Elliptic Curves on Abelian Surfaces

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The purpose of this paper is to present two theorems which give an overview of the set of elliptic curves lying on an abelian surface and to discuss several applications. One of these applications is a classical theorem of Biermann (1883) and Humbert (1893) on the characterization of abelian surfaces containing elliptic curves in terms of the "singular relations" of Humbert. As a by–product one obtains a purely algebraic description of such relations and hence also of *Humbert surfaces*.

1. Introduction

The principal aim of this note is to classify the set of elliptic curves lying on an abelian surface A defined over an algebraically closed field K. Since any elliptic curve on A may be translated to the origin, it is sufficient to classify the *elliptic subgroups* of A. The main result here is that these can be characterized numerically, that is, inside the Néron–Severi group $\mathrm{NS}(A) = \mathrm{Div}(A)/\equiv$ of numerical equivalence classes.

Theorem 1.1 *The map* $E \mapsto cl(E) \in \mathrm{NS}(A)$ *induces a bijective correspondence between the set of elliptic subgroups* $E \leq A$ *of* A *and the primitive classes* $cl(D) \in \mathrm{NS}(A)$ *with* $(D.D) = 0$ *and* $(D.\Theta) > 0$ *for some (hence any) ample divisor* $\Theta \in \mathrm{Div}(A)$*.*

Here and below, we call an element $x \in M$ of a finitely generated free $\mathbb{Z}$-module M *primitive* if the quotient module $M/\mathbb{Z}x$ is torsion-free.

As a first application of the above theorem, we note the following result which was first proved (for $K = \mathbb{C}$) by Bolza [Bo] and by Poincaré [Po] in 1886 (cf. Krazer[Kr], chapter XII, Satz IX, or Lange[La2]):

Corollary 1.2 *If an abelian surface has more than two elliptic subgroups, then it has infinitely many.*

On the other hand, the fact that $\mathrm{NS}(A)$ is a finitely generated group shows that there cannot be "too many" elliptic subgroups on an abelian surface. To make this precise, fix an ample divisor $\Theta \in \mathrm{Div}(A)$ so that we can partially order the curves C on A according to their *degrees*:

$$\deg(C) = \deg_\Theta(C) = (C.\Theta).$$

We then have the following finiteness result:

Corollary 1.3 *An abelian surface A has only finitely many elliptic subgroups $E \leq A$ of bounded degree.*

Of course, there is no reason why a given abelian surface should contain any elliptic subgroups at all, and indeed, those that do are rather special. These were analyzed by Humbert as a special case of his notion of abelian surfaces "satisfying a singular relation".

Although Humbert defined this notion analytically in terms of the period matrix of $A = \mathbb{C}^2/\Gamma$ (and hence only in the case $K = \mathbb{C}$), it is possible to give an *algebraic definition* of this concept which therefore extends to an arbitrary ground field. Since this algebraic definition is perhaps of independent interest, we develop it here in some detail.

To define this notion, consider the quadratic form $\Delta = \Delta_\Theta$ on the Néron–Severi group $\mathrm{NS}(A)$ defined by the formula

$$\Delta(D) = (D.\Theta)^2 - 2\delta(D.D),$$

where $\delta = \frac{1}{2}(\Theta.\Theta)$ denotes the "polarization degree" of the polarized surface (A, Θ). From the Hodge Index Theorem it follows that if Θ is a primitive element of $\mathrm{NS}(A)$, then Δ defines a positive definite quadratic form on the "polarized Néron–Severi group"

$$\mathrm{NS}(A, \Theta) := \mathrm{NS}(A)/\mathbb{Z}\Theta.$$

As will be explained below in section 5, $\Delta(D)$ is essentially the invariant defined by Humbert [Hu1] in his study of Humbert surfaces and hence will be called the *Humbert invariant* of the divisor $D \in \mathrm{Div}(A)$. As a result, the notion of a "singular relation" can be defined purely algebraically in the following way.

Definition 1.4 A polarized abelian surface (A, Θ) is said to *satisfy a singular relation with invariant N* if there exists a primitive class $[D] \in \mathrm{NS}(A, \Theta)$ with Humbert invariant $\Delta(D) = N$.

By using the above Humbert invariant we obtain a second classification of the set of elliptic subgroups of A, at least if (A, Θ) is principally polarized (i.e. $\delta = 1$):

Theorem 1.5 *Let (A, Θ) be a principally polarized abelian surface, and let $n \in \mathbb{N}$ be an integer. Then the map $E \mapsto [E] = \text{class of } E$ in $\mathrm{NS}(A, \Theta)$ induces a one–to–one correspondence between*

1) the set of all elliptic subgroups $E \leq A$ of A of degree $n = \deg(E)$, and

2) the set of primitive classes $[D] \in \mathrm{NS}(A, \Theta)$ with invariant $\Delta(D) = n^2$.

As a first application of this result, one obtains the following algebraic analogue of a classical theorem due to Biermann [Bi] and Humbert [Hu1] which (for $K = \mathbb{C}$) was recently reproved by Murabayashi [Mur]:

Corollary 1.6 (Biermann–Humbert) *A principally polarized abelian surface (A, Θ) has an elliptic subgroup $E \leq A$ of degree n if and only if (A, Θ) satisfies a singular relation with invariant n^2.*

Strictly speaking, the above formula relating the degree of the elliptic subgroup to the Humbert invariant is not stated explicitly in the papers of Biermann and Humbert, but a close look at the proof as presented in the book of Krazer [Kr], chapter XII, Satz V, shows that their arguments yield this formula as well.

The above result can also be interpreted in terms of the *Humbert surfaces* H_N introduced by Humbert [Hu1] (cf. also van der Geer[vdG]):

$$H_N = \{(A, \Theta) \in \mathcal{A}_2 : (A, \Theta) \text{ satisfies a singular relation with invariant } N\}.$$

Here, as usual, $\mathcal{A}_2$ denotes the moduli space of principally polarized abelian varieties.

Corollary 1.7 *A principally polarized abelian surface (A, Θ) has an elliptic subgroup $E \leq A$ of degree n if and only if $(A, \Theta) \in H_{n^2}$.*

Humbert proves in his papers that for $K = \mathbb{C}$ the set H_N is a closed subset of the moduli space $\mathcal{A}_2$ and hence is an algebraic surface. In view of the above corollary, this is also true for any field K when $N = n^2$ and $\mathrm{char}(K) \nmid n$ by the results of [Ka1], [Ka2], where the moduli space of principally polarized abelian varieties containing an elliptic curve is studied. (If n is a prime number, then this also follows from the work of Lange [La1]). Moreover, by combining the results of these papers with Corollary 1.7 we obtain the following interesting structure theorem of the Humbert surfaces with square invariant:

Corollary 1.8 *If $\mathrm{char}(K) \nmid n$, then H_{n^2} is a closed irreducible subvariety of $\mathcal{A}_2$ whose normalization is*

$$\tilde{H}_{n^2} = (X(n) \times X(n))/\langle \tau, G(n) \rangle,$$

where $X(n)$ denotes the affine modular curve parametrizing elliptic curves with a level n-structure, τ is the involution of $X(n) \times X(n)$ which interchanges the two factors, and $G(n) = Sl_2(\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$ which acts in the usual way on the first factor and in a twisted fashion on the second factor.

By specializing to the case that $A = J_X$ is the Jacobian of a curve X of genus 2, the above theorems can be translated to results about the (minimal) *elliptic subcovers* $f : X \to E$ of X, where the minimality means that f does not factor over any non-trivial isogeny of the elliptic curve E. Indeed, as is explained in section 4, it follows from duality theory that the elliptic subgroups $E \leq J_X$ correspond bijectively to isomorphism classes of minimal subcovers of X, and so we obtain from the above theorems:

Theorem 1.9 *Let X be a curve of genus 2 with Jacobian J_X and canonical principal polarization Θ. Then there is a bijective correspondence between the following sets:*

1) the set of isomorphism classes of (minimal) elliptic subcovers $f : X \to E$ of degree $\deg(f) = n$;

2) the set of elliptic subgroups $E \leq J_X$ of J_X of degree $\deg_\Theta(E) = n$;

3) the set of primitive classes $cl(D) \in \mathrm{NS}(J_X)$ of degree $\deg_\Theta(D) = n$ and with self–intersection $(D.D) = 0$;

4) the set of primitive classes $[D] \in \mathrm{NS}(J_X, \Theta)$ with invariant $\Delta(D) = n^2$.

In a similar manner the above corollaries can be translated to assertions about elliptic subcovers. For example, Corollary 1.3 becomes the genus two case of a theorem of Tamme [Ta] which is valid for arbitrary genus:

Corollary 1.10 (Tamme) *There are only finitely many isomorphism classes of elliptic subcovers of X of bounded degree.*

Finally, in section 5 we compare the algebraic theory developed so far with the analytic theory (for $K = \mathbb{C}$). In particular, we justify that the algebraic definitions of Humbert's invariant and of Humbert surfaces as presented above agree with the classical definitions; cf. Remark 5.6. Moreover, by introducing the notion of an "elliptic submodule" of the singular cohomology group $H^1(A, \mathbb{Z})$, we give an analytic reformulation of Theorem 1.1 together with a quick analytic proof.

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2. A numerical characterization of elliptic subgroups

Let A be an abelian surface defined over an algebraically closed field K of arbitrary characteristic. We first review some basic facts of such surfaces.

