The number of genus 2 covers of an elliptic curve

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Abstract: The main aim of this paper is to determine the number $c_{N,D}$ of genus 2 covers of an elliptic curve E of fixed degree $N \geq 1$ and fixed discriminant divisor $D \in Div(E)$. In the case that D is reduced, this formula is due to Dijkgraaf.

The basic technique here for determining $c_{N,D}$ is to exploit the geometry of a certain compactification $\mathcal{C} = \mathcal{C}_{E,N}$ of the universal genus 2 curve over the *Hurwitz space* $H_{E,N}$ which classifies (normalized) genus 2 covers of degree N of E. Thus, a secondary aim of this paper is to study the geometry of $\mathcal{C}$. For example, the structure of its degenerate fibres is determined, and this yields formulae for the numerical invariants of $\mathcal{C}$ which are also of independent interest.

1 Introduction

Let E be an elliptic curve over an algebraically closed field K. The main aim of this paper is to compute the number $c_{N,D} = \#\mathrm{Cov}_{E,N,D}$ of genus 2 covers of E of fixed degree $N \geq 1$ and fixed discriminant divisor $D \in \mathrm{Div}(E)$. Since this number is closely related to the *weighted number* $\bar{c}_{N,D} := \sum_{f \in \mathrm{Cov}_{E,N,D}} \frac{1}{|\mathrm{Aut}(f)|}$ of such covers and since the latter leads to simpler formulae, we determine $\bar{c}_{N,D}$ as well.

Theorem 1 *If* $\mathrm{char}(K) \nmid N!$ *and* $D \in \mathrm{Div}(E)$ *is an effective divisor of degree* 2*, then*

$$(1) \qquad \bar{c}_{N,D} = \frac{N}{3\mu_D}\left(\sigma_3(N) - N\sigma_1(N)\right) - \frac{\mu_D - 1}{24}\left(7\sigma_3(N) - (6N+1)\sigma_1(N)\right)$$

where $\mu_D = 1$ *if* D *is reduced and* $\mu_D = 2$ *otherwise, and where* $\sigma_k(n) = \sum_{d|n} d^k$ *denotes the sum of the* k*th powers of the divisors of* n*. Moreover, if we put* $\sigma_1(N/2) = 0$ *if* N *is odd, then the total number of genus* 2 *covers is given by*

$$(2) \qquad c_{N,D} = \bar{c}_{N,D} + \left(\frac{N}{\mu_D} - (\mu_D - 1)\right)\sigma_1(N/2).$$

Corollary 2 *If* $\mathrm{char}(K) = 0$*, then the generating function* $F_D(q)$ *of the* $\bar{c}_{N,D}$*'s is a quasi-modular form, i.e.* $F_D \in \mathbb{Q}[E_2, E_4, E_6]$*, where the* E_k*'s are the usual Eisenstein series:* $E_k = 1 + c_k \sum_{n\geq 1} \sigma_{k-1}(n) q^n$*. More precisely, we have*

$$F_D(q) := \sum_{N\geq 1} \bar{c}_{N,D} q^N = \frac{1}{25920\mu_D}(5E_2^3 - 3E_2E_4 - 2E_6) - \frac{\mu_D - 1}{5760}(2E_4 + 5E_2^2 + 10E_2 - 17).$$

Note that in the case that $\mu_D = 1$ (i.e. D is reduced), it follows that F_D is pure of weight 6, and so in this case the above formula for F_D reduces to that of Dijkgraaf[Di] (see also [KZ]), except that his weighted sum is defined as $\frac{1}{2}\bar{c}_{N,D}$.

As will be shown in §2, the calculation of the numbers $c_{N,D}$ and $\bar{c}_{N,D}$ can be reduced to the calculation of the number of such covers which are *minimal*, i.e. those covers which do not factor over a non-trivial isogeny of E.

Theorem 3 *The number of minimal genus* 2 *covers of* E *of degree* $N \geq 2$ *and discriminant* D *is given by*

$$\#\mathrm{Cov}^{(\min)}_{E,N,D} = \left(\frac{2}{3\mu_D}(N-1) - \frac{\mu_D - 1}{12N}(7N-6)\right)\overline{sl}(N), \tag{3}$$

where $\overline{sl}(N) = |\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}|$.

The basic technique for proving this theorem is to study the geometry of the *Hurwitz space* $H_{E,N}$ which classifies genus 2 covers of degree N of E which are *normalized* in the sense of [Ka3]. More precisely, in Corollary 14 below we shall relate the above number to the degree of a certain "discriminant map" $\delta = \delta_{E,N} : H_{E,N} \to \mathbb{P}^1$.

The computation of $\deg(\delta)$ will be achieved by relating it to the "basic numerical invariants" (such as the modular height) of a certain relative genus 2 curve $\mathcal{C} = \mathcal{C}_{E,N}$ over the modular curve $X(N)$ of level N. This relative curve, which is also of independent interest, is constructed in §4 as follows.

