for then also the invariant $K_{min}^2 \ge K_\alpha^2$ of the associated minimal model Z_{min} is positive, and so Z_{min} and $\tilde{Z}_\alpha$ are of general type (cf. Beauville [2], X.1).

To show that $K_\alpha^2 > 0$, we use the inequality (39), which yields

$$K_\alpha^2 \ \ge\ (2g-2)(r + 4(\bar{g}-1)) \ >\ 0$$

because $g_{\bar{x},\alpha} \ge \bar{g} \ge 1$. Thus $\tilde{Z}_\alpha$ is a surface of general type.

Finally, we note that $\tilde{Z}_\alpha$ is minimal. Indeed, the only rational curves on $\tilde{Z}_\alpha$ are the curves $C_{s,j}$, for if $C \ne C_{s,j}$ were a rational curve on $\tilde{Z}_\alpha$, then $\tilde{\psi}_i(C)$ would be a rational curve covering $\bar{X}$, which is impossible by Lüroth. But each $C_{s,j}$ has self-intersection $C_{s,j}^2 \le -2$, so is not a (-1)-curve. Thus, there are no (-1)-curves on $\tilde{Z}_\alpha$, which means that $\tilde{Z}_\alpha$ is minimal.

In the above theorem, we determined the Kodaira dimension of all the surfaces except for those appearing in case c). This will be done next.

Theorem 4.2 *If $g = 1$ and $\bar{g} = 0$, then $\tilde{Z}_\alpha$ is an elliptic surface with $p_g \le 1$ and $K_\alpha^2 \le 0$, and we have the following three possibilities:*

a) If $p_g = 0$ and $K_\alpha^2 < 0$, then $\tilde{Z}_\alpha$ is rational ($\kappa = -\infty$).

b) If $p_g = 0$ and $K_\alpha^2 = 0$, then $\tilde{Z}_\alpha$ is a minimal Enriques surface ($\kappa = 0$).

c) If $p_g = 1$, then $\tilde{Z}_\alpha$ is a minimal K3-surface ($\kappa = 0$).

Remark 4.3 All the cases listed in the theorem can actually occur. Cases a) and c) occur in the modular curve case, i.e. $X = X(N)$ with $N = 6$; cf. [17]. Moreover, case b) also exists as we shall show now.

Example 4.4 An Enriques Surface.
Let X be an elliptic curve with origin P_0, and let $G = \langle \sigma, \tau \rangle$, where σ denotes the minus map $\sigma(P) = -P$, and τ is the translation map by some point P_1 of order 2. Then $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so there is an involution $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(\sigma) = \tau$. We claim that then $p_g(\tilde{Z}_\alpha) = 0$, $K_\alpha^2 = 0$, so that $\tilde{Z}_\alpha$ is an Enriques surface by Theorem 4.2.

To see this, note first that $\bar{g} = 0$ since σ has a fixed point. Furthermore, we have $r = 4$ and $e_{\bar{x}} = 2$ for all $\bar{x} \in R$. Now $G_x = \langle \sigma \rangle$ for $x \in \pi^{-1}(\bar{x})$, for two (say $\bar{x} = \bar{x}_1, \bar{x}_2$) of the four points $\bar{x} \in R$, and hence $g_{\bar{x},\alpha} = 1$ for these by our choice of α. We thus have $r_{\bar{x}_i,\alpha} = 0$ for $i = 1, 2$ and $r_{\bar{x}_i,\alpha} = 4$ for $i = 3, 4$. It thus follows from the definitions that $\mathbb{G}_\alpha = \mathbb{S}_\alpha = 0$ and $\mathbb{L}_\alpha = 8$, and hence by Theorem 3.7 we obtain $p_g = \mathbb{G}_\alpha - \mathbb{S}_\alpha = 0$ and $K_\alpha^2 = 8(1+\mathbb{G}_\alpha) - \mathbb{L}_\alpha + 12\mathbb{S}_\alpha = 0$, which proves the claim.

Note that if we take $\alpha = id$, then $\mathbb{G}_{id} = 1$, $\mathbb{S}_{id} = 0$ and $\mathbb{L}_\alpha = 16$, so $p_g(\tilde{Z}_{id}) = 1, K_{id}^2 = 0$ and hence $\tilde{Z}_\alpha$ is a *K3*-surface in this case.

The proof of Theorem 4.2 is based on the following facts.

Proposition 4.5 *Suppose $g \geq 1$ and $\bar{g} = 0$. Then:*

a) If $H^0(\tilde{Z}_\alpha, -K_\alpha) \neq 0$, then $g = 1$ and either $K_\alpha = 0$ or $K_\alpha^2 < 0$.

b) The second plurigenus $P_2 = P_2(\tilde{Z}_\alpha)$ satisfies the inequality

$$P_2 \;\geq\; K_\alpha^2 + 1 + p_g,$$

except when $g = 1$ and $K_\alpha = 0$, i.e. except when $\tilde{Z}_\alpha$ is a K3-surface.

Before we prove this Proposition, let us note the following important consequence.

