Elliptic Subcovers of Hyperelliptic Curves

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1 Introduction

Let C be a curve over an arbitrary field K. An *elliptic subcover* is a finite K-morphism $f : C \to E$ to an elliptic curve E/K such that its base-change $f_{\overline{K}} : C_{\overline{K}} \to E_{\overline{K}}$ to the algebraic closure does not factor over a non-trivial isogeny of $E_{\overline{K}}$. If $f' : C \to E'$ is another elliptic subcover, then f' is said to be *equivalent* to f if there is an isomorphism $\varphi : E \xrightarrow{\sim} E'$ such that $f' = \varphi \circ f$. Let $\mathcal{E}(C)$ denotes the set of all equivalence classes of elliptic subcovers of C.

If $C(K) \neq \emptyset$, then it is well-known that there is a natural bijection between the set $\mathcal{E}(C)$ and the set $\mathcal{S}(J_C)$ of elliptic subgroups of the Jacobian J_C of C.

The purpose of this paper is to show that this result is also true when $C(K) = \emptyset$, provided that C is a hyperelliptic curve whose genus is even. Thus, we prove in §4:

Theorem 1.1. *Let C/K be a hyperelliptic curve of genus $g_C \equiv 0\,(2)$. Then the rule $f \mapsto f^*J_E$ induces a bijection*

$$\Phi_C : \mathcal{E}(C) \xrightarrow{\sim} \mathcal{S}(J_C).$$

Note that this result does not extend to hyperelliptic curves of odd genus, as counterexamples show; cf. Example 2.5 and Remark 2.4.[1]

Theorem 1.1 implies:

Corollary 1.2. *If C is as above, then every finite morphism $f : C \to E$ to an elliptic curve E/K factors over an elliptic subcover $f_1 : C \to E_1$, and f_1 is uniquely determined by f up to equivalence. In particular, f is an elliptic cover if and only if f does not factor over a non-trivial K-isogeny.*

This result can also be reformulated in terms of elliptic subfields of the function field $F = \kappa(C)$ of C:

Corollary 1.3. *If C is as above, and E/K is an elliptic curve, then a finite morphism $f : C \to E$ is an elliptic cover if and only if $f^*\kappa(E)$ is a maximal elliptic subfield of $F = \kappa(C)$. Thus, the map $f \mapsto f^*\kappa(E)$ induces a bijection between $\mathcal{E}(C)$ and the set of $M_1(F)$ of a maximal elliptic subfields of F.*

We note in passing that throughout we could have replaced "elliptic curve" by "curve of genus 1". Indeed, the following result is an easy consequence of Theorem 1.1.

Corollary 1.4. *If E/K is a curve of genus 1 which admits a finite cover $f : C \to E$ by a hyperelliptic curve C/K of even genus, then $E(K) \neq \emptyset$, and so E/K is an elliptic curve.*

[1]This answers a question posed by the referee.

In the case that $\mathrm{char}(K) \neq 2$ and $g_C = 2$, Theorem 1.1 was already proven in [4]. The proof there uses the key fact that in that case one can "normalize" an elliptic subcover $f : C \to E$, and that normalized subcovers possess a Galois descent. Here in §4 we generalize this concept to hyperelliptic curves of higher genus (and to the case of an arbitrary base field K) and show that a similar result holds when g_C is even.

Theorem 1.5. *Let C/K be a hyperelliptic curve whose genus is even. If $f : C \to E$ is an elliptic subcover, then there exists an $x \in E(K)$ such that $T_x \circ f$ is normalized.*

By combining this result with the fact that normalized covers posses a Galois-descent (cf. Corollary 4.4), it is easy to complete the proof of Theorem 1.1.

2 Elliptic subcovers

Let C/K be a (smooth, projective, geometrically connected) curve of genus $g \geq 2$ over an arbitrary field K. We review here some basic facts about elliptic subcovers. For technical reasons, however, it is useful consider a slightly more general class of subcovers.

Definition 2.1. A *quasi-elliptic subcover* of C/K is a finite K-morphism $f : C \to E$ to a curve E/K of genus 1 such that its base-change $f_{\overline{K}} : C_{\overline{K}} \to E_{\overline{K}}$ to the algebraic closure $\overline{K}$ of K does not factor over an non-trivial isogeny of $E_{\overline{K}}$. If, in addition, E has a K-rational point, then we call f an *elliptic subcover.*

As we shall see below, in many cases quasi-elliptic subcovers are automatically elliptic subcovers (cf. Proposition 2.3 and Corollary 1.4). However, there do exist quasi-elliptic subcovers which are not elliptic; cf. Example 2.5 below.

For much of what follows, it is important to note that quasi-elliptic covers can be characterized by the condition that the induced map $f^* : J_E \to J_C$ on the Jacobians is a closed immersion. Here and below, we use the notations and basic facts about Jacobians as in [4]. (See also [9].)

Proposition 2.2. *Let $f : C \to E$ be a morphism to a curve E/K of genus 1. Then f is a quasi-elliptic subcover if and only $f^* : J_E \to J_C$ is a closed immersion. If this is the case, then f is separable.*

Proof. The first assertion follows immediately from [2], Proposition 4.2, because $(f_{\overline{K}})^* = (f^*)_{\overline{K}}$ is a closed immersion if and only if f^* is a closed immersion. To prove the second assertion, suppose the contrary, i.e., that f is not separable. Then also $f_{\overline{K}}$ is not separable, and hence the function field $F := \kappa(C_{\overline{K}})$ of $C_{\overline{K}}$ is an inseparable extension of $F_0 = f^*\kappa(E_{\overline{K}})$, and so $F_0 \subset F^p$, where $p = \mathrm{char}(K) \neq 0$; cf. [11], Prop. III.9.2. Thus $F_1 := F_0^{1/p} \subset F$ is an elliptic subfield of F, and so $f_{\overline{K}}$ factors as $f_{\overline{K}} = f_1 \circ f_2$, where $f_1 : E_1 \to E_{\overline{K}}$ is purely inseparable of degree p. Thus $f_{\overline{K}}^* = f_2^* \circ f_1^*$ factors over the (inseparable) isogeny f_1^* of degree p, and hence f^* cannot be a closed immersion, contradiction. Thus f is separable. $\square$

As in the introduction (or as in [2]), two quasi-elliptic subcovers $f_i : C \to E_i$ are said to be *equivalent* if there exists an isomorphism $\varphi : E_1 \xrightarrow{\sim} E_2$ (of genus 1 curves) such that $f_2 = \varphi \circ f_1$. We denote the set of equivalence classes of quasi-elliptic subcovers of C/K by $\tilde{\mathcal{E}}(C) = \tilde{\mathcal{E}}(C/K)$. Clearly, the set $\mathcal{E}(C)$ of equivalence classes of elliptic subcovers of C/K is naturally a subset of $\tilde{\mathcal{E}}(C)$. We then have the following fact which generalizes Corollary 4.3 of [2].

Proposition 2.3. *The rule $f \mapsto f^* J_E$ defines an injection*

$$\Phi_C : \tilde{\mathcal{E}}(C) \hookrightarrow \mathcal{S}(J_C)$$

of $\tilde{\mathcal{E}}(C)$ into the set $\mathcal{S}(J_C)$ of elliptic subgroups $E \le J_C$ of J_C. Moreover, if C/K has a divisor of degree 1, then every quasi-elliptic cover is an elliptic cover, and Φ_C is a bijection.

Remark 2.4. If K is a finite field, then C/K always has has a divisor of degree 1 by a theorem of F. K. Schmidt; cf. [11], Corollary V.1.11.

Moreover, if C/K has an *elliptic* subcover $f : C \to E$ of *odd* degree, then C/K has a divisor of degree 1 (because $f^*(0_E)$ has odd degree and the canonical divisor of C has even degree), and hence it follows from Proposition 2.3 that every quasi-elliptic subcover is elliptic and that Φ_C is a bijection.

However, quasi-elliptic covers which are not elliptic do exist for curves over $\mathbb{Q}$ or $\mathbb{R}$, as the following general Example 2.5 shows. Thus, for these examples the restriction of the map Φ_C to $\mathcal{E}(C)$ is not surjective. In particular, Theorem 1.1 does not extend to hyperelliptic curves $C/\mathbb{Q}$ of odd genus.