Since the canonical class ω_A of A is trivial, the Riemann–Roch theorem for a divisor $D \in \mathrm{Div}(A)$ assumes the form

$$\chi(D) := h^0(D) - h^1(D) + h^0(-D) = \frac{1}{2}(D.D), \tag{2.1}$$

where, as usual, $h^q(D) = \dim_K H^q(A, \mathcal{L}(D))$. Here we have also used Serre duality: $h^2(D) = h^0(-D)$. Moreover, if $D \geq 0$ is effective, then the adjunction formula becomes

$$p_a(D) - 1 = \frac{1}{2}(D.D). \tag{2.2}$$

We shall also frequently make use of the following fact. For any $x \in A$, the translated divisor $T_x(D)$ is numerically equivalent to D,

$$T_x(D) \equiv D, \tag{2.3}$$

because $T_x(D)$ is visibly algebraically equivalent to D.

We now want to classify the elliptic curves lying on A. The first step in this direction is accomplished by the following auxiliary result:

Proposition 2.1 *Let C_1 and C_2 be two irreducible curves lying on A. Then $(C_1.C_2) \geq 0$, and $(C_1.C_2) = 0$ if and only if $C_1 = T_x(C_2) = E$, for some $x \in A$ and some elliptic curve E on A.*

Proof. This depends on the following well-known fact:

(2.4) *There is no rational curve C lying on A.*

Indeed, if $i : C \to A$ denotes the inclusion map and $n : \tilde{C} \to C$ the normalization of C, then $i \circ n : \tilde{C} \to A$ factors over the Jacobian $J_{\tilde{C}}$ of $\tilde{C}$, and so we must have $g(\tilde{C}) = \dim J_{\tilde{C}} > 0$, and hence C is not rational.

If $C_1 \neq C_2$ are distinct curves, then clearly $(C_1.C_2) \geq 0$, so assume $C_1 = C_2 = C$. Since C is not rational, we have by the adjunction formula that $(C.C) = 2p_a(C) - 2 \geq 0$, which proves the first assertion.

To prove the second, assume that $(C_1.C_2) = 0$ and choose $x \in A$ such that $T_x(C_2) \cap C_1 \neq \phi$. Since $(T_x(C_2).C_1) = (C_2.C_1) = 0$, this forces $T_x(C_2) = C_1$ because both are irreducible curves. Thus, by the adjunction formula we have $2p_a(C_1) - 2 = (C_1.C_1) = (C_1.T_x(C_2)) = (C_1.C_2) = 0$, so $p_a(C_1) = 1$. If C_1 were not smooth, then C_1 is rational, which contradicts (2.4), so $C_1 = E$ is an elliptic curve.

Corollary 2.2 *a) If $D_1, D_2 \geq 0$ are effective divisors, then $(D_1.D_2) \geq 0$, and*

$$(D_1.D_2) = 0 \iff D_i = \sum n_{ij} T_{x_{ij}}(E),\ i = 1,2, \text{ for some elliptic curve } E. \tag{2.5}$$

b) If $D \in Div(A)$ is any divisor on A, then $(D.D) > 0$ if and only if either D or $-D$ is ample.

Proof. a) Write $D_i = \sum n_{ij} C_{ij}$, where $n_{ij} > 0$ and each C_{ij} is an irreducible curve. Then $(D_1.D_2) = \sum_{i,j} n_{1i} n_{2j} (C_{1i}.C_{2j}) \geq 0$ by Proposition 2.1. Moreover, $(D_1.D_2) = 0$ if and only if $(C_{1i}.C_{2j}) = 0$, for all i, j. By induction on i, j it follows from Proposition 2.1 again that $C_{ij} = T_{x_{ij}}(C_{11})$ and $C_{11} = E$, which proves the assertion.

b) The converse being trivial, assume that $(D.D) > 0$. From the Riemann–Roch formula we have $h^0(D)+h^0(-D) \geq \frac{1}{2}(D.D) > 0$, so the linear equivalence class of either D or $-D$ contains an effective divisor. Without loss of generality we may thus assume that $D \geq 0$.

To prove that D is ample, by the Nakai–Moizeshon criterion it is enough to show that $(D.C) > 0$ for each irreducible curve C on A. Now by part a) this is automatic unless $C = E$ is an elliptic curve and $D = \sum_x n_x T_x(E)$. But then $(D.D) = 0$, contradiction. Thus $(D.C) > 0$ and hence D is ample.

The above proposition gives a characterization of elliptic curves E as *divisors* on A. The next and crucial step in their classification is to characterize their *numerical equivalence classes* $cl(E) \in \mathrm{NS}(A)$ within the Néron–Severi group $\mathrm{NS}(A) = \mathrm{Div}(A)/\equiv$ itself. While the above corollary gives a partial result in this direction in that it shows that the "elliptic classes" $cl(mE) \in \mathrm{NS}(A)$, where E is an elliptic curve and $m \in \mathbb{N}$ a positive integer, are precisely the isotropic classes of the *effective cone* $\mathrm{NS}^+(A) = \{cl(D) : D \geq 0\}$, considerable work is still necessary to eliminate the effectivity condition "$D \in \mathrm{NS}^+(A)$" in this characterization.

As we shall see below, the elliptic classes can be characterized numerically up to a sign. To be able to determine this sign, we shall fix from now on an ample divisor $\Theta \in \mathrm{Div}(A)$ so that the the *degree* of a divisor $D \in \mathrm{Div}(A)$ with respect to Θ is defined:

$$\deg(D) = \deg_\Theta(D) = (D.\Theta). \tag{2.6}$$

Clearly, all elliptic classes have positive degree (with respect to any ample divisor Θ), and hence their sign is determined in this way. are We then have the following result which should be viewed as the main cornerstone in the classification of the elliptic curves on A.

Proposition 2.3 *Let $D \in \mathrm{Div}(A)$ be a divisor on A. Then $D \equiv mE$ for some elliptic curve E and some $m \in \mathbb{Z}$ if and only if $(D.D) = 0$. Moreover, $m > 0$ if and only if $\deg_\Theta(D) > 0$.*

Proof. The second assertion and one direction of the first assertion are clear, for if $D \equiv mE$, then clearly $(D.D) = m^2(E.E) = 0$ and $\deg(D) = m(E.\Theta)$. It is therfore enough to prove the converse of the first assertion.

Thus, assume $(D.D) = 0$. If $\deg_\Theta(D) = 0$, then by the Hodge Index Theorem we have $D \equiv 0$, so we are done by taking $m = 0$. Thus, assume $\deg(D) \neq 0$; by replacing D by $-D$ if necessary we may assume that $\deg(D) > 0$.

Next, if $D \equiv D'$, for some D' with $h^0(D') > 0$, then by Corollary 2.2 we have that $D \equiv mE$, so we may henceforth assume that

$$h^0(D') = 0, \ \forall D' \equiv D; \tag{2.7}$$

we will eventually see that this leads to a contradiction.

Step 1: There exists an elliptic curve E such that $(D.E) = 0$.

Consider the closed subgroup $K(D) = \{x \in A : T_x^*(D) \sim D\}$; cf. Mumford [Mu], p. 60. If $K(D)$ is finite, then $(D.D) \neq 0$ by the Riemann–Roch Theorem of Mumford[Mu], p. 150, and if $K(D) = A$ then $D \in \mathrm{Pic}^0(A)$, so $D \equiv 0$ (cf. Milne[Mi], Remark 9.3, p. 118), and hence $\deg(D) = 0$, contradiction. We thus have that $\dim K(D) = 1$, and so there is an elliptic subgroup $E \leq K(D)$.

Put $\mathcal{L}' = \mathcal{L}(D)_{|E}$. Then $T_x^*\mathcal{L}' = (T_x^*\mathcal{L}(D))_{|E} \simeq (\mathcal{L}(D))_{|E} = \mathcal{L}'$, for all $x \in E$, and so $\mathcal{L}' \in \mathrm{Pic}^0(A)$. Thus $(D.E) = \deg(\mathcal{L}') = 0$, which completes the first step.

Step 2: There exists $\tilde{D} \equiv D$ such that $\mathcal{L}(\tilde{D})_{|E} \simeq \mathcal{O}_E$.

Since $\Theta' = \Theta_{|E}$ is ample on E, the map $\phi_{\Theta'} : E \to \mathrm{Pic}^0(E)$ is surjective, and so there exists $x \in E$ such that $T_x^*(\mathcal{L}(\Theta')) \otimes \mathcal{L}(\Theta')^{-1} \simeq \mathcal{L}(D)_{|E}$. Thus, if we put $\tilde{D} = D + \Theta - T_x^*\Theta$, then $\tilde{D} \equiv D$ and $\mathcal{L}(\tilde{D})_{|E} \simeq \mathcal{O}_E$.

Step 3: There is a surjective homomorphism $f : A \to E'$ to an elliptic curve E' such that $\mathrm{Ker}(f) = E$.

This follows either from the existence of quotients of smooth group schemes over a field (cf. SGA3, Exp. VI_A.5.4) or, more elementarily, from Poincaré's complete reducibility theorem together with the existence of quotients for finite group schemes (cf. Mumford [Mu], §12).

Step 4: $\tilde{D} \sim f^*(D')$, for some $D' \in \mathrm{Div}(E')$.

By Hartshorne[Ha], Ex. III.12.4, it is enough to show that $\tilde{D}_t \sim 0$, for all $t \in E'$. Moreover, since $\deg(\tilde{D}_t) = (D.E) = 0$, it is enough to show that $h^0(\tilde{D}_t) \geq 1$, for all $t \in E'$.