Corollary 4.6 *If $g \geq 1$ and $K_\alpha^2 > 0$ then $\tilde{Z}_\alpha$ is of general type ($\kappa = 2$).*

Proof. By Theorem 4.1b), d) we may assume that $\bar{g} = 0$. Then, since $K_\alpha \neq 0$, we have by Proposition 4.5b) that $P_2 \geq K^2 + 1 + p_g > 0$, so $\tilde{Z}_\alpha$ cannot be rational. Thus, since $K_\alpha^2 > 0$, it follows that $\tilde{Z}_\alpha$ is of general type (cf. proof of Theorem 4.1).

The proof of Proposition 4.5 depends on the following simple fact.

Lemma 4.7 *Suppose that $\bar{g} = 0$ and that $D > 0$ is an effective (non-zero) divisor on $\tilde{Z}_\alpha$ with $(D.\tilde{C}_{\bar{x},i}) \leq 0$ for some $\bar{x} \notin R$ and for $i = 1, 2$. Then D is a linear combination of the exceptional curves $C_{s,j}$ and hence $D^2 < 0$ and $(D.\tilde{C}_{\bar{x},i}) = 0$ for $\bar{x} \notin R$ and $i = 1, 2$.*

Proof. Since $\bar{D} := \tilde{\psi}_*(D)$ is an effective divisor on $\bar{Y} = \mathbb{P}^1 \times \mathbb{P}^1$ with $(\bar{D}.\bar{C}_{\bar{x},i}) = (D.\tilde{C}_{\bar{x},i}) \leq 0$, we must have $\bar{D} = 0$. Thus D consists entirely of curves blown down under $\tilde{\psi}$. This proves the first statement, and the second follows since the intersection pairing is negative definite on the space spanned by the $C_{s,j}$'s. The last is clear by the projection formula since $\bar{D} = 0$.

Proof of Proposition 4.5. a) By hypothesis, there is an effective divisor $D \geq 0$ such that $D \sim -K_\alpha$. Let $\bar{x} \notin R$. Then $\tilde{C}_{\bar{x},i}^2 = 0$, so by the adjunction formula $(D.\tilde{C}_{\bar{x},i}) = -(K_\alpha.\tilde{C}_{\bar{x},i}) = -(2g-2)$. From Lemma 4.7 we therefore see that $g \leq 1$, hence $g = 1$ and that moreover $D^2 < 0$ if $D \neq 0$. Since $D \sim -K$, the assertion follows.

b) We may assume that $K_\alpha \geq 0$, for otherwise the statement follows from the trivial inequality $P_2 \geq p_g$. Then by part a) we have either $g = 1$ and $K_\alpha = 0$ or $h^0(-K_\alpha) = 0$. In the former case $\tilde{Z}_\alpha$ is clearly (by definition) a *K3*-surface; recall that $q = \bar{g} = 0$. In the latter case we have by Riemann–Roch (and Serre duality) on $\tilde{Z}_\alpha$:

$$P_2 = h^0(2K_\alpha) = h^0(2K_\alpha) + h^0(-K_\alpha) \geq \frac{1}{2}(2K_\alpha).(2K_\alpha - K_\alpha) + 1 + p_a = K_\alpha^2 + 1 + p_g,$$

which proves the desired inequality.

Proof of Theorem 4.2. The hypotheses imply that $\tilde{\psi}_i : \tilde{Z}_\alpha \to \bar{X}$ is an elliptic fibring; cf. 2.1b). The second assertion follows from the inequality $p_g(\tilde{Z}_\alpha) \le p_g(Y) = 1$, and the third follows from Corollary 4.6 since we already know that $\kappa \le 0$ by Theorem 4.1c).

a) Suppose not; then by Castelnuovo's criterion we must have $P_2 \ge 1$ and hence $\kappa = 0$. Let $D \in |2K_\alpha| \ne \emptyset$. Then, as in the proof of 4.5, we have $(D.\tilde{C}_{\bar{x},i}) = 2(2g-2) = 0$, for all $\bar{x} \notin R$, and so by Lemma 4.7 it follows that all components of D are of the form $C_{s,j}$; in particular, D does not contain any (-1)-curves. But this means that $\tilde{Z}_\alpha$ is minimal, for any (-1)-curve is contained in some $D \in |2K_\alpha|$ because $P_2 \ge 1$. But there is no minimal surface with $P_2 \ge 1$, $K^2 < 0$ and $\kappa = 0$ (cf. Beauville[2], VIII.2); contradiction. It thus follows that $\tilde{Z}_\alpha$ must be rational.

b) Since $p_g = 0$, $K_\alpha \ne 0$, so from Proposition 4.5 we obtain $P_2 \ge 0+1+0 = 1$. This shows that $\tilde{Z}_\alpha$ cannot be rational, and hence $\kappa = 0$. Thus, since $p_g = 0$ and $q = 0$, this means that $\tilde{Z}_\alpha$ is an Enriques surface; cf. Beauville[2], VIII.2.

c) If $K_\alpha \ne 0$, then Proposition 4.5 yields $P_2 \ge 0+1+1 = 2$, which is impossible for a surface with $\kappa = 0$. Thus $K_\alpha = 0$ and hence $\tilde{Z}_\alpha$ is a *K3*-surface.