Example 2.5. Let $K \subset \mathbb{R}$ be a subfield of $\mathbb{R}$, and consider the curve E/K whose function field is $F_0 := K(x, y)$, where $y^2 = -(x^4 + 1)$. Since the double cover $\bar{\varphi} : E \to \mathbb{P}^1$ is unramified at the point P_∞ at infinity and has relative residue degree 2 there (because $-1 \notin (K^\times)^2$), we see that E/K is a curve of genus 1 with $E(K) = \emptyset$.

Next, consider integers $n \ge 2$ and $d \ge 1$ with $\gcd(n, d) = 1$, and let $\bar{F} = \bar{F}_{n,d} = K(x, z)$ be the function field with equation $z^n = \prod_{i=1}^{d}(x - a_i)$, where the $a_i \in K$ are distinct. The associated curve $\bar{C} = \bar{C}_{n,d}$ has genus $g_{\bar{C}} = (n-1)(d-1)$ and has a subcover $\bar{f} = \bar{f}_{n,d} : \bar{C}_{n,d} \to \mathbb{P}^1$ of degree n; cf. [11], Prop. VI.3.1.

Finally, let $C = C_{n,d}$ be the curve with function field $F = \bar{F}_{n,d} F_0$. Since $\bar{F} \cap \bar{F}_0 = K(x)$ (because the points P_{a_i} are totally ramified in $\bar{F}$ and are unramified in F_0), we see that C is the normalization of $\bar{C} \times_{\mathbb{P}^1} E$ and hence we have morphisms $f = f_{n,d} : C \to E$ and $\varphi : C \to \bar{C}$ of degrees n and 2, respectively, such that $\bar{f} \circ \varphi = \bar{\varphi} \circ f$. Note that $f : C \to E$ is a quasi-elliptic subcover because the points on E over P_{a_i} are totally ramified in F, so $f_{\overline{K}}$ cannot factor over a nontrivial isogeny of $E_{\overline{K}}$.

Since the cover $\bar{\varphi}_{\overline{K}} \circ f_{\overline{K}} : C_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$ is abelian of degree $2n$ and has $d + 1$ points of $\mathbb{P}^1$ which are ramified of degree n and 4 points which are ramified of degree 2, we see that the genus of C is $g_C = 1 + (n-1)(d+1)$. In particular, when $d = 1$, the curve $C = C_{n,1}$ is hyperelliptic (because $g_{\bar{C}} = 0$) of odd genus $g_C = 1 + 2(n-1)$, for $n = 2, 3, \ldots$, so we have a hyperelliptic example for every odd genus $g \ge 3$.

Proof of Proposition 2.3. Note first that this map is well-defined because if $f_1 \sim f_2$ are two equivalent quasi-elliptic subcovers, then $f_2 = \varphi \circ f_1$, for some isomorphism $\varphi : E_1 \xrightarrow{\sim} E_2$, and hence $f_2^* J_{E_2} = f_1^* \varphi^* J_{E_2} = f_1^* J_{E_1}$, so both define the same elliptic subgroup of J_C.

We first consider the case that C/K has a divisor $D \in \mathrm{Div}(C)$ of degree 1. Now if $f : C \to E$ is a quasi-elliptic subcover, then f is in fact an elliptic subcover because $f_* D \in \mathrm{Div}(E)$ is also a divisor of degree 1, and so $f_* D \sim P_0 \in E(K)$ by Riemann-Roch. Thus, E is an elliptic curve and f is an elliptic subcover.

We next show that Φ_C is bijective by constructing an inverse map $\Psi_C = \Psi_{C,D}$. For this, let $j_D : C \to J_C$ be the closed immersion given by $j_D(P) = cl(P - D)$, for $P \in C(\overline{K})$, and let $E \le J_C$ be an elliptic subgroup with inclusion $h_E : E \hookrightarrow J_C$. Put

$$\Psi(h_E) \;:=\; [-1]_E \circ j_E^{-1} \circ \hat{h}_E \circ \lambda_C \circ j_D : C \;\to\; E, \tag{1}$$

where $j_E = j_{0_E} : E \xrightarrow{\sim} J_E$ is the canonical isomorphism and $\lambda_C : J_C \to \hat{J}_C$ denotes the canonical polarization of J_C. Then we have that

$$\Psi(h_E)^* \;=\; h_E \circ j_E^{-1}. \tag{2}$$

To verify this, let $\lambda_E : J_E \xrightarrow{\sim} \hat{J}_E$ be the canonical theta-polarization, and let $\kappa_E : E \xrightarrow{\sim} \hat{\hat{E}} = \hat{J}_E$ be the duality isomorphism. Since $\kappa_E = -\lambda_E \circ j_E$, and $j_D^* = -\lambda_C$ (cf. [4], p. 45), we see that $\Psi(h_E)^* = j_D^* \hat{\lambda}_C \hat{\hat{h}}_E (j_E^*)^{-1} \circ [-1]_{J_E} = (-\lambda_C)\hat{\lambda}_C \kappa_{J_C} h_E \kappa_E^{-1}(\lambda_E) = -h_E \kappa_E^{-1} \lambda_E = h_E \circ j_E^{-1}$, as claimed.

We thus see that the rule $E \mapsto$ (equivalence class of $\Psi(h_E)$) defines a map $\Psi_C = \Psi_{C,D} : \mathcal{S}(J_C) \to \mathcal{E}(C) = \tilde{\mathcal{E}}(C)$. Indeed, (2) shows that $\Psi(h_E)^*$ is a closed immersion, so $\Psi(h_E) : C \to E$ is an elliptic subcover by Proposition 2.2. Moreover, the equivalence class of $\Psi(h_E)$ does not depend on the choice of h_E because if $h_E \circ \alpha : E \hookrightarrow J_C$ is another inclusion with image E, then $\Psi(h_E \circ \alpha) = [-1]_E \circ j_E^{-1} \circ \hat{\alpha} \circ \hat{h}_E \circ \lambda_C \circ j_D = (j_E^{-1} \circ \hat{\alpha} \circ j_E) \circ \Psi(h_E)$ is equivalent to $\Psi(h_E)$.

We now show that Φ_C and Ψ_C are inverses of each other. For this, suppose first that $E \le J_C$ is an elliptic subgroup. Then $\Phi(h_E)^* J_E = \Phi(h_E)^*(j_E(E)) = h_E(E) = E$ by (2), and so $\Phi_C(\Psi_C(E)) = E$.

On the other hand, if $f : C \to E$ is a quasi-elliptic subcover, then by the above $\exists P_0 \in E(K)$, so we have an isomorphism $j_E = j_{P_0} : E \xrightarrow{\sim} J_E$ defined by $P \mapsto cl(P - P_0)$. Then $h_E := f^* \circ j_E : E \hookrightarrow J_C$ is a closed immersion by Proposition 2.2, and $f_1 := \Psi(h_E) : C \to E$ is an elliptic subcover, as was shown above. Since $f_1^* \overset{(2)}{=} h_E \circ j_E^{-1} = f^*$, it follows that $\exists x \in E(K)$ such that $f_1 = T_x \circ f$ (cf. equations (47) and (48) of [4]) and so $f \sim f_1$. Thus $\Psi_C(\Phi_C(f)) =$ (class of f), which shows that the maps Φ_C and Ψ_C are inverses of each other, and so both are bijections.

It thus remains to show that for any C/K, the map Φ_C is always injective. For this, let $f_i : C \to E_i$ be two quasi-elliptic subcovers with $\Phi_C(f_1) = f_1^* J_{E_1} = f_2^* J_{E_2} = \Phi_C(f_2)$. Then also $\Phi_{C_{K'}}((f_1)_{K'}) = \Phi_{C_{K'}}((f_2)_{K'})$, for any extension K'/K. Choose K'/K Galois such that $C(K') \neq \emptyset$. Since $\Phi_{C_{K'}}$ is a bijection by what was shown above, we see that $(f_1)_{K'} \sim (f_2)_{K'}$, so $(f_2)_{K'} = \varphi' \circ (f_1)_{K'}$, for some isomorphism

$\varphi' : (E_1)_{K'} \xrightarrow{\sim} (E_2)_{K'}$. Now since $(f_1)_{K'}$ is surjective and hence an epimorphism, it follows that φ' is uniquely determined by $(f_1)_{K'}$ and $(f_2)_{K'}$, and hence φ' is invariant under $\mathrm{Gal}(K'/K)$. Thus, $\varphi' = \varphi_{K'}$ for some isomorphism $\varphi : E_1 \xrightarrow{\sim} E_2$, and then $f_2 = \varphi \circ f_1$. Thus, $f_1 \sim f_2$, which means that Φ_C is injective. □

We observe that the above proof actually yields the following slightly more precise assertion. For this, let

$$\mathcal{E}(C, E) = \{f : C \to E \text{ is an elliptic cover}\}$$

denote the set of elliptic subcovers of C/K with target a fixed elliptic curve E/K. Note that the group $E(K)$ of K-rational points of E/K acts on $\mathcal{E}(C, E)$ on the left via translations (i.e., via $(x, f) = T_x \circ f$).