Since $H = H_{E,N}$ is a fine moduli space (cf. [Ka3]), there exists a universal (normalized) genus 2 cover $f_{univ} : C_{univ} \to E_H = E \times H$ of degree N; in particular, C_{univ}/H is a smooth relative curve of genus 2. Since H is an open subset of $X = X(N)$, the *minimal model* $\mathcal{C}$ of C_{univ} is a relative genus 2 curve over X which contain C_{univ} as an open subset. Thus, by construction, the curve $\mathcal{C}$ has good reduction at all points of H, and it turns out that the set $X \setminus H$ is the *degenerate locus* of $\mathcal{C}/X$, i.e. the set of points of X where $\mathcal{C}/X$ has bad reduction.

The precise structure of the degenerate fibres of $\mathcal{C}/X$ is given in Theorem 22. To prove this theorem, we first calculate the *modular height* $h_{\mathcal{C}/X}$ of $\mathcal{C}/X$ (cf. Theorem 17) and then combine the "mass-formula" of [Ka1], [Ka2] with *Mumford's formula* [Mu] (and with Noether's formula). This also leads to a formula for the self-intersection number $(\omega^0_{\mathcal{C}/X})^2$ of the relative dualizing sheaf $\omega^0_{\mathcal{C}/X}$:

Theorem 4 *If* $\mathrm{char}(K) \nmid N!$, *then the minimal model* $\mathcal{C}$ *of* C_{univ} *over* $X = X(N)$ *is a stable curve of genus* 2 *which has* $\frac{1}{12N}(5N+6)\overline{sl}(N)$ *singular fibres, and each such singular fibre has a unique singular point. Moreover, the modular height* $h_{\mathcal{C}/X}$ *and the self-intersection number of the relative dualizing sheaf* $\omega^0_{\mathcal{C}/X}$ *are given by*

$$h_{\mathcal{C}/X} = \frac{1}{12}\overline{sl}(N) \quad \text{and} \quad (\omega^0_{\mathcal{C}/X})^2 = \frac{1}{12N}(7N-6)\overline{sl}(N). \tag{4}$$

Remark. The above theorem, which is a special case of Theorems 17, 21, 22 and Corollary 24, is also of independent interest because the above formula (4) shows that the relative curves $\mathcal{C}/X(N)$ are curves of "maximal height", as is explained in more detail in Remark 18 below. In this connection one should also mention the recent preprint of E. Viehweg and K. Zuo[VZ] in which the authors give a remarkable characterization of abelian varieties (of arbitrary dimension) of "maximal height".

Now the degree of $\delta : H_{E,N} \to \mathbb{P}^1$ is related to the invariant $(\omega^0_{\mathcal{C}/X})^2$ by the formula

$$\deg(\delta) = \frac{N}{36}(9(\omega^0_{\mathcal{C}/X})^2 - W^2_{\mathcal{C}/X}), \tag{5}$$

in which $W_{\mathcal{C}/X}$ denotes the Weierstrass divisor (cf. §6). Since the self-intersection number $W^2_{\mathcal{C}/X}$ of the Weierstrass divisor can be calculated by the adjunction formula (cf. (38)), this, together with (4), yields the desired formula for $\deg(\delta)$.

Finally, the proof of (5) (= equations (35) and (39) below) follows from a study of the different divisor $D_{\mathcal{C}/X} := \mathrm{Diff}(\bar{f}_{univ}) \in \mathrm{Div}(\mathcal{C})$; cf. §5. In particular, we use the (generalized) Riemann-Hurwitz formula (cf. Corollary 35) and the degeneration structure of $\mathcal{C}/X$ to show that the sheaf $\mathcal{L}(D_{\mathcal{C}/X})$ associated to $D_{\mathcal{C}/X}$ is the relative dualizing sheaf $\omega^0_{\mathcal{C}/X}$ of $\mathcal{C}/X$ (cf. Theorem 26). Formula (5) follows from this because the Weierstrass divisor $W_{\mathcal{C}/X}$ is closely related to the tri-canonical sheaf $(\omega^0_{\mathcal{C}/X})^{\otimes 3}$; cf. Proposition 29 and Corollary 30.

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2 Normalized Genus 2 Covers

Let E/K be an elliptic curve over an algebraically closed field K of characteristic $\neq 2$, and let $E[2] = \{P_0, P_1, P_2, P_3\}$ denote its group of 2-torsion points, where $P_0 = 0_E$

denotes the identity point. Moreover, let $\pi_E : E \to E/\langle[-1]_E\rangle \simeq \mathbb{P}^1$ denote the quotient map, and put $\bar{P}_i = \pi_E(P_i)$, for $i = 0, \ldots, 3$.