Since Theorems 4.1 and 4.2 completely classify the cases that $g \le 1$ or that $\bar{g} \ge 1$, we are left with the case that $g \ge 2$ and $\bar{g} = 0$. Here, however, it seems to be impossible to give a classification based on the behaviour of g alone. Nevertheless, one might expect that $\tilde{Z}_\alpha$ is of general type if $m = |G|$ is small relative to g. Since the latter condition is true if $r = |R|$ is large, the following theorem substantiates this expectation to some extent.

Theorem 4.8 *$\tilde{Z}_\alpha$ is of general type if any one of the following conditions holds:*

a) $r \ge 8$;

b) $r \ge 6$ and G is not cyclic;

c) $r \ge 4$, $g \ge 2$ and $g_{\bar{x},\alpha} \ge 1$, for all $\bar{x} \in R$.

Proof. In all three cases we shall apply Corollary 4.6 and hence have to verify that $K_\alpha^2 > 0$ in these cases. Note that the hypotheses imply that $g \ge 2$ because $2g-2 \ge m(-2+\frac{r}{2}) > 0$ if $r \ge 5$.

a) By Corollary 3.8, specifically by inequality (39), we have

$$K_\alpha^2 \ge (2g-2)(r-4) + 2\sum_{\bar{x}\in R}(e_{\bar{x}}-2)(g_{x,\alpha}-1) \ge (2g-2)(r-4) - 2\sum_{\bar{x}\in R}(e_{\bar{x}}-2).$$

Let $r_n = \#\{\bar{x} \in R : e_{\bar{x}} = n\}$. Then $\sum_{\bar{x}\in R}(e_{\bar{x}}-2) = \sum_{1<d|m} r_d(d-2)$, so

$$\begin{aligned}\sum_{\bar{x}\in R}(e_{\bar{x}}-2) &\le \sum_{1<d|m} r_d\frac{m}{d}(d-2) = -\sum_{1<d|m} mr_d + 2\sum_{1<d|m} mr_d\left(1-\frac{1}{d}\right)\\ &= -rm + 2[(2g-2) - m(2\bar{g}-2)].\end{aligned}$$

Since we may assume that $\bar{g} = 0$ (cf. Theorem 4.1d)), we obtain

$$K_\alpha^2 \ge (2g-2)(r-8) + 2m(r-4) > 0.$$

b) If G is not cyclic, then $r_m = 0$, so we have $r_d \leq \frac{m}{2d} r_d$ for all $d|m$, and hence in this case the argument of a) leads to the inequality

$$K_\alpha^2 \geq (2g-2)(r-6) + m(r-4) > 0.$$

c) From Corollary 3.8 we see directly that $K_\alpha^2 \geq 0$. Suppose $K_\alpha^2 = 0$. Then (39) is an equality, so $r = 4$ and also $e_{\bar{x}} = 2$ if $\pi_{\bar{x},\alpha}$ is ramified by the last assertion of Corollary 3.8. Now if $e_{\bar{x}} = 2$ for all $\bar{x} \in R$, then $2g - 2 = m(-2 + 4 \cdot \frac{1}{2}) = 0$, contradiction. Thus, there is at least one $\bar{x} \in R$ with $e_{\bar{x}} \geq 3$. Then $\pi_{\bar{x},\alpha}$ must be unramified, so $g_{\bar{x},\alpha} > 1$. But then (39) yields $K_\alpha^2 > 0$, as claimed.

Remark 4.9 In Theorem 4.8, the main criterion employed for proving that Z_α is of general type was that $K_\alpha^2 > 0$; cf. Corollary 4.6. Now if $\tilde{Z}_\alpha$ were *minimal*, then this condition would be also necessary; however, the surfaces $\tilde{Z}_\alpha$ are rarely minimal, as specific examples show; cf. [17]. We also have the following example of a more general nature:

The surface $\tilde{Z}_\alpha$ is never minimal if $\alpha = id$, $r = 3$ and $\bar{g} = 0$.

To see this, we shall show that $\tilde{\Gamma}$:= the proper transform of (the image of) the diagonal $\Gamma = \varphi(\{(x,x) \in Y : x \in X\})$ is an exceptional (-1)-curve.

Since the diagonal on Y is invariant under the untwisted diagonal action, we have that Γ is isomorphic to $G\backslash X$ and hence is smooth of genus $\bar{g}$, and the same is true for $\tilde{\Gamma}$. Furthermore, since $\tilde{\psi}_*\tilde{\Gamma} = \bar{\Gamma}$ is the diagonal of $\bar{Y}$, the projection formula and 3.13 imply

$$1 = (\bar{C}_{\bar{x},i}.\bar{\Gamma}) = e_{\bar{x}}(\tilde{C}_{\bar{x},i}.\tilde{\Gamma}) + \sum_{s \in S_\alpha} \sum_{j=1}^{r_s} a_{s,i,j}(C_{s,j}.\tilde{\Gamma}).$$