Corollary 2.6. *If E/K is an elliptic curve, then the rule $f \mapsto f^*$ induces a injection*

$$\Phi_{C,E} : E(K)\backslash\mathcal{E}(C, E) \hookrightarrow \mathcal{I}(J_E, J_C),$$

where $\mathcal{I}(J_E, J_C)$ denotes the set of injective homomorphisms of J_E into J_C. Moreover, if C/K has a divisor of degree 1, then $\Phi_{C,E}$ is a bijection.

We next observe:

Corollary 2.7. (a) *If $\Phi_C(\tilde{\mathcal{E}}(C)) = \mathcal{S}(J_C)$, then every finite morphism $f : C \to E$ to a curve E/K of genus 1 factors over a quasi-elliptic subcover $f_1 : C \to E_1$. Moreover, the equivalence class of f_1 is uniquely determined by f.*

(b) *If $\Phi_C(\mathcal{E}(C)) = \mathcal{S}(J_C)$, then $\tilde{\mathcal{E}}(C) = \mathcal{E}(C)$, i.e., every quasi-elliptic subcover of C is an elliptic subcover. Moreover, a finite morphism $f : C \to E$ to an elliptic curve E/K is an elliptic subcover if and only if f does not factor over a non-trivial isogeny.*

Proof. (a) Let $A = \mathrm{Im}(f^*) \leq J_C$. Then A is an elliptic subgroup of J_C and $f^* = h_A \circ \nu$, where $\nu : J_E \to A$ is an isogeny and $h_A : A \hookrightarrow J_C$ the inclusion. Since Φ_C is a bijection, there exists a quasi-elliptic subcover $f_1 : C \to E_1$ such that $f_1^* J_{E_1} = A$, and so there is an isomorphism $\alpha : J_{E_1} \xrightarrow{\sim} A$ such that $f_1^* = h_A \circ \alpha$. Put $\nu' = \alpha^{-1} \circ \nu : J_E \to J_{E_1}$. Now choose a Galois extension K'/K such that $C(K') \neq \emptyset$. Then $E_{K'}$ and $(E_1)_{K'}$ are elliptic curves and there exists (after choosing origins) an isogeny $\nu_1 : (E_1)_{K'} \to E_{K'}$ such that $\nu_1^* = \nu'_{K'}$. Now consider $f'' = \nu_1 \circ (f_1)_{K'} : C_{K'} \to E_{K'}$. Since $(f'')^* = (f_1)^*_{K'} \circ \nu_1^* = f^*_{K'}$, there exists (as in the above proof) an $x \in E(K')$ such that $f_{K'} = T_x \circ f''$. Thus $f_{K'} = \varphi' \circ (f_1)_{K'}$, where $\varphi' = T_x \circ \nu_1 : (E_1)_{K'} \to E_{K'}$. Then (as in the above proof) we see that $\varphi' = \varphi_{K'}$ for a unique $\varphi : E_1 \to E$ and that $f = \varphi \circ f_1$. Thus f factors over the quasi-elliptic subcover f_1.

To verify the last assertion, note that if $f = \varphi_1 \circ f_2$ another such factorization, then $f_2^* J_{E_2} = f^* J_E = f_1^* J_{E_1}$, and so $f_1 \sim f_2$ by the injectivity of Φ_C.

(b) The first assertion follows immediately from the injectivity of Φ_C. Next, suppose that $f : C \to E$ does not factor over an isogeny. By part (a) we have a factorization $f = \nu \circ f_1$, where $f_1 : C \to E_1$ is a quasi-elliptic, hence an elliptic subcover of C. Since $\nu : E_1 \to E$ is, up to a translation, an isogeny, we have by our hypothesis on f that ν is an isomorphism, and hence $f : C \to E$ is an elliptic subcover. Since the converse implication is obvious, this proves the last assertion. $\square$

Remark 2.8. If $f : C \to E$ is a quasi-elliptic subcover, then $F_E := f^*\kappa(E)$ is a genus 1 subfield of $F = \kappa(C)$ which is *maximal* in the sense that F_E is not properly contained in any other genus 1 subfield of F. Thus, the map $f \mapsto f^*\kappa(E)$ induces an injection of $\mathcal{E}(C)$ into the set $\mathrm{Max}_1(F)$ of maximal genus 1 subfields of F. Moreover, this map is a bijection whenever Φ_C is a bijection, as the above Corollary 2.7 shows.

3 The hyperelliptic case

We now specialize the discussion of quasi-elliptic subcover to the case that C/K hyperelliptic curve of genus $g_C \geq 2$. Thus, C has a unique involution $\sigma_C \in \mathrm{Aut}(C)$ such that the quotient curve $\overline{C} := C/\langle\sigma_C\rangle$ has genus 0. (See [7] for various equivalent versions of the hyperellipticity condition.) We let $\pi_C : C \to \overline{C} = C/\langle\sigma_C\rangle$ denote the associated hyperelliptic subcover.

Note that if $g_C = 2$, then $\overline{C} \simeq \mathbb{P}^1_K$ because in that case the canonical morphism $\kappa_C : C \to \mathbb{P}^{g-1}_K = \mathbb{P}^1_K$ is surjective. However, for $g_C > 2$, it may happen that $\overline{C}(K) = \emptyset$.

The following result, due to Tamme[12], is a basic tool in the study of quasi-elliptic subcovers of hyperelliptic curves. Note that Kuhn proves only a much weaker statement in [6] (and only for $g_C = 2$ and $\mathrm{char}(K) \neq 2$).

Proposition 3.1 (Tamme). *If $f : C \to E$ is a finite morphism to a curve E of genus 1, then there exists a unique involution σ_f on E such that*

$$f \circ \sigma_C = \sigma_f \circ f. \tag{3}$$

Moreover, if $\pi_f : E \to E/\langle\sigma_f\rangle$ is the associated degree 2 subcover, then $\overline{E} := E/\langle\sigma_f\rangle$ has genus 0 and there is a unique morphism $\varphi_f : \overline{C} \to \overline{E}$ such that

$$\varphi_f \circ \pi_C = \pi_f \circ f. \tag{4}$$

Proof. For convenience of the reader, we present a proof in terms of covers (rather than in terms of function fields as in [12]).

First note that the last assertions are immediate consequences of (3). Indeed, (3) shows that $\pi_f \circ f$ is σ_C-invariant, and so by universal property of the quotient map π_C, there is a unique morphism $\varphi_f : \overline{C} \to \overline{E}$ satisfying (4). Moreover, by Lüroth we have that $g_{\overline{E}} = g_{\overline{C}} = 0$.

Next we observe that if σ_f exists, then it is uniquely determined by (3) because f is an epimorphism. Thus, it is enough to construct σ_f after a suitable Galois base-extension K'/K, for (as in the proof of Proposition 2.3) σ_f will be automatically Galois-invariant and hence defined over K.