Suppose $h^0(\tilde{D}_{t_0}) = 0$ for some $t_0 \in E'$. Then by semi-continuity we have $h^0(\tilde{D}_t) = h^1(\tilde{D}_t) = 0$ for almost all $t \in E'$, and so $R^1 f_*\mathcal{L}(\tilde{D})$ is a skyscraper sheaf on E'. Using the Leray spectral sequence $E_2^{pq} = H^p(E', R^q f_*\mathcal{L}(D)) \Rightarrow H^{p+q}(A, \mathcal{L}(D))$, we obtain the exact sequence

$$0 \to H^1(E', f_*\mathcal{L}(\tilde{D})) \to H^1(A, \mathcal{L}(\tilde{D})) \to H^0(E', R^1 f_*\mathcal{L}(\tilde{D})) \to H^2(E', f_*\mathcal{L}(\tilde{D}))$$

of low-degree terms. But since $H^2(E', f_*\mathcal{L}(D)) = 0$ because $\dim E' = 1$ and since $h^1(\tilde{D}) = 0$ by hypothesis (2.7), we must have $R^1 f_*\mathcal{L}(\tilde{D}) = 0$. On the other hand, since $H^2(A_t, D_t) = 0, \forall t \in E'$, we have by base-change (cf. Mumford[Mu], p. 63) that

$$R^1 f_* \mathcal{L}(\tilde{D}) \otimes \kappa(t) \xrightarrow{\sim} H^1(A_t, D_t)$$

is an isomorphism, which means that $h^1(D_t) = 0$, for all $t \in E'$. But this contradicts the choice of $\tilde{D}$, which was constructed in step 2 so that $h^0(D_t) = h^1(D_t) = 1$ for $t = 0$. Thus $h^0(\tilde{D}_t) \geq 1, \forall t \in E'$, which concludes step 4.

To conclude the proof of the proposition, we first note that by steps 2 and 4 we have $D \equiv \tilde{D} \sim f^*(D') \equiv mE$, with $m = \deg(D')$. Moreover, since $m(E.\Theta) = (D.\Theta) > 0$, we have that $m > 0$, whence the proposition.

Corollary 2.4 *A divisor class $cl(D) \in \mathrm{NS}(A)$ lies in the effective cone (i.e. $cl(D) \in \mathrm{NS}^+(A)$) if and only if $(D.D) \geq 0$ and $\deg_\Theta(D) > 0$.*

Proof. If $D \equiv D' \geq 0$, then $(D.D) \geq 0$ by Corollary 2.2a) and $\deg(D) > 0$ since $D' \neq 0$. Conversely, assume that $(D.D) \geq 0$ and that $(D.\Theta) > 0$. If $(D.D) > 0$, then by (the proof of) Corollary 2.2b) we have that D is ample and effective, and if $(D.D) = 0$, then $D \equiv mE$ is an effective class by Proposition 2.3.

As another application of Proposition 2.3, we can now see that classes belonging to elliptic curves are primitive in the sense of the following definition.

Definition 2.5 An element $0 \neq x \in \Gamma$ of a torsionfree abelian group Γ is called *primitive* if the quotient group $\Gamma/\mathbb{Z}x$ is torsionfree (cf. eg. Barth/Peters/van der Ven [BPV], p. 13) or, equivalently, if x satisfies the following condition:

(2.8) $$x = my, y \in \Gamma, m \in \mathbb{N} \Rightarrow m = 1.$$

Corollary 2.6 *The class $cl(E) \in \mathrm{NS}(A)$ of an elliptic curve E on A is primitive.*

Proof. Suppose $E \equiv mD$ for some $D \in \mathrm{Div}(A)$ and $m > 0$. Then $(D.D) = 0$ and $\deg(D) > 0$, so by Proposition 2.3 $D \equiv m'E'$ for some $m' > 0$ and some elliptic curve E'. But since $(E.D) = 0$, we must have $E \equiv E'$ by Proposition 2.1, and so we obtain $E \equiv mD \equiv mm'E$. Since $E \not\equiv 0$, it follows that $m = m' = 1$, so E is primitive.

Remark 2.7 The last corollary can also be proved without recourse to Proposition 2.3 by using the following criterion of primitivity:

$$cl(D) \in \mathrm{NS}(A) \text{ is primitive } \iff A[N] \not\subset K(D), \forall N > 1,$$

where $A[N] = \mathrm{Ker}(N \cdot id_A)$ denotes the subgroup scheme of N-torsion points of A; this follows from Mumford[Mu], Th. 23.3, p. 231, together with the fact that $\mathrm{Pic}^0(A)$ is divisible. Applying this to $D = E$, the primitivity follows easily once one has that

$$K(E) = E,$$

which can be proved by using the map $f : A \to A/E$ and functorial properties of $K(D)$.

We are now ready to prove the numerical characterization of elliptic subgroups announced in Theorem 1.1 of the introduction:

Theorem 2.8 *The map* $E \mapsto cl(E) \in \mathrm{NS}(A)$ *induces a one–to–one correspondence between the set of elliptic subgroups* $E \leq A$ *of* A *and the primitive classes* $cl(D) \in \mathrm{NS}(A)$ *with* $(D.D) = 0$ *and* $\deg(D) > 0$.

Proof. First note that it follows from the above Corollary 2.6 that $cl(E)$ has the asserted properties. Next we observe that the map $E \mapsto cl(E)$ is injective. Indeed, if $E \neq E'$ are two distinct elliptic subgroups, then $E \neq T_x(E'), \forall x \in A$, so $(E.E') > 0$ by Proposition 2.2 and hence $cl(E) \neq cl(E')$.

Finally, the map $E \mapsto cl(E)$ is surjective, for if $cl(D) \in \mathrm{NS}(A)$ is a primitive class with $(D.D) = 0$ and $\deg(D) > 0$, then by Proposition 2.3 we have $D = mE$ for some elliptic curve E and $m > 0$. But since $cl(D)$ is primitive, we must have $m = 1$, and so the assertion is proved.

Corollary 2.9 (Bolza–Poincaré) *If* A *has more than two elliptic subgroups, then it has infinitely many.*

Proof. Let E_1, E_2 and E_3 be three distinct elliptic subgroups of A. We first observe that their classes in $\mathrm{NS}(A)$ are linearly independent. Indeed, if $aE_1 + bE_2 + cE_3 \equiv 0$, and (say) $a \neq 0$, then $0 = a^2(E_1)^2 = (abE_2 + bcE_3)^2 = 2bc(E_2.E_3)$, so either $b = 0$ or $c = 0$. But if (say) $b = 0$, then $aE_1 \equiv -cE_3$, which is impossible since $(E_2.E_3) > 0$ but $(E_2.E_2) = 0$. Thus, $cl(E_1), cl(E_2)$ and $cl(E_3)$ are linearly independent.

Now write $a = (E_1.E_2) > 0, b = (E_1.E_3) > 0, c = (E_2.E_3) > 0$ and $D = cE_1 + bE_2 - aE_3$. Then $(D.E_1) = (D.E_2) = 0$ and $(D.D) = a(b+c) > 0$. Thus, for every $x \in \mathbb{Q}$, the rational divisor

$$D(x) = xE_1 + \frac{(D.D)}{2x}E_2 - D = (x-c)E_1 + \left(\frac{(D.D)}{2x} - b\right)E_2 + aE_3$$

satisfies $(D(x).D(x)) = 0$. Since E_1, E_2, and E_3 are linearly independent, we see that by multiplying each $D(x)$ by a suitable integer, we obtain infinitely many primitive classes $D'(x)$ satisfying $(D'(x).D'(x)) = 0$ and $(D'(x).\Theta) > 0$, and hence infinitely many elliptic curves.

Corollary 2.10 *There are only finitely many elliptic subgroups* $E \leq A$ *of bounded degree* $\deg_\Theta(E) \leq k$.

Proof. Since $\mathrm{NS}(A)$ is finitely generated (cf. [Mu], p. 178), it follows from the Hodge Index Theorem that there are only finitely many effective classes of bounded degree (cf. [Ha], Ex. V.1.11a) for details), and so the assertion follows from Theorem 2.8.

3. Humbert's invariant and Humbert surfaces

As before, let Θ be an ample divisor on an abelian surface A, so that the pair $(A, cl(\Theta))$ defines a polarized abelian surface. Without loss of generality, we may and will assume that its class $cl(\Theta) \in \mathrm{NS}(A)$ is primitive in the sense of Definition 2.5; we then say that $(A, cl(\Theta))$ is a *primitively polarized* abelian surface of degree

$$\delta = \chi(\Theta) = \frac{1}{2}(\Theta.\Theta).$$

Note that if Θ is a principal polarization (i.e. $\delta = 1$), then Θ is automatically primitive.

If $D \in \mathrm{Div}(A)$ is any divisor, then define its *Humbert invariant* $\Delta(D)$ by

$$\Delta(D) = \Delta_\Theta(D) = \deg(D)^2 - 2\delta(D.D), \tag{3.1}$$

where, as before, $\deg(D) = \deg_\Theta(D) = (D.\Theta)$.

Since $\Delta(D + m\Theta) = \Delta(D)$, for all $D \in \mathrm{Div}(A)$ and $m \in \mathbb{Z}$, it follows that

$$\Delta(D) = -(D^*.D^*), \tag{3.2}$$

where $D^* = 2\delta D - \deg(D)\Theta$. Thus, since $(D^*.\Theta) = 0$, it follows from the Hodge index theorem that

$$\Delta(D) \geq 0, \quad \text{and} \quad \Delta(D) = 0 \Longleftrightarrow D \equiv \frac{\deg(D)}{2\delta}\Theta. \tag{3.3}$$

We therefore see that Δ defines a positive definite (non-degenerate) quadratic form on the "polarized Néron–Severi group"

$$\mathrm{NS}(A, \Theta) = \mathrm{NS}(A)/\mathbb{Z}cl(\Theta)$$

by the rule

$$\Delta([D]) = \Delta(D),$$

where $[D]$ denotes the class of $D \in \mathrm{Div}(A)$ in $\mathrm{NS}(A, \Theta)$.