Since all these terms are non-negative and $e_{\bar{x}} \geq 2$ if $\bar{x} \in R$, it follows that

$$(\tilde{\Gamma}.\tilde{C}_{\bar{x},i}) = 0, \ \forall \bar{x} \in R \qquad \text{and} \qquad (C_{s,j}.\tilde{\Gamma}) = \delta_{s,s_{\bar{x}}}, \quad \forall s, j,$$

where $s_{\bar{x}} = \varphi(x,x)$ for $x \in \pi^{-1}(\bar{x})$. It thus follows from (44) that $(D(\tilde{\psi}).\tilde{\Gamma}) = r$ because $a_{s_{\bar{x}},i,1} = r_{s_{\bar{x}}} = 1$. Thus, the canonical class formula 3.10 gives $(K_\alpha.\tilde{\Gamma}) = (K_{\bar{Y}}.\bar{\Gamma}) + (D(\tilde{\psi}).\tilde{\Gamma}) = 4\bar{g} - 4 + r$, and so the adjunction formula yields $\tilde{\Gamma}^2 = 2 - 2\bar{g} - r$. Thus, $\tilde{\Gamma}$ is a (-1)-curve if $r = 3$ and $\bar{g} = 0$.

4.2 Criteria for general type

In order to be able to treat some of the cases for which the criterion of Corollary 4.6 may fail (cf. Remark 4.9), we introduce here the following invariant $d(S,C)$ which often helps to resolve the issue.

Notation 4.10 Let C be a curve on a smooth surface S. We put

$$d(S,C) := 2\,(p_g(S) + q(S) - p_a(C)) + (C.C) = 2(p_g(S) + q(S) - 1) - (K_S.C),$$

where the last equality follows from the adjunction formula.

This invariant satisfies the following properties.

Proposition 4.11 *a) If $f : S \to S'$ is a birational morphism of smooth surfaces and C is an irreducible curve on S such that $f(C)$ is not blown down to a point, then*

$$d(S,C) \le d(S', f(S)).$$

b) Let $f : S \to B$ be a (minimal) elliptic fibration with multiple fibres $m_1F_1, \ldots, m_tF_t$ where $t \ge 0$. If $g(B) = q(S)$, then for any irreducible curve C on S we have

$$d(S,C) = 2(p_g + q - 1) - \deg_f(C)\left(p_g + q - 1 + \sum_{i=1}^{t}\left(1 - \frac{1}{m_i}\right)\right),$$

where $\deg_f(C) = (C.f^*(P))$, for any $P \in B$. In particular, if C is neither a section of f nor a component of a fibre of f, then $d(S,C) \le 0$, and even $d(S,C) < 0$ if f has multiple fibres.

Proof. a) By the factorization lemma and induction, it is enough to verify this when f is the blow-down map of a (-1)-curve E. Then $K_S = f^*K_{S'} + E$, and hence we have $(K_S.C) = (K_{S'}.f_*(C)) + (E.C)$. Thus, since p_g and q are birational invariants, $d(S,C) - d(S', f_*(C)) = -(E.C) \le 0$ because $C \ne E$.

b) By Kodaira's formula (cf. BPV[1], V.12.3) we have $K_S = f^*(D) + \sum_{i=1}^{t}(m_i - 1)F_i$, where $\deg(D) = \chi(\mathcal{O}_S) - 2\chi(\mathcal{O}_B) = p_g + q - 1$ (since $g(B) = q(S)$). Thus $(K_S.C) = (C.f^*(P))(p_g + q - 1 + \sum(m_i - 1)\frac{1}{m_i})$, and so the formula follows. The second assertion is clear, since the we have $\deg_f(C) \ge 2$ in this case.

Theorem 4.12 *Let S be a smooth regular surface with $\kappa(S) \ge 1$. Then S is of general type if any one of the following conditions holds:*

a) There is an irreducible curve C on S of genus $p_a(C) \ge 2$ and with $d(S,C) \ge 1$.

b) There exist two intersecting elliptic curves E_1 and E_2 on S with $d(S,E_i) \ge 1$.

c) There exist $t \ge 1$ distinct elliptic curves $E_1, \ldots, E_t$ on S with $d(S,E_i) \ge 1$ for $1 \le i \le t$ such that

$$K_S^2 > \sum_{i=1}^{t} E_i^2. \tag{46}$$

Proof. Let $\bar{S}$ denote the minimal model of S and $\sigma : S \to \bar{S}$ the associated birational map. Suppose that S, hence $\bar{S}$, is not of general type. Then $\kappa = 1$, so there is an elliptic fibration $\bar{\pi} : \bar{S} \to B$ to some smooth curve B (cf. Beauville[2], IX.2) of genus $g(B) = q(\bar{S})$ or $g(B) = q(S) - 1$ (cf. [2], Ex. IX.1). Since $q(S) = 0$ by hypothesis, we must have $g = q = 0$. We shall now show that none of the three cases under consideration is possible.

a) First note that since $p_a(C) > 0$, $\bar{C} := \sigma(C)$ is not a point, and so we have $p_a(\bar{C}) \ge p_a(C)$. Thus, $\bar{C}$ cannot be a section of $\bar{\pi}$, for otherwise $p_a(\bar{C}) = 0$, nor a component of a fibre, for otherwise $p_a(\bar{C}) \le 1$ (cf. BPV[1], p. 150). It thus follows from Proposition 4.11b) that $d(S,C) \le 0$, which contradicts the inequality $d(\bar{S},\bar{C}) \ge d(\bar{S},\bar{C}) \ge 1$ which we obtain from Proposition 4.11a) and the hypothesis.

