Modular Diagonal Quotient Surfaces

Introduction

The modular diagonal quotient surfaces $Z'_{N,\varepsilon}$ occur naturally as the (coarse) moduli spaces associated to the moduli problem which classifies isomorphisms between the mod N Galois representations attached to elliptic curves:

$$\overline{\rho}_{E,N} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(E[N]) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Such moduli problems arise in connection with Mazur's Question and/or Frey's Conjecture (see below). As a result, this modular interpretation gives a (partial) geometric interpretation of these questions and conjectures.

In addition, there is a close analogy between these surfaces and the Hilbert modular surfaces studied by F. Hirzebruch and others (cf. van der Geer[3]).

The Definition of Modular Diagonal Quotient Surfaces

1. Algebraic Description

Fix an integer $N \geq 1$ and $\varepsilon \in (\mathbb{Z}/N\mathbb{Z})^\times$, and let

$X = X(N) = \Gamma(N)\backslash\mathfrak{H}^*$	denote the modular curve of level N,
$G = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$,	viewed as a subgroup of the automorphism group of X,
$\alpha_\varepsilon \in \mathrm{Aut}(X)$	the automorphism of G defined by conjugation with Q_ε;

more precisely, $\alpha_\varepsilon(g) = Q_\varepsilon g Q_\varepsilon^{-1}$, where $Q_\varepsilon = \left(\begin{smallmatrix}\varepsilon & 0\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$. In addition, let

$Y_N = X(N) \times X(N)$	be the product surface of X by itself,
$\Delta_\varepsilon = \{(g, \alpha_\varepsilon(g)) : g \in G\}$	the "twisted diagonal" subgroup defined by α_ε,
$Z_{N,\varepsilon} = \Delta_{\alpha_\varepsilon}\backslash Y$	the (twisted) diagonal quotient surface defined by α_ε.

Notes: 1) The modular diagonal quotient surfaces as defined above may be viewed as a special case of the general diagonal quotient surfaces studied in Kani/Schanz[7].

2) The inclusions of subgroups $\{1\} \leq \Delta_{\alpha_\varepsilon} \leq G \times G$ induce finite morphisms (both of degree $m = |G|$)

$$Y_N \xrightarrow{\Phi} Z_{N,\varepsilon} \xrightarrow{\Psi} Y_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

Moreover, by using Shimura's canonical model $X(N)_{/\mathbb{Q}}$, it can be shown that all these varieties have canonical models over $\mathbb{Q}$ (and even over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$).

3) The above curves and surfaces are all projective. For the above-mentioned moduli problem, however, it is also useful to consider the following affine variants. Let

$X'(N) = \Gamma(N)\backslash\mathfrak{H} = X(N) \setminus \{\text{cusps}\}$	denote the affine modular curve of level N,
$Y'_N = X'(N) \times X'(N)$	be the product surface of $X'(N)$ by itself,
$Z'_{N,\varepsilon} = \Delta_\alpha\backslash Y'_N$	the (affine) diagonal quotient surface.

In particular, $Z'_{N,\varepsilon} \subset Z_{N,\varepsilon}$ is an affine subvariety of $Z_{N,\varepsilon}$ and the complement $Z^{inf}_{N,\varepsilon} = Z_{N,\varepsilon} \setminus Z'_{N,\varepsilon}$ is a union of finitely many curves (the curves at infinity).

2. Analytic Description

As usual, let $\Gamma(N) = \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$ denote the principal congruence subgroup of level N, and let $\pm\Gamma(N) = \langle \Gamma(N), \pm 1\rangle$. Then $(Z'_{N,\varepsilon})_{/\mathbb{C}}$ may also defined by

$$(Z'_{N,\varepsilon})_{/\mathbb{C}} = \tilde{\Delta}_{N,\varepsilon} \backslash (\mathfrak{H} \times \mathfrak{H}),$$

where $\tilde{\Delta}_{N,\varepsilon} \leq \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ is the unique subgroup containing $(\pm\Gamma(N)) \times (\pm\Gamma(N))$ such that

$$\Delta_{N,\varepsilon} := \tilde{\Delta}_{N,\varepsilon}/(\pm\Gamma(N)) \times (\pm\Gamma(N))),$$

is the twisted diagonal subgroup as defined above via the identification $G \simeq \mathrm{SL}_2(\mathbb{Z})/(\pm\Gamma(N))$.

The above analytic description of $Z'_{N,\varepsilon}$ points to the analogy with Hilbert modular surfaces (which are quotients of $\mathfrak{H} \times \mathfrak{H}$ by other discrete subgroups of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$.) Indeed, many of the analytic properties of these surfaces are analogous to those of the usual Hilbert modular surfaces, and hence the $Z_{N,\varepsilon}$'s may also be called degenerate Hilbert modular surfaces of discriminant N^2. (Terminology of C.F. Hermann[5]).