Thus, we may assume (after replacing K by a suitable K') that $C(K) \neq \emptyset$. Fix $P \in C(K)$. Then $f(P) \in E(K)$, and so there exists an isomorphism $j = j_{f(P)} : E \xrightarrow{\sim} J_E$ such that $j(f(P)) = 0_{J_E}$. Then $j \circ f(P) = 0_{J_E}$ and so by the Albanese property of (C, j_P) there exists a unique homomorphism $h = h_f : J_C \to J_E$ such that $h \circ j_P = j \circ f$. Next we observe that

$$j_P \circ \sigma_C \;=\; T_x \circ [-1]_{J_C} \circ j_P \quad \text{with } x = j_P(\sigma_C(P)),$$

as is easy to check. Thus $j \circ f \circ \sigma_C = h \circ j_P \circ \sigma_C = h \circ T_x \circ [-1]_{J_C} \circ j_P = T_{h(x)} \circ h \circ [-1]_{J_C} \circ j_P = T_{h(x)} \circ [-1]_E \circ h \circ j_P = T_{h(x)} \circ [-1]_E \circ h \circ j_P = T_{h(x)} \circ [-1]_E \circ j \circ f = j \circ \sigma_f \circ f$, where $\sigma_f = j^{-1} \circ T_{h(x)} \circ [-1]_E \circ j$, and so (3) holds because j is an isomorphism. Note that the construction of σ_f shows that $\sigma_f^2 = 1$ (and $\sigma_f \neq 1$), so σ_f is an involution. □

As usual, the different divisor $W_C := \mathrm{Diff}(\pi_C)$ of the hyperelliptic cover is called the *Weierstrass divisor* of the hyperelliptic curve C/K. In [1] and [4], this divisor played a fundamental role in the analysis of elliptic subcovers of a genus 2 curve C, and this turns out to be the case here as well. A key result (which replaces many of the delicate arguments in [1], [4] and [6]) is the following simple fact.

Proposition 3.2. *If $f : C \to E$ is a finite separable morphism to a curve E/K of genus 1, then there exists an effective divisor $D_f \in \mathrm{Div}(E)$ of degree $2\deg(f) - g_C - 1$ such that*

(5) $$f_*W_C \;=\; f_*\mathrm{Diff}(\pi_C) \;=\; \deg(f)\mathrm{Diff}(\pi_f) - 2D_f,$$

where π_f is as in Proposition 3.1 *and* $\mathrm{Diff}(\pi_f) \in \mathrm{Div}(E)$ *is its different. Moreover,* $\mathrm{supp}(f_*W_C) \subset \mathrm{supp}(\mathrm{Diff}(\pi_f))$.

To prove this, we will use the following general fact.

Lemma 3.3. *Let $f_i : C \to C_i$ and $f_i' : C_i \to C'$, $i = 1, 2$ be four separable curve covers such that $f_1' \circ f_1 = f_2' \circ f_2$ and $\deg(f_1) = \deg(f_2')$. If f_1 and f_2 are disjoint, i.e., if $f_1^*\kappa(C_1) f_2^*\kappa(C_2) = \kappa(C)$, then we have that*

(6) $$(f_2')^*(f_1')_*D_1 \;=\; (f_2)_*f_1^*D_1, \quad \textit{for all } D_1 \in \mathrm{Div}(C_1).$$

Moreover, there is an effective divisor $D_2 \in \mathrm{Div}(C_2)$ such that

(7) $$(f_2')^*(f_1')_*\mathrm{Diff}(f_1') \;=\; (f_2)_*\mathrm{Diff}(f_2) + 2D_2,$$

and hence we have

(8) $$(f_2)_*\mathrm{Diff}(f_1) \;=\; \deg(f_2)\mathrm{Diff}(f_2') - 2D_2.$$

Proof. It clearly suffices to verify (6) in the case that K is algebraically closed. Let $\tilde{f} : \tilde{C} \to C'$ be a Galois cover which factors over $f := f_1' \circ f_1 = f_2' \circ f_2$, so $\tilde{f} = f \circ \varphi$, for some $\varphi : \tilde{C} \to C$. Put $\tilde{f}_i = f_i \circ \varphi$ and write $H_i = \mathrm{Aut}(\tilde{f}_i)$, $G = \mathrm{Aut}(\tilde{f})$ and $H = \mathrm{Aut}(\varphi)$. By the disjointness and degree conditions (and Galois theory) it follows that $H = H_1 \cap H_2$ and $G = \langle H_1, H_2 \rangle$. Moreover, since $|G| = d_1 d_2 d$, where $d_i = \deg(f_i)$ and $d = \deg(\varphi)$ and since $|H_1 H_2| = \frac{|H_1||H_2|}{|H_1 \cap H_2|} = \frac{(dd_1)(dd_2)}{d} = dd_1 d_2$, we have that $G = H_1 H_2 = H_2 H_1$. Thus, for any $\tilde{D} \in \mathrm{Div}(\tilde{C})$, we obtain

$$(\tilde{f}_2)^*(\tilde{f}_2)_*(\tilde{f}_1)^*(\tilde{f}_1)_*\tilde{D} = \sum_{h \in H_2} h_* \sum_{h' \in H_1} (h')_*\tilde{D} = \sum_{h \in H_2} \sum_{h' \in H_1} (hh')_*\tilde{D} = d \sum_{g \in G} g_*\tilde{D} = d\tilde{f}^*\tilde{f}_*\tilde{D}.$$

Taking $\tilde{D} = \tilde{f}_1^* D_1$ yields $(\tilde{f}_2)^*(\tilde{f}_2)_*(\tilde{f}_1)^*(dd_1 D_1) = (\tilde{f}_2)^*(\tilde{f}_2)_*(\tilde{f}_1)^*(\tilde{f}_1)_*(\tilde{D}) = d\tilde{f}^*\tilde{f}_*\tilde{D} = d\tilde{f}^*(f_1')_*(dd_1 D_1)$. Thus, since $(\tilde{f}_2)^*$ is injective, it follows from this that $(\tilde{f}_2)_*(\tilde{f}_1)^*(D_1) = d(f_2')^*(f_1')_*(D_1)$. This clearly implies (6) because $(\tilde{f}_2)_*(\tilde{f}_1)^*(D_1) = (f_2)_*\varphi_*\varphi^*(f_1)^*(D_1) = d(f_2)_*(f_1)^*(D_1)$.

To verify (7), let $U' = \mathrm{Spec}(A)$ be an open affine subset of C' which contains all the ramification points of f, and put $U_i = f_i^{-1}(U')$ and $U = f^{-1}(U')$. Then $U_i = \mathrm{Spec}(A_i)$ and $U = \mathrm{Spec}(B)$, where A_i and B are the integral closures of A in $F_i = \kappa(C_i)$ and in $F = \kappa(C)$, respectively. Since F_1 and F_2 are linearly disjoint over $F' = \kappa(C')$ (as we saw above), it follows that $F_1 \otimes_{F'} F_2 \simeq F_1 F_2 = F$. Thus, $A' := A_1 \otimes_A A_2$ is an A_2-sublattice of B, and so by [10], p. 49, we have that their discriminants are related by the formula

$$\mathfrak{d}_{A'/A_2} \;=\; \mathfrak{d}_{B/A_2}\mathfrak{a}^2,$$

where $\mathfrak{a} = \chi(A', B) \neq 0$ is some A_2-ideal. From the definition of the discriminant we see that $\mathfrak{d}_{A_1 \otimes A_2/A_2} = \mathfrak{d}_{A_1/A} A_2$, and so we obtain by [10], p. 50, the relation

$$N(\mathfrak{D}_{A_1/A})A_2 \;=\; N(\mathfrak{D}_{B/A_2})\mathfrak{a}^2,$$

where $N()$ denotes the norm and $\mathfrak{D}_{A_1/A}$ the different. Since this is just the ideal-theoretic version of (7), we see that assertion (7) has been proved.

Finally, to verify (8), note first that $\mathrm{Diff}(f_1) = \mathrm{Diff}(f_2) + f_2^*\mathrm{Diff}(f_2') - f_1^*\mathrm{Diff}(f_1')$ because $\mathrm{Diff}(f_1) + f_1^*\mathrm{Diff}(f_1') = \mathrm{Diff}(f_1' \circ f_1) = \mathrm{Diff}(f_2' \circ f_2) = \mathrm{Diff}(f_2) + f_2^*\mathrm{Diff}(f_2')$. Thus

$$\begin{aligned} (f_2)_*\mathrm{Diff}(f_1) &\overset{(6)}{=} (f_2)_*\mathrm{Diff}(f_2) + n_2\mathrm{Diff}(f_2') - (f_2')^*(f_1')_*\mathrm{Diff}(f_1') \\ &\overset{(7)}{=} (f_2)_*\mathrm{Diff}(f_2) + n_2\mathrm{Diff}(f_2') - ((f_2)_*\mathrm{Diff}(f_2) + 2D_2), \end{aligned}$$

which proves (8). □

Proof of Proposition 3.2. We apply Lemma 3.3 with $f_1 = \pi_C$, $f_1' = \varphi_f$, $f_2 = f$, $f_2' = \pi_f$, where φ_f and π_f are as in (4). By Lüroth, $\kappa(E) \not\subset \kappa(\overline{C})$, so π_C and f are disjoint. Thus, the hypotheses of Lemma 3.3 hold, and so by (8) there exists a divisor $D \geq 0$ such that (5) holds. Since $\deg(f_*W_C) = \deg(W_C) = 2g_C + 2$ and $\deg(\mathrm{Diff}(\pi_f)) = 4$, it follows from (5) that $\deg(D) = \frac{1}{2}(4\deg(f) - (2g_C + 2)) = (2\deg(f) - 1) - g_C$.