b) If $\bar{E}_i := \sigma(E_i)$ is not a component of a fibre of $\bar{\pi}$, then the proof of a) shows that we have a contradiction. Thus, $\bar{E}_i$ must be a component of a fibre and hence must be equal to a fibre by Kodaira's classification of elliptic fibres

([1], loc. cit.). It thus follows that $\bar{E}_1$ and $\bar{E}_2$ are disjoint, hence so are E_1 and E_2, which contradicts the hypothesis.

c) By part b) we know that each $\bar{E}_i = \sigma(E_i)$ is a fibre of $\bar{\pi}$, so the $\bar{E}_i$ are disjoint smooth elliptic curves on $\bar{S}$ with $(\bar{E}_i.\bar{E}_i) = 0$ for $1 \le i \le t$.

Write $\sigma = \sigma_n \circ \cdots \circ \sigma_1$, where $\sigma_i : S_{k-1} \to S_k$ is a blowup of one point $P_k \in S_k$; thus $S_0 = S$ and $S_n = \bar{S}$. Put $E_{i,k} = \sigma_k \circ \cdots \circ \sigma_1(E_i)$; note that all these curves are again disjoint smooth elliptic curves. It thus follows that the point P_k can lie on at most one elliptic curve $E_{i,k}$; if this is the case, then we get

$$E_{i,k}^2 = (\sigma_k^*(E_{i,k}).E_{i,k-1}) = ((E_{i,k-1} + C_k).E_{i,k-1}) = E_{i,k-1}^2 + 1,$$

where $C_k = \sigma_k^{-1}(P_k)$ denotes the exceptional curve. Thus, if we put $E(k) = E_{1,k} + \ldots + E_{t,k}$, then we have shown that $E(k-1)^2 \ge E(k)^2 - 1$, for $1 \le k \le n$, and so

$$m := \sum_{i=1}^{t} E_i^2 = E(0)^2 \ge E(n)^2 - n = -n.$$

On the other hand, since the self-intersection numbers of the canonical divisors on S and $\bar{S}$ are related by the formula $K_{\bar{S}}^2 = K_S^2 + n$, we obtain from the hypothesis (46) the inequality $K_{\bar{S}}^2 > m + n \ge 0$, which shows that $\bar{S}$ is of general type.

4.3 Criteria for special type

In this section we want to establish some criteria which are useful for determining that a given surface is of *special type*, i.e. not of general type. This is based on the following property of surfaces of general type which is also interesting in itself.

Proposition 4.13 *Let S be a surface of general type with minimal model $f : S \to \bar{S}$, and let D be a reduced, connected divisor with $D^2 > 0$. Then we have*

$$K_{\bar{S}}^2 \le \frac{(K_{\bar{S}}.f_*D)^2}{(f_*D)^2} \le \frac{(K_S.D)^2}{D^2}, \tag{47}$$

and equality holds throughout if and only if $K_{\bar{S}}^2 \cdot D \equiv (K.D)^2 \cdot f^*K_{\bar{S}}$ and D does not contain any blow-down curve of f.

Remark 4.14 a) As the proof below shows, one can weaken the hypothesis on D slightly by allowing D to be sum of divisors of the above type. However, one cannot drop the hypothesis entirely, for otherwise one could apply it to $D \in |K_S|$ (assuming $p_g > 0$ and $K_S^2 > 0$), which, when S is not minimal, leads to the contradiction $K_{\bar{S}}^2 \le K_S^2 < K_{\bar{S}}^2$.

b) If $\tilde{Z}_\alpha$ is a diagonal quotient surface of general type, then the above proposition yields the following upper bound on the first Chern class of the minimal model $\bar{Z}_\alpha$ of $\tilde{Z}_\alpha$:

$$K_{\bar{Z}_\alpha}^2 \le \frac{8(g-1)^2}{m} \quad \text{and hence} \quad K_{\bar{Z}_\alpha}^2 - K_{\tilde{Z}_\alpha}^2 \le \mathbb{L}_\alpha + 12\mathbb{S}_\alpha. \tag{48}$$

Indeed, if we take $D = \tilde{C}_{\bar{x},1} + \tilde{C}_{\bar{x},2}$, where $\bar{x} \notin R$, then $D^2 = 2m$ and $(K_S.D) = 4(g-1)$, and so the first bound follows. The second follows from the first by using the formula for K_S of Theorem 3.7 together with equation (12).

The proof of the above proposition depends on the following useful fact.

Lemma 4.15 *Let $f : S \to S'$ be a birational morphism of smooth surfaces. Then for any divisor $D \in \mathrm{Div}(S)$ we have*

$$(f_*D)^2 \ \geq \ D^2, \tag{49}$$

and equality holds if and only if D does not meet any blow-down curve C of f, i.e. $(D.C) = 0$ for every C such that $f(C)$ is a point. Moreover, if D is reduced and connected and if $f_*D \neq 0$ (e.g. if $D^2 > 0$), then we also have

$$(f_*D.K_{S'}) \ \leq \ (D.K_S). \tag{50}$$

Moreover, if D does not contain any (-1)-curve on S, then equality holds in (50) if and only if D neither meets nor contains any blow-down curve of f.