If $f : C \to E$ is any genus 2 cover of E, then by Riemann-Hurwitz its *discriminant* $\mathrm{Disc}(f) = f_*\mathrm{Diff}(f)$ is an effective divisor of degree 2. As was mentioned in the introduction, we want to count the number of genus 2 covers $f : C \to E$ of fixed degree $N = \deg(f)$ and fixed discriminant $D = \mathrm{Disc}(f)$, i.e. we want to determine the cardinality of the set

$$\mathrm{Cov}_{E,N,D} := \{f : C \to E : g_C = 2, \deg(f) = N, \mathrm{Disc}(f) = D\}/\simeq$$

of isomorphism classes of genus 2 covers of E of degree N and discriminant D. Here, as usual, two covers $f_i : C_i \to E$ are called *isomorphic* ($f_1 \simeq f_2$) if there is an isomorphism $\varphi : C_1 \xrightarrow{\sim} C_2$ such that $f_1 = f_2 \circ \varphi$.

As we shall see presently, the study of this set can be reduced to the study of *normalized* genus 2 covers which were introduced in [Ka3]. To define these, recall first that f is called *minimal* if f does not factor over a non-trivial isogeny of E. Then we say that f is *normalized* if it minimal and if the norm (or direct image) f_*W_C of the hyperelliptic (Weierstrass) divisor W_C has the form

$$f_*W_C = 3P_0 + P_1 + P_2 + P_3, \quad \text{respectively} \quad f_*W_C = 2(P_1 + P_2 + P_3), \tag{6}$$

if N is odd, respectively, if N is even; cf. [Ka3], §2. Recall from [Ka3], Proposition 2.2, that every minimal cover can be normalized by replacing f by a suitable translate $T_P \circ f$ (with $P \in E(K)$) and that each normalized cover f satisfies the relation

$$[-1]_E \circ f = f \circ \sigma_C, \tag{7}$$

where σ_C denotes the hyperelliptic involution on C. Note that this equation shows in particular that $\mathrm{Disc}(f)$ is *symmetric*, i.e. that $[-1]^*\mathrm{Disc}(f) = \mathrm{Disc}(f)$. Thus $\mathrm{Disc}(f) = D_P := P + [-1]^*P$, for some $P \in E(K)$; note that D_P is reduced if and only if $P \notin E[2]$.

Example 5 Let E be the elliptic curve defined by the equation $y^2 = (x-a)(x-b)(x-c)$, with $abc \neq 0$ and let C be the genus 2 curve defined by $s^2 = (t^2-a)(t^2-b)(t^2-c)$. Then map $f : C \to E$, defined by $f^*x = t^2$ and $f^*y = s$, is clearly a normalized genus 2 cover of E of degree 2 with $\mathrm{Disc}(f) = D_P$, where $P = (0, \sqrt{-abc}) \in E(K)$.

It turns out that every normalized cover of degree 2 is of the above form; more precisely, we have:

Proposition 6 *If $P \in E(K) \setminus E[2]$, then there is (up to isomorphism) a unique normalized genus 2 cover $f : C \to E$ of degree 2 with* $\mathrm{Disc}(f) = D_P$. *On the other hand, if $P \in E[2]$, then there is no degree 2 cover $f : C \to E$ with* $\mathrm{Disc}(f) = D_P$.

Proof. First note that if $f : C \to E$ is any degree 2 cover, then each ramified fibre of f has a unique ramification point of ramification index 2, and hence $\mathrm{Disc}(f)$ is necessarily reduced. This proves the second assertion.

To prove the first, put $\bar{P} = \pi_E(P)$, and choose a generator x of the function field of $\mathbb{P}^1$ such that its principal divisor is $(x) = \bar{P} - \bar{P}_0$. Then E is given by an equation as in Example 5 for suitable $a, b, c \neq 0$, and $P = (0, \sqrt{-abc})$. Thus, the above example shows that such a normalized degree 2 cover $f : C \to E$ exists.

To prove that f is unique up to isomorphism, suppose that $f' : C' \to E$ is another cover of this type. Now if $\pi_C : C \to C/\langle\sigma_C\rangle \simeq \mathbb{P}^1$ denotes the hyperelliptic cover of C, then equation (7) shows that $\pi_E \circ f$ factors over π_C, so that $\pi_E \circ f = \bar{f} \circ \pi_C$, for a unique double cover $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$. Similarly, we have $\pi_E \circ f' = \bar{f}' \circ \pi_{C'}$, for another double cover $\bar{f}' : \mathbb{P}^1 \to \mathbb{P}^1$.

It is easy to see that both $\bar{f}$ and $\bar{f}'$ are ramified at $\bar{P}_0$ and $\bar{P}$, and so it follows (since K is algebraically closed) that there exists an $\alpha \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\bar{f} = \bar{f}' \circ \alpha$. Now since $\bar{f}^{-1}(\{\bar{P}_1, \bar{P}_2, \bar{P}_3\}) = \pi_C(\mathcal{W}_C)$, where $\mathcal{W}_C$ denotes the set of (six) Weierstrass points on C, and since similarly $(\bar{f}')^{-1}(\{\bar{P}_1, \bar{P}_2, \bar{P}_3\}) = \pi_{C'}(\mathcal{W}_{C'})$, we see that $\alpha(\pi_C(\mathcal{W}_C)) = \pi_{C'}(\mathcal{W}_{C'})$, and this implies that α lifts to an isomorphism $\tilde{\alpha} : C \to C'$ such that $\alpha \circ \pi_C = \pi_{C'} \circ \tilde{\alpha}$. Thus $\bar{f}' \circ \pi_{C'} \circ \tilde{\alpha} = \bar{f} \circ \pi_C$, and so $\pi_E \circ f' \circ \tilde{\alpha} = \pi_E \circ f$.