Theorem 3.1 *Suppose (A, Θ) is a principally polarized abelian variety. Then for any positive integer $k \in \mathbb{N}$, the map $E \mapsto [E] \in \mathrm{NS}(A, \Theta)$ induces a bijection between:*

1) the set of elliptic subgroups $E \leq A$ of degree $k = \deg(E)$, and

2) the set of primitive classes $[D] \in \mathrm{NS}(A, \Theta)$ with Humbert invariant $\Delta(D) = k^2$.

Proof. We first show that $[E]$ is primitive in $\mathrm{NS}(A, \Theta)$. If not, there are $m, n \in \mathbb{Z}$ such that $E + m\Theta = nD$. Since $cl(E)$ is primitive (in $\mathrm{NS}(A)$) by Corollary 2.6, we must have $(m, n) = 1$. Thus, since $n^2D^2 - 2mn(D.\Theta) + m^2\Theta^2 = E^2 = 0$, it follows that $n \mid \Theta^2 = 2$. If $2 \mid n$, then $4 \mid m^2\Theta^2 = 2m^2$, which is impossible. Thus $n = \pm 1$, and so $[E]$ is primitive.

Next we show that the map $E \mapsto [E]$ is injective. Thus, assume $[E] = [E']$, or $E \equiv E' + m\Theta$, where E, E' are elliptic subgroups of A and $m \in \mathbb{Z}$. We then obtain $0 \geq -2(E.E') = (E - E')^2 = m^2\Theta^2 \geq 0$, so equality holds throughout, which means that $E = E'$, as asserted.

It remains to show that the map is surjective. Thus, suppose $D \in \mathrm{Div}(A)$ is such that $\Delta(D) = k^2$ and $[D]$ is primitive. Put $d = \deg(D) = (D.\Theta)$. Then $d^2 = \Delta(D) + 2(D.D) = k^2 + 2(D.D)$, so $d^2 \equiv k^2 \pmod 2$, and hence $d \equiv k \pmod 2$. Thus $D' = D - \frac{d-k}{2}\Theta \in [D]$; note that D' is primitive in $\mathrm{NS}(A)$ since $[D'] = [D]$ is primitive in $\mathrm{NS}(A, \Theta)$. Now $\deg(D') = d - \frac{d-k}{2}\Theta^2 = k$ and so $2(D'.D') = \deg(D')^2 - \Delta(D') = k^2 - k^2 = 0$. Thus, by Proposition 2.8 we have $D' \equiv E$, for some elliptic subgroup E, and so $[E] = [D'] = [D]$, which proves the surjectivity.

Essentially the same proof shows that the above theorem can be generalized to arbitrary primitively polarized abelian surfaces as follows.

Theorem 3.2 *Suppose (A, Θ) is a primitively polarized abelian surface of degree δ. Then for any positive integer $k \in \mathbb{N}$ with $(k, \delta) = 1$, the map $E \mapsto [E] \in \mathrm{NS}(A, \Theta)$ induces a bijection between:*

1) the set of elliptic subgroups $E \leq A$ of degree $k = \deg(E)$, and

2) the set of primitive classes $[D] \in \mathrm{NS}(A, \Theta)$ with $\Delta(D) = k^2$ and $\deg(D) \equiv k \pmod{2\delta}$.

As a first application of Theorem 3.1, we can prove an algebraic analogue of a theorem of Biermann [Bi] and Humbert [Hu1] on the characterization of (principally polarized) abelian surfaces containing an elliptic curve in terms of the "singular relations" of Humbert, which we define algebraically as follows.

Definition 3.3 A primitively polarized abelian surface (A, Θ) is said to satisfy a *singular relation with invariant* N if there exists a primitive class $[D] \in \mathrm{NS}(A, \Theta)$ with Humbert invariant $\Delta([D]) = \Delta_\Theta([D]) = N$.

Remark 3.4 Note that (A, Θ) satisfies a singular relation (for some N) if and only if $rk(\mathrm{NS}(A)) \geq 2$. Moreover, N is uniquely determined if and only if $rk(\mathrm{NS}(A)) = 2$; in fact, in the case that $rk(\mathrm{NS}(A)) \geq 3$ there are infinitely many singular relations (and hence N's).

Using this terminology, the above–mentioned result of Biermann and Humbert may be formulated as follows.

Corollary 3.5 (Biermann–Humbert) *A principally polarized abelian surface (A, Θ) has an elliptic subgroup $E \leq A$ of degree n if and only if (A, Θ) satisfies a singular relation with invariant n^2.*

Proof. This is immediate by Theorem 3.1.

Remark 3.6 As was mentioned above, this corollary is just an algebraic analogue of the actual result proved by Biermann [Bi], pp. 981–983 and Humbert [Hu1], Theorem 15 (p. 308). To obtain the original result, one still has to relate the present algebraic terminology to the analytic terminology of Humbert; this is done in section 5; cf. Remark 5.6.

The above result can also be interpreted in terms of the Humbert surfaces introduced by Humbert[Hu1]. Although we require below only the case of a principal polarization (i.e. $\delta = 1$), it is just as easy to give the definition in the general case, as is done in van der Geer[vdG], ch. IX.2.

Definition 3.7 Let $\mathcal{A}_2(\delta)$ denote the moduli space consisting of all isomorphism classes of primitively polarized abelian varieties (A, Θ) of degree δ. Then for each integer $N \geq 1$ the subset

$$H_N = \{(A, \Theta) \in \mathcal{A}_2(\delta) : (A, \Theta) \text{ satisfies a singular relation with invariant } N\}$$

is called the *Humbert surface* of invariant N in $\mathcal{A}_2(\delta)$.

Remark 3.8 Once we have shown that the above algebraic definition of a singular relation coincides with that of Humbert, it follows also that this algebraic definition of a Humbert surface coincides with the analytic definition of Humbert (cf. van der Geer[vdG], p. 211). As a result, it follows from the work of Humbert and others that for $K = \mathbb{C}$ the set H_N is a closed subset of the moduli space $\mathcal{A}_2(\delta)$ and hence is an algebraic surface. While the corresponding result for ground fields K of positive characteristic has not yet been proved in general, there is little doubt that it is correct, provided one restricts to the case that $\text{char}(K) \not| N$. For a partial result in this direction, see Corollary 3.10 below.

With the above definition, Corollary 3.5 can be rephrased as follows.

Corollary 3.9 *A principally polarized abelian surface* (A, Θ) *has an elliptic subgroup* $E \leq A$ *of degree* n *if and only if* $(A, \Theta) \in H_{n^2}$*.*

The above corollary may be viewed as the umbilical cord connecting the Humbert surfaces with square invariant to the moduli spaces $\mathcal{A}_{g,1}(N)$ studied in [Ka1] and [Ka2]. The latter is the moduli space parametrizing isomorphism classes of principally polarized abelian varieties (A, Θ) of dimension g containing an elliptic subgroup $E \leq A$ of degree N:

$$\mathcal{A}_{g,1}(N) = \{\langle A, \Theta, E\rangle : \langle A, \Theta\rangle \in \mathcal{A}_g, E \leq A, \deg_\Theta(E) = N\}.$$

For $g = 2$ and $\text{char}(K) \not| N$ this moduli space can be explicitly determined; in fact, we have by [Ka1], [Ka2] that

$$\mathcal{A}_{2,1}(N) = (X(N) \times X(N))/G(N),$$

where $X(N)$ denotes the (affine) modular curve parametrizing elliptic curves with a level N structure (of fixed determinant), and $G(N) = Sl_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$

which acts in the usual manner on the first factor and in a certain twisted manner on the second factor. To be precise, this twisted action is given by $(g, x) \mapsto g^\varepsilon.x$, where $\varepsilon \in Gl_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$ is a fixed element with $\det(\varepsilon) = -1$, $g^\varepsilon = \varepsilon^{-1} g \varepsilon$, and $(g, x) \mapsto g.x$ denotes the usual action of $G(N)$ on $X(N)$.

Thus, since the image of the "forget map"

$$f_N : \mathcal{A}_{2,1}(N) \to \mathcal{A}_2$$

is precisely H_{N^2} by Corollary 3.9, we obtain from [Ka1] and [Ka2]:

Corollary 3.10 *If* $\mathrm{char}(K) \nmid N$, *then* H_{N^2} *is a closed irreducible subvariety of* $\mathcal{A}_2$ *whose normalization is* $\mathcal{A}_{2,1}(N)/\langle\tau\rangle$, *where* τ *is the involution on* $\mathcal{A}_{2,1}(N)$ *which is induced by the involution of* $X(N) \times X(N)$ *which interchanges the factors.*

4. Application to elliptic subcovers of curves

Let C be a (connected, smooth, projective) curve defined over K, and let J_C denote its Jacobian variety. As in Milne[Mi], we view J_C as a Picard variety; i.e. J_C represents the Picard functor Pic^0, so in particular we have a canonical identification $\mathrm{Pic}^0(C) = J_C(K)$. Thus, if $f : C \to C'$ is any covering of curves, then by functoriality we have an induced homomorphism

$$f^* : J_{C'} \to J_C$$

between their Jacobian varieties. By autoduality of the Jacobian (i.e. by the universal property of J_C as an Albanese variety) we also have an induced homomorphism

$$f_* : J_C \to J_{C'}$$

which is characterized by the relation

$$j_{f(P)} \circ f = f_* \circ j_P, \text{ for any } P \in C, \tag{4.1}$$

where $j_P : C \to J_C$ denotes the canonical embedding of C in its Jacobian defined by the point P, i.e. $j_P(Q) = cl(Q - P) \in \mathrm{Pic}^0(C) = J_C(K)$. These two homomorphisms are actually dual to each other in the sense that we have

$$(f^*)^\wedge \circ \lambda_C = \lambda_{C'} \circ f_*, \tag{4.2}$$

$$(f_*)^\wedge \circ \lambda_{C'} = \lambda_C \circ f^*, \tag{4.3}$$

where $\lambda_C = \phi_{\Theta_C} : J_C \to \hat{J}_C$ denotes the canonical principal polarization of J_C induced by the theta–divisor Θ_C and, for any homomorphism $h : A \to B$ of abelian varieties, $\hat{h} : \hat{B} \to \hat{A}$ denotes the dual map.