Proof. Write f as $f = f_n \circ \ldots \circ f_0$, where $f_k : S_{k-1} \to S_k$ is a blow-down map with respect to a (-1)-curve E_{k-1} on S_{k-1}. We now induct on n. If $n = 1$, then for any divisor D we have $f^*f_*D = D + (D.E_0)E_0$, from which we obtain by the projection formula that $(f_*D)^2 = (D.f^*f_*D) = D^2 + (D.E_0)^2$. From this the inequality (49) is immediate, as is the assertion about when equality holds. Now suppose that $D \geq 0$ is reduced and connected and that $f_*D \neq 0$. Since $(f_*D.K_{S'}) = (D.K_S) - (D.E_0)$, the inequality (50) will follow once we have shown that $(D.E_0) \geq 0$. The latter is clear if E_0 is not a component of D, so assume the contrary, i.e. $D = D_1 + E_0$. Note that $E_0 \not\leq D_1$ since D is reduced. Moreover, $D_1 \neq 0$ since $f_*D \neq 0$. Thus, since D is connected we have $(D_1.E_0) \geq 1$, and so $(D.E_0) = (D_1.E_0) + E_0^2 \geq 0$, as desired. Now suppose $n > 1$, and write $f' = f_n \circ \ldots \circ f_2$. Then by induction $(f_*D)^2 = (f'_*f_{0*}D)^2 \leq (f_{0*}D)^2 \leq D^2$, and so the first assertion follows readily. To prove the second, we note that if D is reduced and connected and if $f_*D \neq 0$, then $\bar{D} := f_{0*}D$ is also reduced and connected, and $f'_*\bar{D} \neq 0$, so by induction we have $(f_*D.K_{S'}) = (f'_*\bar{D}.K_{S'}) \leq (\bar{D}.K_{S_1}) \leq (D.K_S)$, which proves the inequality (50).

Finally, to prove the last statement, assume that some blow-down curve C meets D. If C is not a component of D, then the assertion is clear by the above, so we may assume that every blow-down curve C which meets D is already a component. This, however, is impossible, for by hypothesis C cannot be a (-1)-curve, so there exists a blow-down map $f'' : S \to S''$ (over which f factors) such that on S'' there is a (-1) curve C'' which meets the (-2)-curve $f(C)$. By our assumption, C'' must be a component of $f(D)$. Replacing C by the proper transform of C'', we see (by induction) that eventually there has to be a (-1)-curve contained in D, contrary to our hypothesis.

Proof of Proposition 4.13. The second inequality follows directly from Lemma 4.15, as does the fact that we have equality here if and only if D does not meet any blow-down curve.

Since $K_{\bar{S}}^2 > 0$, the Hodge Index Theorem (cf. [1], IV.2.15), applied to $\bar{D} := K_{\bar{S}}^2 \cdot f_*D - (K_{\bar{S}}.f_*D)^2 \cdot K_{\bar{S}}$, yields $\bar{D}^2 = (K_{\bar{S}}^2)^2(f_*D)^2 - (K_{\bar{S}}.f_*D)^2 K_{\bar{S}} \leq 0$, which proves the first inequality of (47). Furthermore, equality holds here if and only if $\bar{D} \equiv 0$. Thus, if equality holds in both places, then D cannot meet (nor contain) any blow-down curve, and then $f^*f_*D = D$, and hence the second assertion follows since the converse is trivial.

In order to be able to derive a useful criterion from the above result, we need to find reduced connected divisors D on S for which $(K_S.D) < D^2$. Such divisors are often obtained by considering (-2)-joins of two curves which are defined as follows.

Definition 4.16 Let S be a smooth surface. As usual, a (-2)-*chain* on S is a reduced connected divisor $E = E_1 + \ldots + E_s$ consisting of (-2)-curves E_i such that $(E_i.E_{i+1}) = 1$ for $1 \le i \le s-1$ and $(E_i.E_j) = 0$ if $|i-j| > 1$.

If C_1 and C_2 are two distinct irreducible curves on S, then a (-2)-*join of* C_1 *to* C_2 *of breadth* $t \ge 0$ is a reduced, connected divisor

$$D = C_1 + C_2 + E_1 + \ldots + E_t,$$

where the $E_i = E_{i1} + \ldots + E_{is_i}$ are (-2)-chains joining C_1 to C_2; i.e. $(C_1.E_i) = (C_1.E_{i1}) > 0$ and $(C_2.E_i) = (C_2.E_{is_i}) > 0$. If all the chains E_i meet each C_j transversely and of all the E_i are mutually disjoint, then we say that the (-2)-join is *simple*.