To prove the last assertion, note that if $P \in \mathrm{supp}(f_*(W_C))$, then there exists $P' \in \mathrm{supp}(W_C)$ such that $f(P') = P$. Then $\sigma_C(P') = P'$ and so $\sigma_f(P) = \sigma_f(f(P')) = f(\sigma(P')) = f(P') = P$, which implies that $P \in \mathrm{supp}(\mathrm{Diff}(\pi_f))$, as claimed. □

Remark 3.4. By using the above Proposition 3.2, it is possible to give a quick proof of the "Normalization Lemma" of [4], without recourse to the (more complicated) results of [1] and [6]. More precisely, if $\mathrm{char}(K) \neq 2$ and if $f : C \to E$ is an elliptic subcover of degree n of a genus 2 curve C, then there exists a unique $P_f \in E(K)$ such that

$$f_*W_C = (2,n)\mathrm{Diff}(\pi_f) - (-1)^n 2P_f. \tag{9}$$

Indeed, since P_f is uniquely determined by (9), it is enough to verify this over a sufficiently large Galois extension K'/K, and so we may assume that $W_C = Q_1 + \ldots + Q_6$ and $\mathrm{Diff}(\pi_f) = P_1 + \ldots + P_4$ consist of 6 and 4 distinct rational points, respectively. Then by (5) we see that

$$v_{P_i}(f_*W_C) \equiv n \pmod 2, \quad \text{for } 1 \leq i \leq 4.$$

Thus, if n is odd, then (9) follows immediately because we then have that $v_{P_i}(f_*W_C) \geq 1$ for all i, and so there can be at most one P_i with $v_{P_i}(f_*W_C) \geq 3$. Thus $f_*W_C = 3P_i + \sum_{j\neq i} P_j = 2P_i + \mathrm{Diff}(\pi_f)$. If n is even, then a similar analysis shows that (9) holds because we have that $v_{P_i}(f_*W_C) \leq 3$, for all i. (To see this, suppose that $v_{P_i}(f_*W_C) \geq 4$. Then there exist (say) $Q_1, \ldots, Q_4 \in f^{-1}(P_i)$, and then $cl(Q_k - Q_1) \in \mathrm{Ker}(f_*)[2]$, for $k = 1, \ldots, 4$. Since $\mathrm{Ker}(f_*) \simeq E'$ is an elliptic curve, this forces $Q_4 - Q_1 \sim Q_3 - Q_1 + Q_2 - Q_1$. Thus $Q_1 + Q_4 \sim Q_2 + Q_4$, which is impossible since $h^0(Q_1 + Q_4) = 1$.)

4 Normalized subcovers

In [4], the notion of a *normalized subcover* was introduced in order to rigidify elliptic subcovers of genus 2 curves (if $\mathrm{char}(K) \neq 2$). Here we generalize this notion to higher genus (hyperelliptic) curves (and also include the case of characteristic 2).

Definition 4.1. An elliptic cover $f : C \to E$ of a hyperelliptic curve C is called *pre-normalized* if $\sigma_f = [-1]_E$, i.e., if

$$[-1]_E \circ f \;=\; f \circ \sigma_C. \tag{10}$$

Moreover, f is called *max-normalized* if f is pre-normalized and if

$$v_{0_E}(f_*W_C) > v_P(f_*W_C)\deg_i(P), \quad \forall P \in \mathrm{supp}(\mathrm{Diff}(\pi_E)) \setminus \{0_E\}, \tag{11}$$

where $\pi_E : E \to E/\langle[-1]_E\rangle \simeq \mathbb{P}^1_K$ is the quotient map, and $\deg_i(P) = [\kappa(P) : K]_i$ denotes the inseparable degree of P. (Thus, $\deg_i(P) = 1$ if K is perfect.) Similarly, f is called *min-normalized* if f is pre-normalized and if

$$v_{0_E}(f_*W_C) < v_P(f_*W_C)\deg_i(P), \quad \forall P \in \mathrm{supp}(\mathrm{Diff}(\pi_E)) \setminus \{0_E\}. \tag{12}$$

Finally, an elliptic cover is called *normalized* if it is either max-normalized or min-normalized.

We observe the following basic properties of normalized subcovers.

Proposition 4.2. *Let $f : C \to E$ be an elliptic K-subcover, and let K'/K be an extension field. Then f is pre-normalized or max-normalized or min-normalized if and only if $f_{K'}$ has the corresponding property.*

Proof. By properties of faithfully flat base-change, the assertion about pre-normalized subcovers is clear. To prove the rest of the assertions, it suffices to verify them in the case that K' is algebraically closed. Indeed, suppose that the assertions have been proven in this case, and let K'' be an algebraic closure of K'. Then from our supposition (applied to K''/K and to K''/K') it follows that f is max-normalized if and only if $f_{K''} = (f_{K'})_{K''}$ is max-normalized if and only if $f_{K'}$ is max-normalized. The argument for min-normalized is similar.

Thus, assume that K' is algebraically closed, and let $\beta = \beta_{K'/K} : C_{K'} \to C$ be the base-change map. If $D_{K'} = \beta^* D$ is the pullback of a divisor $D \in \mathrm{Div}(E)$, then

$$v_{P'}(D_{K'}) = v_{\beta(P)}(D)\deg_i(\beta(P')), \quad \text{for all } P' \in C(K'), \tag{13}$$

$$\mathrm{supp}(D_{K'}) = \beta^{-1}(\mathrm{supp}(D)). \tag{14}$$

Thus, since β is surjective and since $\beta^{-1}(0_E) = 0_{E_{K'}}$, it is clear from the definitions (and the fact that $(f_* W_C)_{K'} = (f_{K'})_* W_{C_{K'}}$) that (11) or (12) holds for f if and only if the corresponding property holds for $f_{K'}$. □

Proposition 4.3. *Let $f_i : C \to E$ be two elliptic subcovers with $f_1^* = f_2^*$.*

(a) *If f_1 and f_2 are both pre-normalized, then there exists a unique $x \in E[2](K)$ such that $f_1 = T_x \circ f_2$.*

(b) *If f_1 and f_2 are both max-normalized (or both are min-normalized), then $f_1 = f_2$.*

Proof. (a) Since $f_1^* = f_2^*$, there exists $x \in E(K)$ such that $f_1 = T_x \circ f_2$; cf. Corollary 2.6. (Note that T_x and hence x is unique because f_2 is an epimorphism.) Then $[-1]_E \circ T_x \circ f_2 = [-1]_E \circ f_1 = f_1 \circ \sigma_C = T_x \circ f_2 \circ \sigma_C = T_x \circ [-1]_E \circ f_2$, and so $[-1]_E \circ T_x = T_x \circ [-1]$ because f_2 is an epimorphism. Thus $T_{-x} = T_x$, so $2x = 0$ or $x \in E[2](K)$.

(b) By part (a) there exists $x \in E[2](K)$ such that $f_1 = T_x \circ f_2$. Note that $x \in \mathrm{supp}(\mathrm{Diff}(\pi_E))$. Thus, if $x \neq 0_E$, then $v_{0_E}((f_2)_* W_C) > v_x((f_2)_* W_C)$. Now $(f_1)_* W_C = (T_x)_*(f_2)_* W_C$, so $v_{0_E}((f_1)_* W_C) = v_x((f_2)_* W_C) < v_{0_E}((f_2)_* W_C) = v_x((f_1)_* W_C)$. Since this contradicts the hypothesis that f_1 is max-normalized, we must have $x = 0$, and so $f_1 = f_2$. The proof for min-normalized maps is similar. □

Thus, max/min-normalized subcovers are uniquely determined by their induced maps on the Jacobians. This fact allows us to descend max/min-normalized subcovers as follows.