We now show that this implies that either $f = f' \circ \tilde{\alpha}$ or that $f = f' \circ \tilde{\alpha} \circ \sigma_C$. For this, let τ and τ' be the nontrivial involutions associated to f and f', respectively. Then $\tau_1 := \tilde{\alpha}^{-1}\tau'\tilde{\alpha} \in \mathrm{Gal}(\pi_E \circ f) = \{1, \sigma_C, \tau, \sigma_C\tau\}$. Thus $\tau_1 = \tau$ or $\tau_1 = \sigma_C\tau$ and hence either $f \circ \tilde{\alpha}^{-1}$ or $f \circ \sigma_C \circ \tilde{\alpha}^{-1}$ is τ'-invariant, and so there exists an automorphism $\rho \in \mathrm{Aut}(E)$ such that $\rho \circ f' = f \circ \tilde{\alpha}^{-1}$ (or $= f \circ \sigma_C \circ \tilde{\alpha}^{-1}$). Since $\pi_E \circ f' \circ \tilde{\alpha} = \pi_E \circ f$, we see that $\pi_E \circ \rho = \pi_E$, and so $\rho = id_E$ or $\rho = [-1]_E$. Thus, using (7), we see that either $f = f' \circ \tilde{\alpha}$ or $f = f' \circ \tilde{\alpha} \circ \sigma_C$. Thus $f' \simeq f$, and so f is unique up to isomorphism.

Notation. Let $\mathrm{Cov}^{(\mathrm{norm})}_{E,N,D}$ (respectively, $\mathrm{Cov}^{(\mathrm{min})}_{E,N,D}$) denote the subset of isomorphism classes $cl(f) \in \mathrm{Cov}_{E,N,D}$ for which f is normalized (respectively, for which f is minimal).

The connection between minimal and normalized covers is clarified by the following result.

Proposition 7 *(a) For any $P \in E(K)$, the map $f \mapsto T_P \circ f$ induces a bijection*

$$\tau_P : \mathrm{Cov}_{E,N,T_P^*D} \xrightarrow{\sim} \mathrm{Cov}_{E,N,D}$$

which maps $\mathrm{Cov}^{(\mathrm{min})}_{E,N,T_P^*D}$ onto $\mathrm{Cov}^{(\mathrm{min})}_{E,N,D}$.

(b) For any effective divisor $D \in \mathrm{Div}^{(2)}(E)$ of degree 2, there is a point $P \in E(K)$ such that T_P^*D is symmetric.

(c) If $D \in \mathrm{Div}^{(2)}(E)$ is symmetric, then

$$\mathrm{Cov}^{(\min)}_{E,N,D} = \dot{\bigcup}_{P_i \in E[2]} \tau_{P_i}(\mathrm{Cov}^{(\mathrm{norm})}_{E,N,T^*_{P_i}D}).$$

Proof. (a) Since $T^*_P(\mathrm{Disc}(T_P \circ f)) = \mathrm{Disc}(f)$, this is immediate.

(b) Write $D = P' + P''$ with $P', P'' \in E(K)$. Since K is algebraically closed, there is a point $P \in E(K)$ such that $2P \sim P' + P''$. Then $T^*_P(P'') \sim (P'' - P) + P_0 \sim P - P' + P_0 \sim 2P_0 - T^*_P(P') \sim [-1]^*_E T^*_P(P')$, and so $T^*_P(P'') = [-1]^*_E T^*_P(P')$. Thus $T^*_P D = T^*_P(P') + [-1]^*_E T^*_P(P')$ is symmetric.

(c) It is clear by part (a) that the right hand side is contained in the left hand side. Conversely, suppose $cl(f) \in \mathrm{Cov}^{(\min)}_{E,N,D}$. Then by Proposition 2.2 of [Ka3], there exists a point $P \in E(K)$ such that $f' = T_P \circ f$ is normalized, and so we know by (7) that $\mathrm{Disc}(f') = T^*_P(\mathrm{Disc}(f)) = T^*_P(D)$ is symmetric. But D was also symmetric by hypothesis, so we must have that $[-1]^*_E P = P$, i.e. that $P \in E[2]$. Thus $f = T_P \circ f'$, with $f' \in \mathrm{Cov}^{(\mathrm{norm})}_{E,N,T^*_P(D)}$, and so both sides are equal.

The study of arbitrary covers can be reduced to the study of minimal covers by means of the following (well-known) fact.