The above proposition implies that on a surface of general type, the breadth t of a (-2)-join has a bound in terms of invariants of C_1 and C_2 alone, and so the existence of a (-2)-join of sufficiently large breadth on S shows that the surface cannot be of general type. More precisely, we have

Proposition 4.17 *Let S be a surface of general type with minimal model $f : S \to \bar{S}$, and let D be a (-2)-join of breadth t of two curves C_1 and C_2 on S. If $k(C_1, C_2) := (K_S.(C_1 + C_2))$ and $i(C_1, C_2) := (C_1 + C_2)^2$, then we have*

$$K_{\bar{S}}^2 \cdot (2t + i(C_1, C_2)) \;\le\; k(C_1, C_2)^2, \tag{51}$$

and equality holds if and only if $K_{\bar{S}}^2 D \equiv k(D) f^* K_{\bar{S}}$ and if D is simple and does not contain any blow-down curve of f.

Proof. We first note that by the definition of D we have $(K_S.D) = (K_S.(C_1 + C_2)) = k(C_1, C_2)$ and $D^2 \ge 2t + (C_1 + C_2)^2$, and that equality holds if and only if D is simple. Thus, since the inequality is trivial if $2t + i(C_1, C_2) \le 0$, the assertion follows immediately from Proposition 4.13.

As an application of the above, we analyze the following situation which occurs in the modular curve case $X = X(N)$; cf. [17].

Corollary 4.18 *Suppose S is a surface of general type and $D = C_1 + C_2 + E_1 + E_2$ is a (-2)-join of breadth 2 of two (-3)-curves C_1 and C_2 on S with $(C_1.C_2) \ge 2$. Then:*

a) D neither meets nor contains any blow-down curve, and the same is true for any (-2)-chain which meets D.

b) Let $E' = C_0 + \tilde{E}'$ be a $(-3, -2, \ldots, -2)$-chain, i.e. C_0 is a (-3)-curve and $\tilde{E}' = \tilde{E}_1 + \ldots + \tilde{E}_t$ is a (-2)-chain such that $(C_0.\tilde{E}') = (C_0.\tilde{E}_1) = 1$. Suppose that E' joins C_1 and C_2. If E' meets a blow-down curve, then every (-2)-chain meeting $D' = D + E'$ is already contained in D'.

Proof. a) If, as before, $f : S \to \bar{S}$ denotes the minimal model, then by (49) and the hypotheses we have $(f_*D)^2 \ge D^2 \ge 2$. Now if some blow-down curve meets (or is contained in) D, then by (50) we have $(K_{\bar{S}}.f_*D) < (K_S.D) = 2$, so $(K_{\bar{S}}.f_*D) \leq 1$. But then $2 \leq K_{\bar{S}}^2(f_*D)^2 \leq (K_{\bar{S}}.f_*D)^2 \leq 1^2$, contradiction.

b) Suppose the blow-down curve C meets E'; without loss of generality we may assume $C^2 = -1$. Then by part a), C can only meet C_0. Moreover, $(C.C_0) = 1$, for otherwise $f(C_0)$ is a singular curve on $\bar{S}$ with $(K_{\bar{S}}.f(C_0)) \leq -1$, which is impossible. But then $f_*(C_0)$ is a (-2)-curve on $\bar{S}$, and so f_*D' is a (-2)-join of $f(C_1)$ and $f(C_2)$ of breadth 3, and hence $(f_*D')^2 \geq 4 = (K_{\bar{S}}.f_*D')^2$. From Proposition 4.17 we thus see that $K_{\bar{S}}^2 = 1$ and that $f_*D' \equiv 2K_{\bar{S}}$.

Now suppose that E'' is a (-2)-chain which meets D' and which is not contained in D'. Then for some component E_i'' of E'' we have $(D'.E_i'') > 0$. Then $\bar{E} := f(E_i'')$ is by part a) a (-2)-curve meeting f_*D', which gives the contradiction $0 < (f_*D'.\bar{E}) = (2K_{\bar{S}}.\bar{E}) = 0$.

Applying the above proposition to the chains of exceptional curves lying on a diagonal quotient surface $\tilde{Z}_\alpha$ yields the following result.

Proposition 4.19 *Let $\bar{x}, \bar{y} \in R$, and let $D = D(\bar{x}, \bar{y})$ be the (-2)-join of $\tilde{C}_{\bar{x},1}$ and $\tilde{C}_{\bar{y},2}$ formed by the (-2)-chains contained in the fibre $\tilde{\psi}^{-1}(\bar{x}, \bar{y})$. Then D is simple and has breadth*

$$t_{\bar{x},\bar{y}} := \sum_{d|e_{\bar{y}}, d \neq e_{\bar{y}}} s_{e_{\bar{y}}-d,\alpha}(\bar{x}, \bar{y}). \tag{52}$$

Moreover, $i(\tilde{C}_{\bar{x},1}, \tilde{C}_{\bar{y},2}) = \tilde{C}_{\bar{x},1}^2 + \tilde{C}_{\bar{y},2}^2 + 2r_{1,\alpha}(\bar{x}, \bar{y})$ and $k(C_{\bar{x},1}, C_{\bar{y},2}) = 2g_{\bar{x},\alpha^{-1}} + 2g_{\bar{y},\alpha} - \tilde{C}_{\bar{x},1}^2 - \tilde{C}_{\bar{y},2}^2 - 4$, where $\tilde{C}_{\bar{x},1}^2$ and $\tilde{C}_{\bar{y},2}^2$ are given by 3.15. Thus, if

$$2t_{\bar{x},\bar{y}} > k(\tilde{C}_{\bar{x},1}, \tilde{C}_{\bar{y},2})^2 - i(\tilde{C}_{\bar{x},1}, \tilde{C}_{\bar{y},2}), \tag{53}$$

then $\kappa(\tilde{Z}_\alpha) \leq 1$. If, in addition, $p_g = 1$ and if $\tilde{C}_{\bar{x},1}$ and $\tilde{C}_{\bar{y},2}$ are (-2)-curves, then $\tilde{Z}_\alpha$ is a (blown-up) K3-surface.