Corollary 4.4. *Let $E \leq J_C$ be an elliptic subgroup with embedding $h : E \hookrightarrow J_C$. If there exists a Galois extension K'/K and a normalized cover $f' : C_{K'} \to E_{K'}$ such that $(f')^* \circ j_{E_{K'}} = h_{K'}$, then there exists a normalized cover $f : C \to E$ such that $f^* \circ j_E = h$.*

Proof. By Galois descent, it is enough to verify that $(f')^\sigma = f'$, for all $\sigma \in \mathrm{Gal}(K'/K)$. Now $((f')^\sigma)^* = (f')^*$ because $((f')^\sigma)^* \circ j_{E_{K'}} = ((f')^* \circ j_{E_{K'}})^\sigma = (h_{K'})^\sigma = h_{K'} = (f')^* \circ j_{E_{K'}}$ (and $j_{E_{K'}}$ is an isomorphism). Thus, the assertion follows from Proposition 4.3(b) once we have shown that $(f')^\sigma$ is also max/min-normalized.

For this, note first that $(f')^\sigma$ is pre-normalized because $[-1]_{E_{K'}} \circ (f')^\sigma = ([-1]_{E_{K'}} \circ f')^\sigma = (f' \circ \sigma_{C_{K'}})^\sigma = (f')^\sigma \circ \sigma_{C_{K'}}$. Now since $(W_{C_{K'}})^\sigma = W_{C_{K'}}$, we see that $(f')^\sigma_*(W_{C_{K'}}) = ((f')_*(W_{C_{K'}}))^\sigma$. Thus, if f' is max-normalized, i.e., if $v_{0_{E_{K'}}}((f')_* W_{C_{K'}}) > v_P((f')_* W_{C_{K'}}) \deg_i(P)$, for all $P \in \mathrm{supp}(\mathrm{Diff}(\pi_{E_{K'}})) \setminus \{0_{E_{K'}}\}$, then also $v_{0_{E_{K'}}}(((f')_* W_{C_{K'}})^\sigma) > v_P(((f')_* W_{C_{K'}})^\sigma) \deg_i(P)$, for all $P \in \mathrm{supp}(\mathrm{Diff}(\pi_{E_{K'}})) \setminus \{0_{E_{K'}}\}$, because $0^\sigma_{E_{K'}} = 0_{E_{K'}}$ and $\mathrm{supp}(\mathrm{Diff}(\pi_{E_{K'}}))^\sigma = \mathrm{supp}(\mathrm{Diff}(\pi_{E_{K'}}))$. Thus $(f')^\sigma$ is also max-normalized, and so the assertion follows. The proof for f' min-normalized is analogous. □

We now turn to investigate the *existence* of normalized subcovers (inside a given equivalence class of subcovers). For this, we first consider the existence of pre-normalized subcovers.

Proposition 4.5. *Let $f : C \to E$ be an elliptic subcover. Then the following conditions are equivalent:*

(i) $\exists x \in E(K)$ *such that $T_x \circ f$ is pre-normalized.*

(ii) $\exists x \in E(K)$ *such that $\sigma_f = T_x^{-1} \circ [-1]_E \circ T_x$.*

(iii) $\mathrm{supp}(\mathrm{Diff}(\pi_f)) \cap E(K) \neq \emptyset$.

Proof. (ii) ⇒ (i): By (ii) and (10) we have that $[-1]_E \circ T_x \circ f = (T_x \circ \sigma_f \circ T_x^{-1}) \circ T_x \circ f = T_x \circ f \circ \sigma_C$, so $T_x \circ f$ is pre-normalized.

(i) ⇒ (iii): By (i) and (10) we have $[-1]_E \circ T_x \circ f = T_x \circ f \circ \sigma_C = T_x \circ \sigma_f \circ f$, and so $[-1]_E \circ T_x = T_x \circ \sigma_f$ because f is an epimorphism. Thus $\sigma_f(-x) = T_x^{-1}([-1]_E(T_x(-x))) = T_{-x}(0_E) = -x$, and so $-x \in \mathrm{Diff}(\pi_f) \cap E(K)$.

(iii) ⇒ (ii): Let $P \in \mathrm{supp}(\mathrm{Diff}(\pi_f)) \cap E(K)$, so $\sigma_f(P) = P$. Put $\sigma_1 = T_P^{-1} \sigma_f \circ T_P$. Then $\sigma_1(0_E) = T_P^{-1}(\sigma_f(P)) = 0_E$, so $\sigma_1 \in \mathrm{Aut}(E)$. Now since $\sigma_f^* = [-1]_{J_E}$ (as is clear from the construction of σ_f in the proof of Proposition 3.1), we thus have that $\sigma_1^* = \sigma_f^* = [-1]_{J_E} = [-1]_E^*$, and so $\sigma_1 = [-1]_E$. Thus (ii) holds with $x = -P$. □

Remark 4.6. Note that unless $E(K)$ is 2-divisible, not every involution σ of E satisfies condition (ii) of Proposition 4.5. Indeed, for any $y \in E(K)$, the element $\sigma_y = T_y \circ [-1]_E$ is clearly an involution of E, but $\sigma_y = T_x^{-1} \circ [-1]_E \circ T_x$ if and only if $y = -2x$. Thus, the conditions of the Proposition 4.5 are not automatically satisfied.

Corollary 4.7. *If $f : C \to E$ be an elliptic subcover, then there exists a $y \in E(K)$ such that $T_y \circ f$ is max-normalized (respectively, min-normalized) if and only if there exists an $x \in \mathrm{supp}(\mathrm{Diff}(\pi_f)) \cap E(K)$ such that*

(15) $v_x(f_* W_C) > v_P(f_* W_C) \deg_i(P)$, *respectively,* $v_x(f_* W_C) < v_P(f_* W_C) \deg_i(P)$,

for all $P \in \mathrm{supp}(\mathrm{Diff}(\pi_f)) \setminus \{x\}$.

Proof. If $T_y \circ f$ is max-normalized, then by (the proof of) Proposition 4.5 we know that $x := -y \in \mathrm{supp}(\mathrm{Diff}(\pi_f)) \cap E(K)$ and that $\sigma_f = T_y^{-1} \circ [-1]_E \circ T_y$. Thus $\pi_E \circ T_y \circ \sigma_f = \pi_E \circ [-1]_E \circ T_y = \pi_E \circ T_y$ and so, by the universal property of quotients, there exists an isomorphism $\alpha : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$ such that $\alpha \circ \pi_f = \pi_E \circ T_y$. Thus $\mathrm{Diff}(\pi_f) = \mathrm{Diff}(\pi_E \circ T_y) = T_y^* \mathrm{Diff}(\pi_E)$, and so $\mathrm{supp}(\mathrm{Diff}(\pi_f)) = T_y^{-1}(\mathrm{supp}(\mathrm{Diff}(\pi_E))) = T_x(\mathrm{supp}(\mathrm{Diff}(\pi_E)))$. Moreover, since $(T_y f)_* W_C = (T_x^{-1})_* f_* W_C = T_x^* f_* W_C$, we have that

$$v_Q((T_y f)_* W_C) \deg_i(Q) \;=\; v_{T_x(Q)}(f_* W_C) \deg_i(T_x(Q)), \quad \forall \text{ prime divisors } Q \text{ on } E.$$

Thus, taking $Q = x$ and $Q = P \in \mathrm{supp}(\mathrm{Diff}(\pi_f)) \setminus \{x\} = T_x(\mathrm{supp}(\mathrm{Diff}(\pi_E)) \setminus \{0_E\})$, it follows from the fact that $T_y f$ is max-normalized that $v_x(f_* W_C) = v_{0_E}((T_y f)_* W_C) > v_{T_x^{-1} P}((T_y f)_* W_C) = v_P(f_* W_C)$, and so the first inequality of (15) holds. Similarly, if $T_y f$ is min-normalized, then the second inequality of (15) holds.