3. Modular Description

The surface $Z'_{N,\varepsilon}$ "classifies" isomorphisms (of determinant ε) between mod N Galois representations attached to elliptic curves E/K:

$$\bar{\rho}_{E/K,N} : G_K \to \mathrm{Aut}(E[N]) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

More precisely, $Z'_{N,\varepsilon}$ "classifies" isomorphism classes of triples (E_1, E_2, ψ), where E_i/K are two elliptic curves and $\psi : E_1[N] \xrightarrow{\sim} E_2[N]$ is an isomorphism of the N-torsion subgroups $E_i[N]$ of determinant ε; the latter means that the Weil pairings are related by the formula

$$e_{E_1,N} \circ (\psi \times \psi) = e^{\varepsilon}_{E_2,N}.$$

In other words, the surface $(Z'_{N,\varepsilon})_{/\mathbb{Q}}$ serves as a coarse moduli scheme for the moduli functor $\mathcal{Z}_{N,\varepsilon} : \underline{\mathrm{Schemes}}_{/\mathbb{Q}} \to \underline{\mathrm{Sets}}$ defined by

$$\begin{aligned}\mathcal{Z}_{N,\varepsilon}(S) \;=\; \{(E_1, E_2, \psi) : \;& E_1,\ E_2 \text{ are elliptic curves over } S \text{ and} \\ & \psi : E_1[N] \xrightarrow{\sim} E_1[N] \text{ is an } S\text{-isomorphism} \\ & \text{with } \det(\psi) = \varepsilon\}/(\text{isomorphisms}).\end{aligned}$$

This modular interpretation can be used to understand:

- Mazur's Question on isomorphisms between mod N Galois representations attached to elliptic curves.
  - $\to$ construction of ∞'ly many such isomorphisms for $N = 11$ (cf. Kani-Rizzo[6])
  - $\to$ explanation of the examples of Kraus/Oesterlé[9], Halberstadt/Kraus[4] for $N = 7$.
- Frey's Conjecture ($\leftrightarrow$ Asymptotic Fermat Conjecture) and Darmon's Conjectures and other related conjectures; cf. Frey[2].
- Moduli problems of curves of genus 2 with elliptic differentials (and related Hurwitz spaces).

The Geometry of Modular Diagonal Quotient Surfaces

1. Singularities

Since Δ_ε has finitely many fixed points on Y_N, the quotient surfaces $Z_{N,\varepsilon}$ have finitely many isolated (cyclic) quotient singularities. (The basic properties of such singularities is explained in Barth/Peters/van de Ven[1], p. 84.) As was shown in Kani-Schanz[8] (cf. also Hermann[5]), the surface $Z_{N,\varepsilon}$ has for $N \geq 5$:

$$\begin{array}{ll} r_0 = h(-4N^2) & \text{singularities lying over } \overline{P}_0 \times \overline{P}_0 \in Y_1 \\ r_1 = h(-3N^2) & \text{singularities lying over } \overline{P}_1 \times \overline{P}_1 \in Y_1 \\ r_\infty = s_\infty(\Gamma_1(N)) & \text{singularities lying over } \overline{P}_\infty \times \overline{P}_\infty \in Y_1 \end{array}$$

(and no other singularities). Here, $h(D)$ denotes the number of (positive-definite) primitive binary quadratic forms of discriminant $D < 0$ and $s_\infty(\Gamma)$ denotes the number of cusps of $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$. In addition, $\overline{P}_0, \overline{P}_1$ and $\overline{P}_\infty \in X(1)$ denote the three ramification points of the G-cover $\pi : X(N) \to X(1) \simeq \mathbb{P}^1$. Thus, $(Z_{N,\varepsilon})_{sing} = S_0 \cup S_1 \cup S_\infty$, where S_k denotes the set of singularities above $\overline{P}_k \times \overline{P}_k$ for $k = 0, 1, \infty$.

By using the above modular description, the finite singularities (i.e. those in $Z'_{N,\varepsilon}$) are precisely the CM-points on the surface $Z_{N,\varepsilon}$ which have extra automorphisms. There are actually two types: the Heegner CM-points S_k^+ and the anti-Heegner CM-points S_k^-, which are defined as follows.