Proposition 8 *If $f : C \to E$ is any cover and $H = H_f := \mathrm{Ker}(f^* : J_E \to J_C)$ is the kernel of the induced homomorphism of the Jacobians, then f factors as $f = \pi_H \circ f'$, where $\pi_H : E'_H \to E$ is the (essentially) unique isogeny such that $\mathrm{Ker}(\pi^*_H) = H$, and where $f' : C \to E'_H$ is minimal. Furthermore, if $f = \pi_H \circ f''$ is another factorization, then $f'' = T_{P'} \circ f'$, for some $P' \in \mathrm{Ker}(\pi_H)$.*

Proof. By the autoduality property of J_C, $f : C \to E$ is minimal if and only if f *maximal* in the sense of Serre[Se], VI, No. 13 (p. 129ff), and so the first assertion follows from Lemma 3 of [Se], (p. 130). The second assertion is an easy consequence of the uniqueness assertion of the autoduality property of J_C.

Corollary 9 *Let $H \le J_E$ be a finite subgroup of order $n|N$ and let $\mathrm{Cov}_{E,N,D,H} \subset \mathrm{Cov}_{E,N,D}$ denote the subset of all isomorphism classes $cl(f) \in \mathrm{Cov}_{E,N,D}$ with $\mathrm{Ker}(f^*) = H$. Fix a point $Q'_1 \in \pi^{-1}_H(Q_1)$, where $D = Q_1 + Q_2$. Then the map $f' \mapsto \pi_H \circ f'$ induces a surjection*

$$\tau_H : \dot{\bigcup_{Q'_2 \in \pi^{-1}_H(Q_2)}} \mathrm{Cov}^{(\min)}_{E'_H, \frac{N}{n}, Q'_1 + Q'_2} \longrightarrow \mathrm{Cov}_{E,N,D,H}$$

which is a bijection if D is reduced. On the other hand, if $D = 2Q_1$, then for any $P' \in \mathrm{Ker}(\pi_H)$ we have

$$\tau^{-1}_H(\tau_H(cl(f'_1)) = \{cl(f'_1), cl(T_{P'} \circ f'_1)\}, \quad \textit{for all } cl(f'_1) \in \mathrm{Cov}^{(\min)}_{E'_H, \frac{N}{n}, Q'_1 + T^*_{P'}(Q'_1)}.$$

In particular, $\#\tau^{-1}_H(\tau_H(cl(f'_1))) = 2$, if $P' \ne 0$, and $\#\tau^{-1}_H(\tau_H(cl(f'_1))) = 1$, if $P' = 0$.

Proof. It is immediate that the rule $\tau_H(cl(f')) := cl(\pi_H \circ f')$ defines a map from the above union to $\mathrm{Cov}_{E,N,D,H}$. To see that τ_H is surjective, let $cl(f) \in \mathrm{Cov}_{E,N,D,H}$. Then by Proposition 8 we have $f = \pi_H \circ f'$, for some minimal $f' : C \to E'_H$. Since $(\pi_H)_*(\mathrm{Disc}(f')) = D$, we have that $\mathrm{Disc}(f') = T^*_{P'}(Q'_1 + Q'_2)$, for some $P' \in \mathrm{Ker}(\pi_N)$ and $Q'_2 \in \pi_H^{-1}(Q_2)$. Put $f'_1 = T_{P'} \circ f'$. Then $\mathrm{Disc}(f'_1) = Q'_1 + Q'_2$, so $cl(f'_1) \in \mathrm{Cov}^{(\min)}_{E'_H,N/n,Q'_1+Q'_2}$ and $\pi_H \circ f'_1 = \pi_H \circ f' = f$, and hence τ_H is surjective.

Assume first that D is reduced. To show that τ_H is injective, suppose that $f'_i : C_i \to E'_H$, $i = 1, 2$ are two covers with $cl(f'_i) \in \mathrm{Cov}^{(\min)}_{E'_H,N/n,D_i}$ such that $\pi_H \circ f'_1 \simeq \pi_H \circ f'_2$. Then there is an isomorphism $\varphi : C_1 \xrightarrow{\sim} C_2$ such that $\pi_H \circ f'_1 = \pi_H \circ f'_2 \circ \varphi$, and so by Proposition 8, there is a point $P' \in \mathrm{Ker}(\pi_H)$ such that $f'_1 = T_{P'} \circ f'_2 \circ \varphi$. Then $T^*_{P'}\mathrm{Disc}(f'_1) = \mathrm{Disc}(f'_2)$. Write $D_i = \mathrm{Disc}(f'_i) = Q'_1 + Q'_{2i}$, where $Q'_{2i} \in \pi_H^{-1}(Q_2)$. Now since $Q'_1, T^*_{P'}(Q'_1) \in \pi_H^{-1}(Q_1)$ and $Q'_{22}, T^*_{P'}(Q'_{21}) \in \pi_H^{-1}(Q_2)$, and since $Q_1 \neq Q_2$, we must have $T^*_{P'}(Q'_1) = Q'_1$ (and $T_{P'}(Q'_{21}) = Q'_{22}$). But this means that $P' = 0_{E'_H}$, and so $f'_1 = f'_2 \circ \varphi$, i.e. $f'_1 \simeq f'_2$. Thus τ_H is injective (and hence bijective).