Here we shall be particularly interested in the case that target curve $C' = E$ has genus 1 (so E is an elliptic curve but we do not fix the identity point) and f is in a certain sense "minimal". More precisely, we shall study the following class of subcovers:

Definition 4.1 An *elliptic subcover* of a curve C is a finite morphism $f : C \to E$ to some a curve E of genus 1 which does not factor over any non–trivial isogeny of E, i.e. the only factorizations $f = f'' \circ f' : C \to E' \to E$, where E' is another curve of genus 1, are those for which f'' is an isomorphism.

If $f' : C \to E'$ is another elliptic subcover, then f' is said to be *equivalent to* f if there is an isomorphism $h : E' \to E$ such that $f = h \circ f'$.

Proposition 4.2 *For a morphism $f : C \to E$ to an elliptic curve E, the following conditions are equivalent:*

(i) f is an elliptic subcover;

(ii) $f_* : J_C \to J_E$ does not factor over a non–trivial isogeny;

(iii) $f^* : J_E \to J_C$ is injective (i.e. a closed immersion).

Moreover, if these conditions are satisfied, then we have

$$\deg(f) = (f^* J_E . \Theta_C). \tag{4.4}$$

Proof. The equivalence of (i) and (ii) follows from the universal (Albanese) property of (J_C, j_P), and the equivalence of (ii) and (iii) follows by duality since f^* is essentially the dual of f_* by (4.1).

It remains to prove (4.4), which is equivalent to the assertion

$$(f^*)^*(\Theta_C) \equiv \deg(f)\Theta_E \tag{4.5}$$

because $(f^* J_E . \Theta_C) = \deg_{J_E}((f^*)^*(\Theta_C))$, and two divisors on J_E are numerically equivalent if and only if they have the same degree.

To prove (4.5), we first note that for any homomorphism $h : A \to B$ of abelian varieties and divisor $D \in \mathrm{Div}(B)$ we have

$$\phi_{h^* D} = \hat{h} \circ \phi_D \circ h;$$

cf. the argument in Mumford[Mu], p. 143. (The assumption that h be an isogeny is not necessary for this formula.) Applying this to $h = f^* : J_{C'} \to J_C$ and $D = \Theta_C$ we obtain by (4.1):

$$\phi_{h^*(D)} = \lambda_{C'} \circ f_* \circ f^* = \deg(f)\lambda_{C'},$$

since $f_* \circ f^* = \deg(f) id_{J_{C'}}$. By the theorem of the square ([Mu], p. 59/60) we have $d\lambda_{C'} = d\phi_{\Theta_{C'}} = \phi_{d\Theta_{C'}}$, where $d = \deg(f)$. Thus $\phi_{h^* D} = \phi_{d\Theta_{C'}}$, or equivalently,

$$(f^*)^*\Theta_C \equiv \deg(f)\Theta_{C'}. \tag{4.6}$$

This proves (4.5) and hence (4.4). Note, however, that the last equation is true for any covering $f : C \to C'$ of curves.

Corollary 4.3 *For a curve C, the map $f \mapsto f^* J_E$ induces a one–to–one correspondence between the set of isomorphism classes of elliptic subcovers $f : C \to E$ of C of degree $n = \deg(f)$ and the set of elliptic subgroups $E \leq J_C$ of its Jacobian J_C of degree $n = (E.\Theta_C)$.*

Proof. It is clear that the assignment $f \mapsto f^*J_E$ is constant on equivalence classes of elliptic subcovers, and hence defines a map Φ from the first set to the second by Proposition 4.2. We shall now construct an inverse.

For this, fix a point $P \in C$ and let $E \leq J_C$ be an elliptic subgroup. Let $\iota : J_E \to J_C$ denote the inclusion map (here we identify E with J_E via $j = j_0 : E \xrightarrow{\sim} J_E$), and define $f = f_{E,P} : C \to J_E$ by

$$f = \lambda_E^{-1} \circ \hat{\iota} \circ \lambda_C \circ j_P.$$

Then $f_* = \lambda_E^{-1} \circ \hat{\iota} \circ \lambda_C$, so

$$(f_*)^\wedge = \hat{\lambda}_C \circ \hat{\hat{\iota}} \circ \hat{\lambda}_E^{-1} = \lambda_C \circ \iota \circ \lambda_E^{-1},$$

using the canonical double–dual identifications $\hat{\hat{J}}_C = J_C$ etc. From equation (4.2) it thus follows that $f^* = \iota$, so f^* is a closed immersion and hence $f : C \to E$ is an elliptic subcover by Proposition 4.2. It is immediate that the map $E \mapsto (f_{E,P} : C \to J_E)$ is an inverse of Φ, and hence Φ is a bijection.

Remark 4.4 As indicated, the above corollary is true for a curve C of arbitrary genus. For genus 2 curves, however, it is possible to give a somewhat different proof; cf. [FK], pp. 154–156.

If we combine Corollary 4.3 with Theorems 2.8 and 3.1 we obtain the following result:

Theorem 4.5 *Let C be a curve of genus 2 with Jacobian J_C and canonical principal polarization Θ. Then there is a bijective correspondence between the following sets:*

1) the set of isomorphism classes of elliptic subcovers $f : C \to E$ of degree $\deg(f) = n$;

2) the set of elliptic subgroups $E \leq J_X$ of J_X of degree $\deg_\Theta(E) = n$;

3) the set of primitive classes $cl(D) \in \mathrm{NS}(J_X)$ of degree $\deg_\Theta(D) = n$ and with self-intersection $(D.D) = 0$;

4) the set of primitive classes $[D] \in \mathrm{NS}(J_X, \Theta)$ with invariant $\Delta(D) = n^2$.

5. Analytic theory

We now want to re-examine at the results developed so far from an analytic viewpoint; in particular, we want to connect the algebraic definition of Humbert's invariant with the classical analytic definition and also give an analytic version of the main theorem.

Thus, suppose $K = \mathbb{C}$ is the field of complex numbers, so each abelian variety A has an analytic uniformization $A = V/U$, where U is a lattice in a $\mathbb{C}$-vector space V.

Summary 5.1 We recall the following basic facts about the cohomology of a complex abelian variety $A = V/U$ of dimension g; cf. Mumford[Mu], ch. 1, or Griffiths/Harris[GH], ch. 2.6:

1. Since V is simply–connected, we have a canonical identification

   $$U = \pi_1(A, 0) = H_1(A, \mathbb{Z}), \tag{5.1}$$

   which induces an isomorphism

   $$H^1(A, \mathbb{Z}) = \mathrm{Hom}(\pi_1(A, 0), \mathbb{Z}) \simeq \mathrm{Hom}(U, \mathbb{Z}).$$

   Via the cup–product, this isomorphism yields a canonical identification

   $$H^n(A, \mathbb{Z}) \simeq \wedge^n H^1(A, \mathbb{Z}) = \mathrm{Alt}^n(U, \mathbb{Z}) \tag{5.2}$$

   of the (singular) integral cohomology group $H^n(A, \mathbb{Z})$ with the group of alternating n-forms $a : U^n \to \mathbb{Z}$; cf. [Mu], p. 3.

2. We can identify V with the tangent space $T(A)$ of A at 0. Thus, if $T = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, and $\overline{T} = \mathrm{Hom}_{\mathbb{C}\text{-antilinear}}(V, \mathbb{C})$, then the canonical decomposition

   $$H^1(A, \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = T \oplus \overline{T} \tag{5.3}$$

   is just the Hodge decomposition of $H^1(A, \mathbb{C})$, which gives rise to canonical isomorphisms

   $$H^{p,q}(A) = H^q(A, \Omega^p) \simeq \wedge^p T \otimes \wedge^q \overline{T}, \tag{5.4}$$

   $$H^n(A, \mathbb{C}) = \wedge^n \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \wedge^n(T \oplus \overline{T}) = \bigoplus_{p+q=n} \wedge^p T \otimes \wedge^q \overline{T}; \tag{5.5}$$

   cf. [Mu], p. 4/5). In particular, if $g = 2$ and $z = (z_1, z_2)$ are Euclidean coordinates of V, then the associated global 1-forms $\{dz_1, dz_2\}$, respectively $\{d\overline{z}_1, d\overline{z}_2\}$, form a $\mathbb{C}$-basis of T, respectively of $\overline{T}$, and

   $$\begin{aligned} H^{1,1}(A) &= \langle dz_1 \wedge d\overline{z}_1, dz_1 \wedge d\overline{z}_2, dz_2 \wedge d\overline{z}_1, dz_2 \wedge d\overline{z}_2 \rangle_{\mathbb{C}} \\ &= \{\omega \in H^2(A, \mathbb{C}) : \omega \wedge dz_1 \wedge dz_2 = \overline{\omega} \wedge dz_1 \wedge dz_2 = 0\}. \end{aligned} \tag{5.6}$$