- The Heegner CM-points are those whose modular description has the form $(E_k, E_k, h_{|E_k[N]})$ with $h \in \mathrm{End}(E_k)$, where $\mathrm{End}(E_k) = \mathbb{Z}[\theta_k]$ with $\theta_k \in \mathrm{Aut}(E_k)$ of order $2k + 2$. (Thus, $\mathrm{End}(E_k) = \mathbb{Z}[i], \mathbb{Z}[\sqrt{-3}]$, for $k = 0, 1$, respectively.) Alternately, these points correspond to the solutions of the congruence equation $q_k(x, y) \equiv \varepsilon \pmod N$ (modulo the action of σ_k). Here $q_k(x, y) = x^2 + kxy + y^2$, for $k = 0, 1$, and $\sigma_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sigma_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.
- The anti-Heegner CM-points are those whose modular description has the form $(E_k, E_k, \psi_P h_{|E_k[N]})$ with $h \in \mathrm{End}(E_k)$. Here ψ_P is the automorphism of $\mathrm{End}(E_k[N])$ which interchanges the two vectors of a normal basis $\{P, \theta_k(P)\}$ of $E_k[N]$. Such points correspond to the solutions of the congruence equation $q_k(x, y) \equiv -\varepsilon \pmod N$ (modulo the action of σ_k).

2. The Desingularization $\tilde{Z}_{N,\varepsilon}$ of $Z_{N,\varepsilon}$

Construction. As is well-known (cf. [1], p. 80ff), a cyclic quotient singularity may be desingularized by replacing it by a chain of $\mathbb{P}^1$'s. The number and self-intersection numbers of these curves depends on the (modified) continued fraction expansion of $\frac{n}{q}$, where (n, q) is the type of the singularity. For example,

$$\begin{array}{lllll} \text{each} & z \in S_0 & \text{is resolved by} & \text{one} & (-2) - \text{curve} \\ \text{each} & z \in S_1^+ & \text{is resolved by} & \text{one} & (-3) - \text{curve} \\ \text{each} & z \in S_1^- & \text{is resolved by} & \text{two} & (-2) - \text{curves (intersecting transversely)} \end{array}$$

However, the resolution of the singularities S_∞ at infinity is more involved and does not have such a simple description (cf. [8].) Thus, if $\mathbb{L}_\varepsilon$ denotes the total number of curves added to the smooth part $(Z_{N,\varepsilon})_{sm}$ of $Z_{N,\varepsilon}$, then we have

$$\mathbb{L}_\varepsilon = r_0 + 2r_1 - s_{1,1,\varepsilon} + \mathbb{L}_{\infty,\varepsilon},$$

where $s_{1,1} = \#S_1^+$ and $\mathbb{L}_\infty$ denotes the total number of irreducible curves needed to resolve the singularities at infinity.

Numerical Invariants. Many of the numerical invariants (such as the Betti, Hodge and Chern numbers) of the desingularization $\tilde{Z}_{N,\varepsilon}$ were worked out in [8] (cf. also [5]). By Theorem 2.6 of [8], these are given by a simple expression in terms of the following basic invariants:

$$m,\ r_0,\ r_1,\ s_{1,1,\varepsilon},\ r_\infty,\ \mathbb{G}_\varepsilon,\ \mathbb{S}_\varepsilon,\ \mathbb{L}_\varepsilon$$

Here $m = |G|$, $r_k = \#S_k$, $s_{1,1,\varepsilon} = \#S_1^+$ and $\mathbb{L}_\varepsilon$ are as defined above. The other two invariants $\mathbb{G}_\varepsilon$ and $\mathbb{S}_\varepsilon$ are defined in terms of the group characters h^1 and ω afforded by G on the modules $H^1(X(N),\mathbb{C})$ and $H^0(X(N),\omega_{X(N)})$, respectively:

$$\mathbb{G}_\varepsilon = \tfrac{1}{4}(h^1,h^1) \quad \text{and} \quad \mathbb{S}_\varepsilon = \tfrac{1}{4}(h^1,h^1) - (\omega, \overline{\omega\circ\alpha_\varepsilon});$$

cf. [8], Remark 2.7 (together with Cor. 1.13). Note that the invariants m, r_k and $\mathbb{G}_\varepsilon$ depend only on N, but not on ε. Moreover, the invariant $\mathbb{G}_\varepsilon$ can be expressed in terms of the r_k's via the formula

$$\mathbb{G}_\varepsilon = \frac{m(N-12)}{144N} - 1 + \frac{1}{8}\phi(N) + \frac{1}{8}r_0 + \frac{1}{6}r_1 + \frac{1}{4}r_\infty$$

On the other hand, the invariant $\mathbb{S}_\varepsilon$ is not so easy to calculate since one has to compute several Dedekind sums (of the form $\mathbb{S}(\varepsilon n^2, N/d)$).