Now suppose that $D = 2Q_1$, and let $cl(f'_1) \in \mathrm{Cov}^{(\min)}_{E'_H,N/n,Q'_1+T^*_{P'}Q'_1}$. Then clearly $\{cl(f'_1), cl(T_{P'} \circ f'_1)\} \subset \tau_H^{-1}(\tau_H(cl(f'_1)))$ because $\pi_H \circ f'_1 = \pi_H \circ T_{P'} \circ f'_1$ and since $\mathrm{Disc}(T_{P'} \circ f'_1) = T^*_{-P'}(\mathrm{Disc}(f'_1) = T^*_{-P'}(Q'_1) + Q'_1$.

To prove the opposite inclusion, let $cl(f'_2) \in \pi_H^{-1}(\pi_H(cl(f'_1)))$. Then, as in the reduced case, there exists $\varphi : C_1 \xrightarrow{\sim} C_2$ and $P'' \in \mathrm{Ker}(\pi_H)$ such that $f'_1 = T_{P''} \circ f'_2 \circ \varphi$. Thus $T^*_{P''}(\mathrm{Disc}(f'_1)) = \mathrm{Disc}(f'_2) = Q'_1 + T^*_{P'_1}(Q'_1)$, for some $P'_1 \in \mathrm{Ker}(\pi_H)$, and hence $T^*_{P''}(Q'_1) + T^*_{P''}T^*_{P'}(Q'_1) = Q'_1 + T^*_{P'_1}(Q'_1)$. If $T^*_{P''}(Q'_1) = Q'_1$, then as before $P'' = 0_{E'_H}$ and so $cl(f'_2) = cl(f'_1)$. Thus, assume $T^*_{P''}(Q'_1) \neq Q'_1$; then $T^*_{P''}(Q'_1) = T^*_{P'_1}(Q'_1)$ and $T^*_{P''}T^*_{P'}(Q'_1) = Q'_1$. This implies that $P'' = P'_1$ and $P'' = -P'$. Thus $T_{P'} \circ f'_1 = f'_2 \circ \varphi$, i.e. $cl(f'_2) = cl(T_{P'} \circ f'_1)$, and so the opposite inclusion holds.

The last assertion is clear from the previous identity, for $cl(T_{P'} \circ f'_1) = cl(f'_1)$ if and only if $P' = 0_{E'_H}$.

To compare the total number $c_{N,D} = \#\mathrm{Cov}_{E,N,D}$ of covers to the weighted number $\bar{c}_{N,D}$, we prove

Lemma 10 *If $f : C \to E$ is any genus 2 cover of degree N, then $|\mathrm{Aut}(f)| \leq 2$. Furthermore, $|\mathrm{Aut}(f)| = 2$ if and only if $|\mathrm{Ker}(f^*)| = \frac{N}{2}$. Thus $c_{N,D} = \bar{c}_{N,D}$, if N is odd, whereas*

$$c_{N,D} = \bar{c}_{N,D} + \frac{1}{2} \sum_{\#H=\frac{N}{2}} \#\mathrm{Cov}_{E,N,D,H}, \quad \textit{if } N \textit{ is even.}$$

Proof. Suppose $G := \mathrm{Aut}(f) \neq \{1\}$, and let $\bar{f} : C \to \bar{C} := G\backslash C$ denote the quotient map. Then $\bar{C}$ is an elliptic curve and f factors as $f = \bar{f}' \circ \bar{f}$. Thus $\mathrm{Ker}((\bar{f}')^*) \leq H := \mathrm{Ker}(f^*)$, and so π_H factors over $\bar{f}'$, and hence $\bar{f}$ factors over the minimal map $f' : C \to$

E'_H (cf. Proposition 8). But then f' is (generically) Galois, so $|\mathrm{Aut}(f')| = \deg(f')$. By rigidity (cf. [Ka3], Proposition 2.1), this happens if and only if $\deg(f') = 2$, or equivalently, if and only if $|H| = N/2$.

Proof of Theorem 1 (using Theorem 3): We first note that it follows from Theorem 3 that for any subgroup $H \le J_E$ with $\#H = n|N$ (and $n \neq N$) we have

$$\#\mathrm{Cov}_{E,N,D,H} = \left(\frac{1}{3\mu_D}(N-n) - \frac{\mu_D - 1}{12N}(7N-6n)\right)\overline{sl}(N/n). \tag{8}$$

To see this, put $c(N,D) := \#\mathrm{Cov}^{(\min)}_{E,N,D}$, which depends only on N and μ_D by (3). Suppose first that D is reduced, i.e. $\mu_D = 1$. Then by Corollary 9 and (3) we have $\#\mathrm{Cov}_{E,N,D,H} = nc(N/n, D') = \frac{2}{3}n(\frac{N}{n}-1)\overline{sl}(N/n)$, which proves (8) in this case.