3. The Chern class map $c_1 : \mathrm{Pic}(A) = H^1(A, \mathcal{O}^*) \to H^2(A, \mathbb{Z})$ induces an isomorphism

   $$\mathrm{NS}(A) \simeq H^2(A, \mathbb{Z}) \cap H^{1,1}(A) \tag{5.7}$$

   which we will henceforth view as an identification (cf. [BPV], p. 119). Thus, by (5.6) we have (when $g = 2$)

   $$\mathrm{NS}(A) = \{\omega \in H^2(A, \mathbb{Z}) : \omega \wedge dz_1 \wedge dz_2 = 0\}. \tag{5.8}$$

4. If A is an abelian surface, let $\mu \in H^4(A,\mathbb{Z}) = H_0(A,\mathbb{Z}) \simeq \mathbb{Z}$ be the cohomology class defined by the canonical orientation of A. For $\omega_1, \omega_2 \in H^2(A,\mathbb{Z}) = \wedge^2 H^1(A,\mathbb{Z})$ define their "intersection number" $\langle \omega_1, \omega_2 \rangle \in \mathbb{Z}$ by the rule

$$\omega_1 \wedge \omega_2 = \langle \omega_1, \omega_2 \rangle \mu. \tag{5.9}$$

If $D_1, D_2 \in \mathrm{Div}(A)$ are divisors on A, then their usual intersection number is related to the one just defined by the expected formula:

$$(D_1.D_2) = \langle c_1(\mathcal{L}(D_1)), c_1(\mathcal{L}(D_2)) \rangle; \tag{5.10}$$

cf. [BPV], p. 66. Thus, if $\theta = c_1(\mathcal{L}(\Theta))$ denotes the Chern class of an ample divisor Θ of degree $\delta = \frac{1}{2}(\Theta.\Theta)$, then we can define the *degree* and *Humbert invariant* of $\omega \in H^2(A,\mathbb{Z})$ by

$$\deg(\omega) = \langle \omega, \theta \rangle \quad \text{and} \quad \Delta(\omega) = \deg(\omega)^2 - 2\delta \langle \omega, \omega \rangle.$$

We now want write down explicit formulae for these invariants in terms of the coordinates of a period lattice of an abelian surface A. To this end we fix the following notation.

Notation 5.2 Fix a primitive ample divisor Θ on A of degree δ with associated cohomology class $\theta = c_1(\mathcal{L}(\Theta)) \in H^2(A,\mathbb{Z}) = \mathrm{Alt}^2(U,\mathbb{Z})$. Viewing θ as a (non–degenerate) alternating bilinear form $\theta(x,y)$ on U, we can choose a symplectic basis $\{u_1, u_2, u_3, u_4\}$ of U with respect to θ, i.e. a basis such that

$$\theta\left(\sum n_{1j} u_j, \sum n_{2j} u_j\right) = \sum_{j=1}^{2} \delta_j (n_{1j} n_{2j+2} - n_{2j} n_{1j+2}), \tag{5.11}$$

where $1 \le \delta_1 \mid \delta_2$; cf. [GH], p. 304. Thus, in terms of the dual basis $x_1, \ldots, x_4$ of $\mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$ with associated 1-forms $dx_1, \ldots, dx_4 \in H^1(A,\mathbb{C}) = H^1(A,\mathbb{Z}) \otimes \mathbb{C}$ we can express θ in the form

$$\theta = \delta_1 dx_1 \wedge dx_3 + \delta_2 dx_2 \wedge dx_4. \tag{5.12}$$

It thus follows from (5.9) and (5.10) that the orientation class μ is given by

$$\mu = \frac{1}{2\delta} \theta \wedge \theta = -\frac{\delta_1 \delta_2}{2\delta} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = -dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4, \tag{5.13}$$

where the latter equality follows because μ and $-dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ are both generators of $H^4(A,\mathbb{Z}) = \mathbb{Z}$. We note that since Θ was assumed to be primitive in $\mathrm{NS}(A)$, we must have $\delta_1 = 1$, as is evident from (5.7), and hence $\delta_2 = \delta$.

Moreover, we can choose a $\mathbb{C}$-basis $\{e_1, e_2\}$ of V such that the period matrix $\Omega = (\omega_{jk})$, defined by $u_k = \sum_j \omega_{jk} e_j$, has the form

$$\Omega = \begin{pmatrix} 1 & 0 & \tau_1 & \tau_2 \\ 0 & \delta & \tau_2 & \tau_3 \end{pmatrix},$$

where the matrix $\left(\begin{smallmatrix}\tau_1 & \tau_2\\ \tau_2 & \tau_3\end{smallmatrix}\right)$ has positive definite imaginary part; cf. [GH], pp. 304–306. The associated 1-forms $dz_1, dz_2, d\overline{z}_1, d\overline{z}_2 \in H^1(A,\mathbb{C})$ are then related to the basis $dx_1, \ldots, dx_4$ by the formulae (cf. [GH], p. 304):

$$\begin{array}{rclrcl} dz_1 &=& dx_1 + \tau_1 dx_3 + \tau_2 dx_4, & d\overline{z}_1 &=& dx_1 + \overline{\tau}_1 dx_3 + \overline{\tau}_2 dx_4 \\ dz_2 &=& \delta dx_2 + \tau_2 dx_3 + \tau_3 dx_4, & d\overline{z}_2 &=& \delta dx_2 + \overline{\tau}_2 dx_3 + \overline{\tau}_3 dx_4. \end{array}$$

In terms of the above bases, the Humbert invariant Δ may be expressed as follows.

Lemma 5.3 *Let $\omega \in H^2(A,\mathbb{Z})$, and write*

$$\omega = \sum_{i<j} a_{ij} dx_i \wedge dx_j = \frac{1}{2}\sum_{i,j} a_{ij} dx_i \wedge dx_j. \tag{5.14}$$

Then the following formulae are valid:

$$\deg(\omega) = \delta a_{13} + a_{24}, \tag{5.15}$$

$$\langle \omega, \omega \rangle = -2pf((a_{ij})) = -2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}), \tag{5.16}$$

$$\Delta(\omega) = (\delta a_{13} - a_{24})^2 + 4\delta(a_{12}a_{34} + a_{14}a_{23}). \tag{5.17}$$

Moreover, $\omega \in \mathrm{NS}(A)$ if and only if

$$\delta a_{34} - \delta a_{14}\tau_1 + (\delta a_{13} - a_{24})\tau_2 + a_{23}\tau_3 + a_{12}(\tau_1\tau_3 - \tau_2^2) = 0. \tag{5.18}$$

Proof. In view of (5.9), (5.12) and (5.13) we have

$$\omega \wedge \theta = -(\delta a_{13} + a_{24}) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = (\delta a_{13} + a_{24})\mu,$$

which proves (5.15). Moreover, a short calculation yields

$$\omega \wedge \omega = 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4,$$

and so (5.16) follows since, as is well–known, the Pfaffian of a 4×4 skew-symmetric matrix (a_{ij}) is given by the indicated formula. Thus, from (5.15) and (5.16) we obtain

$$\begin{array}{rcl} \Delta(\omega) &=& (\delta a_{13} + a_{24})^2 - 2\delta(-2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})) \\ &=& (\delta a_{13} - a_{24})^2 + 4\delta(a_{12}a_{34} + a_{14}a_{23}), \end{array}$$

which shows that (5.17) is valid.

To prove the last assertion, we first note that

$$\begin{array}{rcl} dz_1 \wedge dz_2 &=& (dx_1 + \tau_1 dx_3 + \tau_2 dx_4) \wedge (\delta dx_2 + \tau_2 dx_3 + \tau_3 dx_4) \\ &=& \delta dx_1 \wedge dx_2 + \tau_2 dx_1 \wedge dx_3 + \tau_3 dx_1 \wedge dx_4 - \delta\tau_1 dx_2 \wedge dx_3 \\ && -\delta\tau_2 dx_2 \wedge dx_4 + (\tau_1\tau_3 - \tau_2^2) dx_3 \wedge dx_4, \end{array} \tag{5.19}$$

and so

$$\begin{array}{rcl} \omega \wedge dz_1 \wedge dz_2 &=& (\delta a_{34} - \delta a_{14}\tau_1 + (\delta a_{13} - a_{24})\tau_2 + a_{23}\tau_3 + a_{12}(\tau_1\tau_3 - \tau_2^2)) \\ && \cdot dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4. \end{array}$$

In view (5.8), we thus see that $\omega \in \mathrm{NS}(A)$ if and only if (5.18) holds.

The above lemma can be rephrased in the following more convenient way.