Proof. Since all chains contained in the fibre are disjoint, D is simple by Lemma 3.12. Now a singularity $s \in S_\alpha$ of type (n, q) gives rise to a (-2)-chain if and only if $\frac{n}{q} = [[2, 2, \ldots, 2]] \Leftrightarrow q = n - 1 \Leftrightarrow \frac{\nu}{d} = \frac{e_{\bar{y}}}{d} - 1 \Leftrightarrow s \in S_{e_{\bar{y}}-d,\alpha}(\bar{x}, \bar{y})$, which proves (52). The formula for $i(\tilde{C}_{\bar{x},1}, \tilde{C}_{\bar{y},2})$ follows from Corollary 3.15, and that for $k(\tilde{C}_{\bar{x},1}, \tilde{C}_{\bar{y},2})$ follows from the adjunction formula, using the fact that $\tilde{C}_{\bar{x},1}$ and $\tilde{C}_{\bar{y},2}$ are smooth curves of genera $g_{\bar{x},\alpha^{-1}}$ and $g_{\bar{y},\alpha}$, respectively (cf. Proposition 2.1). Thus, since $K_{\bar{S}}^2 \geq 1$, the hypothesis (53) implies that the inequality (51) cannot hold, and hence $\tilde{Z}_\alpha$ cannot be of general type by Proposition 4.17.

Now suppose that $p_g = 1$ and that $\tilde{C}_{\bar{x},1}$ and $\tilde{C}_{\bar{y},2}$ are (-2)-curves. Then $\kappa(\tilde{Z}_\alpha) \geq 0$, and $\tilde{Z}_\alpha$ is regular (because $q = \bar{g} \leq g_{\bar{x},\alpha^{-1}} = 0$). Since we already know that $\kappa(\tilde{Z}_\alpha) \leq 1$, it follows from the classification theory that the minimal model $\bar{Z}$ of $\tilde{Z}_\alpha$ is either a K3-surface or an elliptic surface. Since we are done in the former case, we assume the latter, so we have a fibration $f : \bar{Z} \to \mathbb{P}^1$ (cf. proof of Theorem 4.12). Then by Kodaira's formula we have $K_{\bar{Z}} \sim \sum_{i=1}^s (m_i - 1)F_i$, where $m_1F_1, \ldots, m_sF_s$ are the exceptional fibres of f, because $p_g + q - 1 = 0$ (cf. proof of 4.11b)). Since no blow-down curve of $\tilde{Z}_\alpha \to \bar{Z}$ meets D (otherwise $\tilde{Z}_\alpha$ would be rational), we may assume that D lies on $\bar{Z}$. Now since $D^2 > 0$, at least one component, say C, of D is not contained in any fibre of f because the intersection pairing is negative definite on the space generated by the components of the fibres.

Thus $(C.F_i) > 0$, for $1 \le i \le s$, and hence $0 = (K_{\bar{Z}}.C) = \sum_{i=1}^{s}(m_i - 1)(F_i.C)$. But this means that all $m_i = 1$, so $K_{\bar{Z}} \sim 0$, and hence $\bar{Z}$ is a K3-surface in this case as well.

Proof of Theorem 6 of the introduction: We first recall that $\tilde{C}_{\bar{x},1}$ and $\tilde{C}_{\bar{x},2}$ are smooth curves on $\tilde{Z}_\alpha$ of genera $g_{\bar{x},\alpha^{-1}}$ and $g_{\bar{x},\alpha}$, respectively, with intersection numbers

$$\tilde{C}_{\bar{x},1}^2 = -c_{x,\alpha}, \quad \tilde{C}_{\bar{x},2}^2 = -c_{x,\alpha^{-1}} \quad \text{and} \quad (\tilde{C}_{\bar{x},1}.\tilde{C}_{\bar{x},2}) = r_{1,x,\alpha} = r_{1,\alpha}(\bar{x},\bar{x});$$

cf. Proposition 2.1 and Corollary 3.15 (together with the symmetry formula (14)).

a) Here, the hypotheses mean that $\tilde{C}_{\bar{x},1}$ and $\tilde{C}_{\bar{x},2}$ are two intersecting (rational) (-1)-curves, and so the surface is rational (cf. van der Geer [6], VII.2.2).

b) In view of the above intersection numbers, this is just a restatement of Proposition 4.19.

c) This follows from Theorem 4.12 by taking $C = \tilde{C}_{\bar{x},2}$ and $E_1 = \tilde{C}_{\bar{x},1}, E_2 = \tilde{C}_{\bar{x},2}$ in 4.12a) and 4.12b), respectively.

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