Now suppose that $D = 2Q_1$ is not reduced, i.e. $\mu_D = 2$. Then by Corollary 9 we have (with the notation there) that $\#\mathrm{Cov}_{E,N,D,H} = c(N/n, 2Q'_1) + \frac{n-1}{2}c(N/n, D') = (\frac{1}{3}n(\frac{N}{n}-1) - \frac{1}{12N/n}(7\frac{N}{n}-6))\overline{sl}(N/n)$, which proves (8) in this case as well.

We now verify formula (2). If N is odd, then $c_{N,D} = \bar{c}_{N,D}$ by Lemma 10. Thus, assume that N is even. Now if $n = \frac{N}{2}$, then formula (8) yields $\#\mathrm{Cov}_{E,N,D,H} = \left(\frac{N}{3\mu_D} - \frac{\mu_D - 1}{3}\right)\overline{sl}(2) = 2(\frac{N}{\mu_D} - (\mu_D - 1))$. Thus, since J_E has precisely $\sigma_1(N/2)$ subgroups H of order $N/2$, we see that formula (2) follows from Lemma 10.

It thus remains to prove formula (1). For this, put $sl(N) = \#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Since $\overline{sl}(N) = \frac{1}{2}sl(N)$, if $N \ge 3$, whereas $\overline{sl}(2) = sl(2) = 6$, we see from (8) combined with Lemma 10 that the weighted number of covers in $\mathrm{Cov}_{E,N,D,H}$ is

$$\bar{c}_{N,D,H} := \sum_{f \in \mathrm{Cov}_{E,N,D,H}} \frac{1}{|\mathrm{Aut}(f)|} = \left(\frac{1}{6\mu_D}(N-n) - \frac{\mu_D - 1}{24N}(7N-6n)\right) sl(N/n). \tag{9}$$

Let $\bar{c}_{N,D,n}$ denote the right hand side of (9). Since this number only depends on n (and on N, D) and since J_E has precisely $\sigma_1(n)$ subgroups H of order n, we see that

$$\bar{c}_{N,D} = \sum_{\substack{n|N \\ n \neq N}} \sum_{\substack{H \le J_E \\ |H| = n}} \bar{c}_{N,D,H} = \sum_{\substack{n|N \\ n \neq N}} \sigma_1(n)\bar{c}_{N,D,n} = \sum_{n|N} \sigma_1(n)\bar{c}_{N,D,n} + \frac{\mu_D - 1}{24}\sigma_1(N).$$

From this, formula (1) follows immediately once we have established the identities

$$\sum_{n|N} \sigma_1(n) sl(N/n) = \sigma_3(N), \tag{10}$$

$$\sum_{n|N} n\sigma_1(n) sl(N/n) = N^2\sigma_1(N). \tag{11}$$

To verify these identities, let $f_1(N)$ and $f_2(N)$ denote the left hand sides of (10) and (11), respectively. Since both sides of these equations are multiplicative, it is enough to consider the case that $N = p^r$ is a prime power. For $r = 1$ these identities are immediate, and for $r > 1$ they can be verified by induction by using the recursion relations $f_1(p^{r+1}) = p^3 f_1(p^r) + 1$ and $f_2(p^{r+1}) = p^3 f_2(p^r) + p^{2r+2}$, which are also satisfied by $\sigma_3(p^r)$ and $p^{2r}\sigma(p^r)$, respectively.

Proof of Corollary 2 (using Theorem 1). Since $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$ and $E_4 = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n$, it follows from (1) that $F_D(q) = \frac{1}{3\mu_D}\left(\frac{1}{240}\theta(E_4) + \frac{1}{24}\theta^2(E_2)\right) - \frac{\mu_D - 1}{24}\left(\frac{7}{240}E_4 + \frac{6}{24}\theta(E_2) - \frac{1}{24}E_2 - \frac{17}{240}\right)\big)$, where $\theta = q\frac{d}{dq}$. Thus, since $\theta(E_2) = \frac{1}{12}(E_2^2 - E_4)$ and $\theta(E_4) = \frac{1}{3}(E_2E_4 - E_6)$ (cf. [La], p. 161), we have $\theta^2(E_2) = \theta(\frac{1}{12}(E_2^2 - E_4)) = \frac{1}{72}(E_2^3 - 3E_2E_4 + 2E_6)$, and so the given formula for $F_D(q)$ follows.

3 The discriminant map

The main reason for working with normalized genus 2 covers in place of arbitrary covers is that normalized covers can be parameterized by a nice algebraic curve.