Proposition 5.4 *There exists a basis* $v_1, \ldots, v_5$ *of* $H(A,\theta) := H^2(A,\mathbb{Z})/\mathbb{Z}\theta$ *such that, under the resulting identification* $H(A,\theta) \simeq \mathbb{Z}^5$*, the induced quadratic form* Δ *on* $H(A,\theta)$ *assumes the form*

$$\Delta(a,b,c,d,e) = b^2 - 4\delta(ac+de) \tag{5.20}$$

and such that the group $\mathrm{NS}(A,\Theta) = \mathrm{NS}(A)/\mathbb{Z}\Theta$*, viewed as a subgroup of* $H(A,\theta)$ *via (5.7), is given by*

$$\mathrm{NS}(A,\Theta) = \{(a,b,c,d,e) : a\delta\tau_1 + b\tau_2 + c\tau_3 + d(\tau_2^2 - \tau_1\tau_3) + e\delta = 0\}. \tag{5.21}$$

Proof. Put $v_1 := -dx_1 \wedge dx_4, v_2 := -dx_2 \wedge dx_4, v_3 := dx_2 \wedge dx_3, v_4 := -dx_1 \wedge dx_2$, and $v_5 := dx_3 \wedge dx_4$. Then $v_1, \ldots, v_5, \theta$ is a basis of $H^2(A,\mathbb{Z})$; more precisely, each $\omega \in H^2(A,\mathbb{Z})$ can be written in the form

$$\omega = av_1 + bv_2 + cv_3 + dv_4 + ev_4 + f\theta,$$

with $a = -a_{14}, b = \delta a_{13} - a_{24}, c = a_{23}, d = -a_{12}, e = a_{34}$, and $f = a_{13}$ in the notation of (5.14). Substituting these expressions in (5.17) and (5.18) and passing to the respective quotients yields the assertions.

We can now relate the algebraic definition of a singular relation (cf. Definition 3.3) to the analytic definition of Humbert[Hu1], p. 297, 301.

Corollary 5.5 *The primitively polarized abelian surface* (A,Θ) *of degree* δ *satisfies a singular relation with invariant* N *if and only if there are integers* $a,b,c,d,e \in \mathbb{Z}$ *with* $(a,b,c,d,e) = 1$ *such that*

$$b^2 - 4\delta(ac+de) = N, \tag{5.22}$$

and

$$a\delta\tau_1 + b\tau_2 + c\tau_3 + d(\tau_1\tau_3 - \tau_2^2) + e\delta = 0, \tag{5.23}$$

where $\left(\begin{smallmatrix}\tau_1 & \tau_2 \\ \tau_2 & \tau_3\end{smallmatrix}\right)$ *is a period matrix of* (A,Θ).

Proof. Put $\omega = av_1 + \ldots + ev_5 \in H^2(A,\mathbb{Z})$. Then by Proposition 5.4 we have that $\omega \in \mathrm{NS}(A)$ if and only if (5.23) holds. If this is the case, then ω is primitive in $\mathrm{NS}(A)$ if and only if ω is primitive in $H^2(A,\mathbb{Z})$ if and only if $(a,b,c,d,e) = 1$. Moreover, by (5.20) we have $\Delta(\omega) = N$ if and only if (5.22) holds.

Remark 5.6 The above Proposition 5.4 and its Corollary 5.5 show that the Definition of a "singular relation with invariant Δ" as defined by Humbert[Hu1], pp. 297 and 301, coincides with the algebraic definition presented in Definition 3.3 above (for $\delta = 1$). Moreover, since the latter definition does not depend on the choice of the period lattice, it is clear that $\Delta(a,b,c,d,e)$ is an invariant of the polarized abelian variety (A,Θ), and so the extensive calculations of

Humbert[Hu1], pp. 298–301 are superfluous. Finally, the above formulae also show that the definition of a Humbert surface as defined by van der Geer[vdG], p. 211 is essentially identical to the algebraic definition given above (cf. Definition 3.7), except that van der Geer uses a different normalization of the period matrix so that his formulae are slightly different.

We now want to present an analytic proof of Theorem 2.8. This depends on the following interesting "dictionary" presented in Proposition 5.9 below which relates the elliptic subgroups $E \leq A$ of a complex abelian surface A to certain submodules M of $H^1(A, \mathbb{Z})$ which we shall call *elliptic submodules*. These submodules are also closely connected to certain subspaces W_M of $H^{1,0}(A) = H^0(A, \Omega_A)$ and H_M of $H^1(A, \mathbb{C})$ which we shall call *elliptic differentials* and *elliptic subspaces* respectively; they are defined as follows.

Definition 5.7 Let X be a compact algebraic variety. Then an *elliptic submodule* of X is a rank 2 submodule $M \subset H^1(X, \mathbb{Z})$ which is primitive (i.e. $H^1(X, \mathbb{Z})/M$ is torsionfree) and which satisfies

$$W_M := (\mathbb{C} \otimes M) \cap H^{1,0}(X) \neq 0. \tag{5.24}$$

An *elliptic differential* on an algebraic variety X is a one–dimensional $\mathbb{C}$-subspace $W = \mathbb{C}\omega \subset H^{1,0}(X)$ of the space $H^{1,0}(A) = H^0(X, \Omega^1_X)$ of holomorphic differentials such that

$$M_W = (W + \overline{W}) \cap H^1(X, \mathbb{Z}) \tag{5.25}$$

is a submodule of rank 2 of $H^1(X, \mathbb{Z})$.

An *elliptic subspace* of X is a two-dimensional $\mathbb{C}$-subspace $H \subset H^1(X, \mathbb{C})$ which has a Hodge structure of weight 1 (cf. [BPV], p. 36) compatible with that of $H^1(X, \mathbb{C})$, i.e. we have

$$H = (H \cap H^1(X, \mathbb{Z})) \otimes \mathbb{C} = (H \cap H^{1,0}(X)) \oplus (H \cap H^{0,1}(X)). \tag{5.26}$$

The set of all elliptic submodules (respectively, of all elliptic differentials, respectively, of all elliptic subspaces) of X will be denoted by $\mathrm{Ell}(X)$ (respectively, by $\mathrm{Ell}_d(X)$, respectively by $\mathrm{Ell}_{sp}(X)$).

Remark 5.8 a) The above three concepts "elliptic submodule", "elliptic differential" and "elliptic subspace" are essentially equivalent notions. Indeed, if W is an elliptic differential, then $H_W := W \oplus \overline{W} = \mathbb{C} \otimes W_M$ is an elliptic subspace and hence M_W is an elliptic submodule. Similarly, if M is an elliptic submodule, then $H_M := \mathbb{C} \otimes M = W_M \otimes \overline{W}_M$ is an elliptic subspace, so W_M is an elliptic differential. Finally, if H is an elliptic subspace, then $W_H = H \cap H^{1,0}(A)$ and $M_H = H \cap H^1(A, \mathbb{Z})$ are an elliptic differential and an elliptic submodule, respectively. Thus, the maps $M \mapsto W_M$ and $W \mapsto H_W$ define bijections between the sets $\mathrm{Ell}(X)$, $\mathrm{Ell}_d(X)$ and $\mathrm{Ell}_{sp}(X)$ of elliptic submodules, elliptic differentials and elliptic subspaces of A.

b) If X is a complex curve (i.e. X is a compact Riemann surface), then the quotient $\mathrm{Jac}(X) = H^1(X, \Omega^1_X)^*/H_1(X, \mathbb{Z})$ is an abelian variety. (Note that

$\mathrm{Jac}(X) = \hat{J_X}$ is the dual of the Jacobian defined earlier.) Moreover, for each $P \in X$, the morphism $j_P : X \to \mathrm{Jac}(X)$ defined by $j_P(Q)(\omega) = \int_P^Q \omega$ induces an isomorphism

$$j^* = j_P^* : H^1(\mathrm{Jac}(X), \mathbb{C}) \xrightarrow{\sim} H^1(X, \mathbb{C})$$

which does not depend on P and which is compatible with the integral structures and with the Hodge decompositions. Thus, j^* induces a natural identification

$$\mathrm{Ell}(X) = \mathrm{Ell}(\mathrm{Jac}(X))$$

between the set of elliptic submodules of X and those of $\mathrm{Jac}(X)$, and similarly, the sets of elliptic differentials (respectively, elliptic subspaces) of X and $\mathrm{Jac}(X)$ correspond.

We now relate the elliptic submodules of a complex abelian surface A its elliptic subgroups $E \leq A$. For the sake of clarity and since it is it involves no extra work, we shall do this more generally for an abelian variety of arbitrary dimension. In this general case, however, one has work with the *co-elliptic subgroups*, i.e. with abelian subvarieties $B \leq A$ of codimension 1 of A (so $E = A/B$ is an elliptic curve), in place of the elliptic subgroups of A.

Proposition 5.9 *a) Let $M \in \mathrm{Ell}(A)$ be an elliptic submodule of a complex abelian variety $A = V/U$. Using the identification $H^1(A, \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$, put*

(5.27) $$V_M = M^\perp = \{v \in V : m(v) = 0, \forall m \in M\}.$$

Then V_M is a $\mathbb{C}$-subspace of V of codimension 1 and hence $B_M = (V_M + U)/U \leq V/U = A$ is a co-elliptic subgroup of A. Moreover, the Chern class of B_M, viewed as a divisor on A, is given by

(5.28) $$c_1(\mathcal{L}(B_M)) = \omega_M := \omega_1 \wedge \omega_2,$$

where $\{\omega_1, \omega_2\}$ is any suitably ordered basis of M.

b) The above map $M \mapsto B_M$ is a bijective correspondence between the set $\mathrm{Ell}(A)$ of elliptic submodules of A and the set of co-elliptic subgroups of A. Moreover, the inverse map is given by $B \mapsto M_B = p^* H^1(A/B, \mathbb{Z})$, where $p : A \to A/B = E$ denotes the quotient map, and similarly, the map $B \mapsto W_B = p^* H^1(A/B, \Omega^1_{A/E})$ is the inverse of the map $W \mapsto B_{M_W}$.