Conversely, suppose that there exists $x \in \mathrm{supp}(\mathrm{Diff}(\pi_f)) \cap E(K)$ such that the first inequality of (15) holds for all $P \in \mathrm{supp}(\mathrm{Diff}(\pi_f)) \setminus \{x\}$. Then by Proposition 4.5 we know that if $y = -x$, then $T_y \circ f$ is pre-normalized and the previous discussion shows that $T_y \circ f$ is max-normalized. Similarly, if the second equality holds, then $T_y \circ f$ is min-normalized. □

We now come to the main result of this section.

Theorem 4.8. *Suppose that g_C is even. If $f : C \to E$ is an elliptic subcover, then there exists an $x \in E(K)$ such that $T_x \circ f$ is normalized.*

Proof. We first note that it sufficies to verify this fact after Galois base change K'/K. Indeed, suppose that there exists $x \in E(K')$ such that $f' := T_x \circ f_{K'}$ is normalized. Then $(f')^* = (f_{K'})^*$, so $(f')^* \circ j_{E_{K'}} = (f^* \circ j_E)_{K'}$, and hence by Corollary 4.4 there is a normalized cover $f_1 : C \to E$ such that $(f_1)_{K'} = f'$. Now $(f_1^*)_{K'} = (f')^* = f_{K'}^*$ and so by Galois decent $f_1^* = f^*$. Thus, by the equivalences (47) and (48) of [4] it follows that there exists $x \in E(K)$ such that $f_1 = T_x \circ f$, which proves the assertion.

Thus, we may assume that all prime divisors appearing in W_C and in $\mathrm{Diff}(\pi_f)$ are purely inseparable because this is the case after a suitable Galois extension. We now distinguish cases.

Case 1: $\mathrm{char}(K) \neq 2$.

Since π_f and π_C are tame extensions of degree 2, we see that $\mathrm{Diff}((\pi_f)_{\overline{K}})$ and $\mathrm{Diff}((\pi_C)_{\overline{K}})$ are reduced and so all prime divisors of $\mathrm{Diff}(\pi_f)$ and of $\mathrm{Diff}(\pi_C)$ are separable. Thus, by our hypothesis on K we have that $\mathrm{Diff}(\pi_f) = \sum_{i=1}^4 P_i$ where the $P_i \in E(K)$ are distinct. Thus by Proposition 3.2 we see that

$$f_* W_C \;=\; \sum_{i=1}^{4} n_i P_i$$

for some integers $n_i \geq 0$ with $\sum n_i = \deg f_* W_C = 2g_C + 2$. By renumbering the P_i's if necessary, we may assume that $n_1 \geq n_2 \geq n_3 \geq n_4$. Now suppose that it could happen that $n_1 = n_2$ and $n_3 = n_4$. But then, since $n_i \equiv \deg(f) \pmod 2$ by (5), it follows that $n_1 + n_3 \equiv 2\deg(f) \equiv 0 \pmod 2$ and so $2g_C + 2 = 2n_1 + 2n_3 \equiv 0 \pmod 4$, which contradicts the fact that $g_C \equiv 0 \pmod 2$.

Thus we must have that either $n_1 > n_2$ or that $n_4 < n_3$. In the former case we see that the first condition of (15) holds for $x = P_1$ and in the latter case we see that the second condition of (15) holds for $x = P_4$. Thus, by Corollary 4.7 there exists $x \in E(K)$ such that $T_x \circ f$ is normalized.

Case 2: $\mathrm{char}(K) = 2$, E supersingular.

Since here $E[2](\overline{K}) = \{0_{E_{\overline{K}}}\}$, it follows that $\mathrm{Diff}(\pi_{E_{\overline{K}}}) = 4(0_{E_{\overline{K}}})$, which implies that $\mathrm{Diff}((\pi_f)_{\overline{K}}) = 4\overline{P}$, for some $\overline{P} \in E(\overline{K})$. (Indeed, since $E(\overline{K})$ is 2-divisible, there exists $\overline{P} \in E(\overline{K})$ such that $T_{-\overline{P}} \circ f_{\overline{K}}$ is pre-normalized (cf. Remark 4.6 and Proposition 4.5) and then $\mathrm{Diff}((\pi_f)_{\overline{K}}) = T^*_{-\overline{P}}(\mathrm{Diff}(\pi_{E_{\overline{K}}})) = 4\overline{P}$ by the proof of Corollary 4.7.)

It therefore follows that $\mathrm{Diff}(\pi_f) = eP$, where P is a prime divisor of E and $e\deg(P) = e\deg_i(P) = 4$. Thus, by Proposition 3.2 we see that $f_*W_C = \frac{2g_C+2}{\deg(P)}P$, and so, since g_C+1 is odd, it follows that $\deg(P)|2$. Now if $\deg(P) = 2$ then $\mathrm{Diff}(\pi_f) = 2P$ and $f_*W_C = (g_C+1)P$, so by (5) we obtain that $(g_C+1)P = \deg(f)(2P) - 2D$, which is impossible since $g_C + 1$ is odd. Thus $\deg(P) = 1$, so $P \in E(K)$. Since condition (15) holds vacuously here (because $\mathrm{supp}(\mathrm{Diff}(\pi_f)) = \{P\}$), we see by Corollary 4.7 that $T_{-P} \circ f$ is both max-normalized and min-normalized.

Case 3: $\mathrm{char}(K) = 2$, E ordinary.

Since here $E[2](\overline{K}) = \{0_{E_{\overline{K}}}, \overline{P}_1\}$, it follows that $\mathrm{Diff}(\pi_{E_{\overline{K}}}) = 2(0_{E_{\overline{K}}}) + 2\overline{P}_1$. By the same argument as in Case 2 we see thus see that $\mathrm{Diff}((\pi_f)_{\overline{K}}) = 2\overline{P}'_0 + 2\overline{P}'_1$, for some (distinct) $\overline{P}'_0, \overline{P}'_1 \in E(\overline{K})$. Thus $\mathrm{Diff}(\pi_f) = e_0P_0 + e_1P_1$, where P_0, P_1 are distinct prime divisors of E with $e_k\deg(P_k) = e_k\deg_i(P_k) = 2$, for $k = 0, 1$.

Suppose first that $\deg(P_k) = 2$, for $k = 0$ and $k = 1$. Then $\mathrm{Diff}(\pi_f) = P_0 + P_1$ and so $f_*W_C = n_0P_0 + n_1P_1$. Comparing degrees yields $n_0 + n_1 = g_C + 1$. Put $m_k = v_{P_k}(D)$, where D is as in (5). Then by (5) we have that $n_k = \deg(f) - 2m_k$, for $k = 0, 1$, and so $g_C+1 = n_0+n_1 = 2\deg(f)-2(m_0+m_1)$, which is impossible since g_C is even. Thus, there exists at least one P_k with $\deg(P_k) = 1$, say (after renumbering) $\deg(P_0) = 1$. Thus $\mathrm{Diff}(\pi_f) = 2P_0 + e_1P_1$ and $P_0 \in \mathrm{supp}(\mathrm{Diff}(\pi_f)) \cap E(K)$. Write, as before, $f_*W_C = n_0P_0 + n_1P_1$, so $n_0 + n_1\deg(P_1) = 2g_C + 2$.

We now claim that $n_0 \neq n_1\deg(P_1)$. Indeed, if this is the case, then $n_0 = n_1\deg(P_1) = g_C + 1$. But then from (5) we obtain that $g_C + 1 = n_0 = \deg(f)(2) - 2v_{P_0}(D)$, which is impossible. Thus $n_0 \neq n_1\deg(P_1)$, and so either the first or second inequality of (15) holds, and so by Corollary 4.7 there exists $x = -P_0$ such that $T_x \circ f$ is either max-normalized or min-normalized. □

In the case of genus 2 one can say more. The following result generalizes Proposition 2.2 of [4] to the case of characteristic 2, and also gives a different proof of it[2] when $\mathrm{char}(K) \neq 2$.

[2]There is an unfortunate typo in formula (4) of [4] which was corrected in [5].

Theorem 4.9. *Let C be a curve of genus $g_C = 2$, and let $f : C \to E$ be an elliptic subcover of degree N. Then there exists a unique $P_f \in E(K)$ such that*

$$f_*W_C = (2, N)\mathrm{Diff}(\pi_f) - (-1)^N(2P_f). \tag{16}$$

Thus, if N is odd, then there exists a unique $x \in E(K)$ such that $T_x \circ f$ is max-normalized, and if N is even, then there exists a unique $x \in E(K)$ such that $T_x \circ f$ is min-normalized.