Betti numbers: These are the numbers $b_i = \dim_{\mathbb{C}} H^i(\tilde{Z}_{N,\varepsilon},\mathbb{C})$, for $i = 0,\ldots,4$:

$$b_0 = b_4 = 1, \quad b_1 = b_3 = 0, \quad b_2 = 2 + 4\mathbb{G}_\varepsilon + \mathbb{L}_\varepsilon.$$

Hodge numbers: These are the numbers $h^{p,q} = \dim_{\mathbb{C}} H^q(\tilde{Z}_{N,\varepsilon},\Omega^p)$, for $0 \le p,q \le 2$:

$$\begin{array}{lllllll}
h^{0,0} & = & b_0 & = & 1 & & \\
h^{0,1} & = & h^{1,0} & = & \frac{1}{2}b_1 & = & 0 \\
h^{0,2} & = & h^{2,0} & = & p_g & = & \mathbb{G}_\varepsilon - \mathbb{S}_\varepsilon \\
h^{1,1} & = & \multicolumn{5}{l}{2 + 2\mathbb{G}_\varepsilon + 2\mathbb{S}_\varepsilon + \mathbb{L}_\varepsilon} \\
h^{1,2} & = & h^{2,1} & = & \frac{1}{2}b_3 & = & 0 \\
h^{2,2} & = & b_4 & = & 1 & &
\end{array}$$

Chern numbers: These are the numbers

$$c_1^2 = K_\varepsilon^2 = 8 + 8\mathbb{G}_\varepsilon - \mathbb{L}_\varepsilon - 12\mathbb{S}_\varepsilon \quad \text{and} \quad c_2 = \chi_{top} = 4 + 4\mathbb{G}_\varepsilon + \mathbb{L}_\varepsilon.$$

Here K_ε is the canonical class and χ_{top} is the (topological) Euler characteristic of $\tilde{Z}_{N,\varepsilon}$. In addition, the signature of $\tilde{Z}_{N,\varepsilon}$ is given by

$$sign(\tilde{Z}_{N,\varepsilon}) = \frac{1}{3}(c_1^2 - c_2) = 2p_g - h^{1,1} + 2 = -4\mathbb{S}_\varepsilon - \mathbb{L}_\varepsilon.$$

Rough Classification. By Theorem 3 of [8], the Kodaira dimension of $\tilde{Z}_{N,\varepsilon}$ is given by the formula

$$\kappa(\tilde{Z}_{N,\varepsilon}) = \min(2, p_g(\tilde{Z}_{N,\varepsilon}) - 1),$$

where, as before, $p_g(\tilde{Z}_{N,\varepsilon})$ denotes the geometric genus of $\tilde{Z}_{N,\varepsilon}$. In particular, $\tilde{Z}_{N,\varepsilon}$ is a rational surface if and only $p_g = 0$. Similarly, $\tilde{Z}_{N,\varepsilon}$ is of general type if and only if $p_g \geq 3$.

In fact, the rough classification type of the surface $\tilde{Z}_{N,\varepsilon}$ is completely determined by it geometric genus p_g. More precisely, by Theorem 4 of [8] (see also [5]) we have:

- $\tilde{Z}_{N,\varepsilon}$ is a rational surface if and only if $p_g(\tilde{Z}_{N,\varepsilon}) = 0$, and this is the case precisely for $N \leq 5$ or for $(N, \varepsilon) = (6, 1)$, $(7, 1)$ or $(8, 1)$.
- $\tilde{Z}_{N,\varepsilon}$ is a (blown-up) elliptic K3-surface if and only if $p_g(\tilde{Z}_{N,\varepsilon}) = 1$, i.e. if and only if $(N, \varepsilon) = (6, 5)$, $(7, 3)$, $(8, 3)$, $(8, 5)$, $(9, 1)$ or $(12, 1)$.
- $\tilde{Z}_{N,\varepsilon}$ is a (blown-up) elliptic surface with $\kappa = 1$ if and only if $p_g(\tilde{Z}_{N,\varepsilon}) = 2$. This is the case for $(N, \varepsilon) = (8, 7)$, $(9, 2)$, $(10, 1)$, $(10, 3)$ or $(11, 1)$.
- $\tilde{Z}_{N,\varepsilon}$ is a surface of general type if and only if $p_g(\tilde{Z}_{N,\varepsilon}) \geq 3$, or equivalently, if $N \geq 13$ or if $(N, \varepsilon) = (11, 2)$, $(12, 5)$, $(12, 7)$ or $(12, 11)$.

Note: In the above lists, the parameter $\varepsilon \in (\mathbb{Z}/N\mathbb{Z})^\times$ is to be viewed as a representative of the associated square class $\varepsilon((\mathbb{Z}/N\mathbb{Z})^\times)^2$ because $\tilde{Z}_{N,\varepsilon'} \simeq \tilde{Z}_{N,\varepsilon}$, for any $\varepsilon' \in \varepsilon((\mathbb{Z}/N\mathbb{Z})^\times)^2$.

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