Proof. a) By construction, $U \cap V_M$ is a (full) lattice in V_M, which is a real subspace of V of (real) codimension 2, so the first assertion follows once we have shown that V_M is a complex subspace of V. For this, it is enough to show that V_M is stable under the map $I : V \to V$ defined by multiplication by $i = \sqrt{-1}$. Now since T and $\overline{T}$ are the ± 1-eigenspaces of the dual map $I^* \in \mathrm{End}_{\mathbb{R}}(H^1(A, \mathbb{C}))$, it follows that $H_M \cap T$ and $H_M \cap \overline{T}$ are stable under I^*, and hence so is $H' := (H_M \cap T) + (H_M \cap \overline{T})$. Since M is an elliptic submodule we have $H' = H_M$, and so $V_M = M^\perp = (H_M)^\perp$ is stable under I. This proves the first statement; the second one (i.e. (5.28)) will be proved at the end.

b) It is enough to show that the map $B \mapsto M_B$ is the inverse of $M \mapsto B_M$. For this, let $B \leq A$ be a co-elliptic subgroup. Then $B = (\tilde{V} + U)/U$, where $\tilde{V}$ is a $\mathbb{C}$-subspace of V of codimension 1 such that $\tilde{U} = U \cap \tilde{V}$ is a lattice in $\tilde{V}$. Put $V' = V/\tilde{V}$ and $U' = (\tilde{V} + U)/\tilde{V}$. Then $E' = V'/U'$ is an elliptic curve, and we have an exact sequence

$$0 \to B \xrightarrow{i} A \xrightarrow{p} E' \to 0.$$

Using the identification (5.1) we obtain an induced exact sequence

$$0 \to H_1(B, \mathbb{Z}) = \tilde{U} \xrightarrow{i_*} H_1(A, \mathbb{Z}) = U \xrightarrow{p_*} H_1(E', \mathbb{Z}) = U' \to 0,$$

which by dualizing yields the exact sequence

$$0 \to H^1(E', \mathbb{Z}) \xrightarrow{p^*} H^1(A, \mathbb{Z}) \xrightarrow{i^*} H^1(B, \mathbb{Z}) \to 0.$$

We thus see that $M_B = p^* H^1(E', \mathbb{Z})$ is a direct factor of $H^1(A, \mathbb{Z})$ and hence is a primitive submodule. Moreover, since p^* is injective, M_B has rank 2. Finally, since p is a holomorphic map, the induced map p^* respects the Hodge decompositions, and so M_B is an elliptic submodule. It is immediate from the above construction that $B \mapsto M_B$ is inverse to $M \mapsto B_M$, and so b) follows.

Finally, to prove (5.28) we note first that since $B = p^{-1}(D)$ is the pull-back of the divisor $D = 0_{E'}$ on E' consisting of the point $0_{E'}$, it follows that $c_1(\mathcal{L}(B)) = p^* c_1(\mathcal{L}(D))$. Since its $\deg_{E'}(D) = 1$, we have $c_1(D) = \omega_1' \wedge \omega_2'$, for any oriented basis $\{\omega_1, \omega_2\}$ of $H^1(E', \mathbb{Z})$, and so (5.28) follows.

We can supplement this dictionary by defining the *degree* of an elliptic submodule.

Definition 5.10 Let Θ be an ample divisor on A with associated Chern class $\theta = c_1(\mathcal{L}(\Theta)) \in H^2(A, \mathbb{Z}) = \wedge^2 H^1(A, \mathbb{Z})$. If $M \subset H^1(A, \mathbb{Z})$ is elliptic submodule with an oriented basis $\{\omega_1, \omega_2\}$, then its *degree* $\deg_\theta(M)$ is defined by

$$\deg_\theta(M) = \frac{1}{(g-1)!} \int_A \theta^{g-1} \wedge \omega_1 \wedge \omega_2.$$

Note that this is a positive integer because if $D = \Theta_{|B_M}$ then by (5.28)

$$(5.29) \quad \deg_\theta(M) = \frac{1}{(g-1)!}(\Theta^{g-1}.B_M) = \frac{1}{(g-1)!}(D)^{g-1} = \chi(\mathcal{L}(D))$$

by the Riemann–Roch Theorem on B_M. In particular, if A is an abelian surface, then $\deg_\theta(M) = (\Theta.B_M) = \deg_\Theta(B_M)$ as defined above.

Applying this to the case of a Jacobian $A = \mathrm{Jac}(X)$, we obtain the following result, which may be viewed as a sharpening of a classical result due to Weierstraß; cf. Krazer[Kr], ch. XII, Satz I, p. 271:

Corollary 5.11 *If X is a complex curve, then the assignment $(f : X \to E) \mapsto M_f := f^*H^1(E,\mathbb{Z})$ induces a natural bijection between the isomorphism classes of elliptic subcovers of X of degree n and the elliptic submodules $M \subset H^1(X,\mathbb{Z}) \simeq H^1(\mathrm{Jac}(X),\mathbb{Z})$ of degree n.*

Proof. If M is an elliptic submodule of $A = \mathrm{Jac}(X)$, then the projection map $p = p_M : A \to E_M = A/B_M$ does not factor over any isogeny, so $f_M = p_M \circ j_P$ is an elliptic subcover. Clearly the assignment $M \mapsto f_M$ is inverse to $f \mapsto M_f$, so both are bijections. To see that they are degree–preserving, note first that

$$\deg(f_M) = \deg f_M^*(0) = \deg(j_P^*(B_M)) = (X.B_M).$$

By Poincaré's formula (cf. [GH], p. 350), $X \equiv \frac{1}{(g-1)!}\Theta_C^{g-1}$, so we see by (5.29) that $(X.B_M) = \deg_\Theta(B_M) = \deg_\theta(M)$, and hence $\deg(f_M) = \deg_\theta(M)$, as desired.

We now want to formulate an analytic version of Theorem 2.8. This depends on the notion of a "primitively decomposable" p-form, which is defined as follows.

Definition 5.12 Let M be a free $\mathbb{Z}$-module of finite rank. A p-vector $\omega \in \wedge^p M$ is called *primitively decomposable* if there exist $m_1, \ldots, m_p \in M$ with $M/(\sum \mathbb{Z}m_i)$ torsionfree such that $\omega = m_1 \wedge \ldots \wedge m_p$.

We note the following two elementary facts, of which the second is essentially a restatement of Grassmann's relation for $n = 4$ and $p = 2$; cf. Bourbaki[Bou], III.11.13 (p. 611).

Lemma 5.13 *a) Each primitively decomposable $\omega \in \wedge^p M$ is primitive in $\wedge^p M$.*

b) Suppose rank$(M) = 4$ and $p = 2$. Then $\omega \in \wedge^2 M$ is primitively decomposable if and only if ω is primitive and $\omega \wedge \omega = 0$.

We are now ready to give a quick proof of the following analytic version of Theorem 2.8.

Theorem 5.14 *Let A be a complex abelian surface with ample divisor Θ. Then there are natural degree–preserving bijections between the following sets:*

1) the set of elliptic subgroups $E \leq A$;

2) the set $\mathrm{Ell}(A)$ of elliptic submodules;

3) the set of primitively decomposable 2-forms $\omega \in H^2(A,\mathbb{Z}) = \wedge^2 H^1(A,\mathbb{Z})$ with $\omega \wedge dz_1 \wedge dz_2 = 0$ and $\langle \omega, \theta \rangle > 0$;

4) the set of primitive divisor classes $cl(D) \in \mathrm{NS}(A)$ with $(D.D) = 0$ and $\deg_\Theta(D) > 0$.

Proof. The bijection between 1) and 2) was already established in Proposition 5.9.

Next, consider the map $M \in \mathrm{Ell}(A) \mapsto \omega_M$. Clearly, ω_M is a primitively decomposable 2-form. Moreover, $\omega_M \in H^{1,1}(A)$ by (5.28), so $\omega_M \wedge dz_1 \wedge dz_2 = 0$ by (5.6). Thus, ω_M belongs to the set described in 3). Conversely, suppose $\omega = \omega_1 \wedge \omega_2$ is as in 3) and put $M_\omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Since $\omega_1 \wedge \omega_2 \wedge dz_1 \wedge dz_2 = 0$, we must have a relation $a_1\omega_1 + a_2\omega_2 + a_3 dz_1 + a_4 dz_2 = 0$, with $a_1, \ldots, a_4 \in \mathbb{C}$ not all zero. This means that $(\mathbb{C} \otimes M) \cap H^{1,0}(A) \neq 0$, so M_ω is an elliptic submodule. It is easily checked that the map $\omega \mapsto M_\omega$ is inverse to $M \mapsto \omega_M$, so both define bijections between the sets 2) and 3) which are degree–preserving by (5.29).

To construct a bijection between the sets 4) and 3), let D be as in set 4), and put $\omega_D = c_1(\mathcal{L}(D)) \in \mathrm{NS}(A) = H^2(A, \mathbb{Z}) \cap H^{1,1}(A)$. Since ω_D is primitive in $\mathrm{NS}(A)$, it is also primitive in $H^2(A, \mathbb{Z})$. Thus, since $\omega_D \wedge \omega_D = 0$, it follows from Lemma 5.11b) that ω_D is primitively decomposable and hence lies in the set 3). Conversely, each ω in the set 3) lies in $\mathrm{NS}(A) = H^2(A, \mathbb{Z}) \cap H^{1,1}(A)$, and hence is of the form $\omega = c_1(\mathcal{L}(D))$ for some divisor D. By Lemma 5.11a), ω is primitive in $H^2(A, \mathbb{Z})$ and hence also in $\mathrm{NS}(A)$. Moreover, since clearly $\omega \wedge \omega = 0$, we have $(D.D) = 0$. It is thus immediate that the map $cl(D) \mapsto \omega_D$ is the desired (degree–preserving) bijection between the sets 4) and 3).

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