Proof. First note that the last assertions follow immediately from (16) and Corollary 4.7 by taking $x = -P_f$. Thus, it is enough to verify (16).

Now if $\mathrm{char}(K) \neq 2$, then (16) was established in Remark 3.4. Thus, assume that $\mathrm{char}(K) = 2$. If E is supersingular, then by the proof of Theorem 4.8 we see that there exists an $x \in E(K)$ such that $T_x \circ f$ is normalized. Thus, since $\mathrm{Diff}(\pi_E) = 4(0_E)$, we see that $\mathrm{Diff}(\pi_f) = 4P_f$ for a unique $P_f \in E(K)$ and that $f_*W_C = 6P_f$. Thus, (16) holds in this case because $6P_f = 4P_f + 2P_f = 2(4P_f) - 2P_f$.

Now suppose that E is ordinary, so $|E[2](\overline{K})| = 2$. We first verify (16) in the case that $K = \overline{K}$ is algebraically closed. Clearly, (16) follows from the following more precise statement.

Claim: If $K = \overline{K}$, then $\mathrm{Diff}(f) = 2Q_1$ with $Q_1 \in C(K)$. If $P_1 = f(Q)$, then $\mathrm{Diff}(\pi_f) = 2P_1 + 2P_2$ with $P_2 \in E(K)$, and $P_2 \neq P_1$. Furthermore,

$$\#(f^{-1}(P_2) \cap \mathrm{supp}(W_C)) = (2, N) \quad\text{and}\quad \#(f^{-1}(P_1) \cap \mathrm{supp}(W_C)) = w - (2, N), \tag{17}$$

where $w = \#\mathrm{supp}(W_C)$, and hence

$$(f_*W_C) \;=\; (2, N)\mathrm{Diff}(\pi_f) - (-1)^N(2P_1). \tag{18}$$

To verify this, note first that since $\deg(\mathrm{Diff}(f)) = 2$, we have $e_Q \leq 3$ for all $Q \in \mathrm{supp}(\mathrm{Diff}(f))$. Moreover, if $e_Q = 3$, for one Q, then $\mathrm{Diff}(f) = 2Q$ and if $e_Q = 2$, for one Q, then due to wild ramification we have $2Q \leq \mathrm{Diff}(f)$ and so $\mathrm{Diff}(f) = 2Q$ in this case as well. This proves the first statement of the claim.

Next we observe that since $\mathrm{Diff}(f)$ is σ_C-invariant, so is Q_1, and hence $Q_1 \in \mathrm{supp}(W_C)$. Thus $P_1 \in \mathrm{supp}(\mathrm{Diff}(\pi_f))$. Now the Deuring-Shafarevich formula (cf. [8]) applied to π_f shows that $\#\mathrm{supp}(\mathrm{Diff}(\pi_f)) = \gamma_E + 1 = 2$, where γ_E is the Hasse-Witt rank (or 2-rank) of E. Thus, $\mathrm{Diff}(\pi_f) = 2P_1 + 2P_2$ for some $P_2 \neq P_1$, which proves the second statement of the claim.

To verify (17), note first that second equality of (17) is clearly a consequence of the first equality. To prove it, we first observe that since $f^{-1}(P_2)$ is a $\langle\sigma_C\rangle$-set of cardinality N, we have that $\#(f^{-1}(P_2) \cap \mathrm{supp}(W_C)) \equiv N \pmod 2$. Now clearly $w \leq 3$ (since $\deg(W_C) = 6$ and all points in W_C have multiplicity ≥ 2), so the first equality follows once we have shown that $1 \leq \#(f^{-1}(P_2) \cap \mathrm{supp}(W_C)) \leq 2$ or, equivalently, that $\mathrm{supp}(W_C) = \{Q_1, \ldots Q_w\}$ cannot lie completely in a single fibre of f. Indeed, if this were the case, then $cl(Q_i - Q_j) \in \mathrm{Ker}(f_*)[2](K)$, $\forall i, j$. Now since $E' := \mathrm{Ker}(f_*)$ is an elliptic curve and $J_C \sim E \times E'$, we see that $|\mathrm{Ker}(f_*)[2](K)| =$

$2^{\gamma_C-1} < 2^{\gamma_C} = |J_C[2](K)|$, where γ_C denotes the 2-rank of J_C. We thus obtain a contradiction, because the classes $cl(Q_i - Q_j)$ generate $J_C[2](K)$. This proves (17).

We now deduce (18) from (17). Consider first the case that $w = 3$. Then $W_C = 2Q_1 + 2Q_2 + 2Q_3$, so from (17) it follows that $f_*W_C = 2((3-(2,N))P_1 + (2,N)P_2) = (2,N)\mathrm{Diff}(\pi_f) - (-1)^N(2P_1)$, which proves (18) in this case. Next, suppose that $w < 3$. Then $w = 2$ (because otherwise $\mathrm{supp}(W_C)$ would lie completely in a fibre), so $\mathrm{supp}(W_C) = \{Q_1, Q_2\}$ with $Q_2 \in f^{-1}(P_2)$. Since $e_{Q_2}(f) = 1$, it follows that $v_{Q_2}(\mathrm{Diff}(\pi_f \circ f)) = v_{f(Q_2)}(\mathrm{Diff}(\pi_f)) = 2$, and so $v_{Q_2}(W_C) \leq v_{Q_2}(\mathrm{Diff}(\varphi_f \circ \pi_C)) = v_{Q_2}(\mathrm{Diff}(\pi_f \circ f)) = 2$. Thus, $W_C = 4P + 2P_2$, and hence $f_*W_C = 4Q_1 + 2Q_2 = \mathrm{Diff}(\pi_f) + 2Q_1$. This proves (18) also when $w = 2$ because in that case we must have that N is odd. Indeed, in that case $J_C \sim E \times E'$ has 2-rank $w - 1 = 1$ (by Deuring/Shafarevich applied to π_C) and then E' is supersingular (whereas E is ordinary). Thus, N has to be odd, for otherwise we would have an induced isomorphism $E[2] \simeq E'[2]$ which is impossible; cf. [3], Theorem 3.4. Thus, (18) holds in all cases.

We now consider the case of an arbitrary ground field K. Since by the claim $\mathrm{Diff}(f)_{\overline{K}} = 2\overline{Q}_1$ with $\overline{Q}_1 \in C(\overline{K})$, it follows that either $\mathrm{Diff}(f) = 2Q_1$ with $Q_1 \in C(K)$ or that $\mathrm{Diff}(f) = Q_1$ where $\deg(Q_1) = \deg_i(Q_1) = 2$. Since in the latter case that $\kappa(Q_2)/\kappa(f(Q_2))$ must be inseparable of degree ≥ 2, it follows that in both cases $P_1 := f(Q_2) \in E(K)$ is K-rational. Thus, the right hand side of (18) is a divisor which is defined over K, and so it follows from (18) (applied to $f_{\overline{K}}$) that this equation already holds over K. In particular, (16) holds with $P_f = P_1$. □

We can use the above results to prove the results stated in the introduction.

Proof of Theorem 1.1. By Proposition 2.3 it is enough to show that Φ_C is surjective. Thus, let $E \leq J_C$ be an elliptic subgroup with embedding $h : E \to J_C$. Choose a Galois extension K'/K such that $C(K') \neq \emptyset$. Then by (the proof of) Proposition 2.3 we know that there is an elliptic subcover $f' : C_{K'} \to E_{K'}$ such that $(f')^* \circ j_{E_{K'}} = h_{K'}$. By Theorem 4.8 we know that there exists an $x \in E(K')$ such that $f'' := T_x \circ f'$ is normalized. Since $(f'')^* = (f')^*$, it follows from Corollary 4.4 that there is a (normalized) subcover $f : C \to E$ such that $f'' = f_{K'}$, and thus $f^*J_E = h(E)$, so Φ_C is surjective. □

Proof of Corollaries 1.2, 1.3 and 1.4. In view of Theorem 1.1, these follow directly from Corollary 2.7(b) and Remark 2.8. □

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