To explain this in more detail, consider for any extension field L/K the set $\mathcal{H}_{E/K,N}(L) = \mathrm{Cov}^{(\mathrm{norm})}_{E_L,N} = \bigcup_D \mathrm{Cov}^{(\mathrm{norm})}_{E_L,N,D}$ of all isomorphism classes of normalized genus 2 covers of degree N of E_L/L. Then the assignment $L \mapsto \mathcal{H}_{E/K,N}(L)$ extends in a natural way to a *Hurwitz functor*

$$\mathcal{H}_{E/K,N} : \underline{Sch}_{/K} \to \underline{Sets};$$

cf. [Ka3], §3. Moreover, if $N \geq 3$ is invertible in K (which we *assume tacitly* henceforth), then by Theorem 1.1 of [Ka3] we know that $\mathcal{H}_{E/K,N}$ is representable by an open subset of the (projective) modular curve $X(N)$:

Theorem 11 *If $N \geq 3$, then the functor $\mathcal{H}_{E/K,N}$ is finely represented by an open subset $H_{E,N}$ of the modular curve $X(N)/K$ of level N.*

It is interesting to note that this representability result was obtained by purely algebraic techniques; in particular, it did not use (not even implicitly) the Riemann Existence Theorem.

We next observe that the sets $\mathrm{Cov}^{(\mathrm{norm})}_{E,N,D} \subset \mathcal{H}_{E/K,N}(K)$ of normalized genus 2 covers of fixed discriminant D can be identified as the fibres of the *discriminant map* $\delta_{E,N} : H_{E,N} \to \mathrm{Div}^{(2)}(E)^{sym} \simeq \mathbb{P}^1$.

Proposition 12 *(a) The rule $cl(f) \mapsto \mathrm{Disc}(f)$ defines a morphism*

$$\delta_{E,N} : H_{E,N} \to \mathrm{Div}^{(2)}(E)^{sym} \simeq \mathbb{P}^1$$

such that $\mathrm{Cov}^{(\mathrm{norm})}_{E,N,D} \simeq \delta^{-1}_{E,N}(D)$, for all $D \in \mathrm{Div}^{(2)}(E)^{sym}$.

Proof. Let $\mathrm{Div}^{(2)}_{E/K} : \underline{Sch}_{/K} \to \underline{Sets}$ denote the functor which associates to a scheme S the set $\mathrm{Div}^{(2)}_{E/K}(S) = \mathrm{Div}^{(2)}(E_S/S)$ of relative effective Cartier divisors of degree 2 of E_S/S (cf. [BLR], p. 214 and 237). Recall that $\mathrm{Div}^{(2)}_{E/K}$ is represented by $E^{(2)}$, the second symmetric power of E (cf. [Gr1], p. 21).

Now if S is any K-scheme and $f : C \to E_S$ is any normalized genus 2 cover of E_S/S (cf. [Ka3], §3), then $\mathrm{Disc}(f) = f_*\mathrm{Diff}(f) \in \mathrm{Div}^{(2)}_{E/K}(S)$, so the assignment $cl(f) \mapsto \mathrm{Disc}(f)$ defines a map $\delta_S : \mathcal{H}_{E/K,N}(S) \to \mathrm{Div}^{(2)}_{E/K}(S)$. We claim that δ_S commutes with base-chance. Indeed, the formation of $\mathrm{Diff}(f)$ commutes with base-change by general properties of differents (cf. Corollary 34) and hence $\mathrm{Diff}(f) = f_*\mathrm{Diff}(f)$ also commutes with base-change (since f_* does; cf. [Ka3], Lemma 7.3). Thus, $\delta = \{\delta_S\} : \mathcal{H}_{E/K,N} \to \mathrm{Div}^{(2)}_{E/K}$ defines a natural transformation of functors, and hence is represented by a morphism $\delta = \delta_{E,N} : H_{E,N} \to E^{(2)}$.

By (7) we know that $\mathrm{Disc}(f)$ is symmetric with respect to $[-1]$, and so δ maps into the subset $\mathrm{Div}^{(2)}(E)^{sym}$ of symmetric divisors. Now if $\pi_E : E \to E/\langle[-1]\rangle \simeq \mathbb{P}^1$ denotes the quotient map, then the rule $\bar{P} \mapsto \pi_E^*\bar{P}$ induces an isomorphism $\mathbb{P}^1(K) \overset{\sim}{\to} \mathrm{Div}^{(2)}(E)^{sym}$.

The last assertion is clear from the construction since by definition $\mathrm{Cov}^{(\mathrm{norm})}_{E,N,D}$ is the fibre of $\delta_K : \mathcal{H}_{E/K,N}(K) \to \mathrm{Div}^{(2)}_{E/K}(K)$ over D, and hence the set of K-rational points of the geometric fibre can be identified with $\mathrm{Cov}^{(\mathrm{norm})}_{E,N,D}$.

Notation. Let $\mathcal{H}^{sm}_{E/K,N}$ denote the subfunctor of $\mathcal{H}_{E/K,N}$ consisting of covers with smooth discriminants, i.e. $\mathcal{H}^{sm}_{E/K,N}$ is defined by the rule

$$\mathcal{H}^{sm}_{E/K,N}(S) = \{cl(f) \in \mathcal{H}_{E/K,N}(S) : \mathrm{Disc}(f) \text{ is smooth over } S\}, \text{ for all } K\text{-schemes } S.$$

Theorem 13 *(a) The functor $\mathcal{H}^{sm}_{E/K,N}$ is represented by an open subscheme $H^{sm}_{E,N}$ of $H_{E,N}$, and the restriction $\delta^{sm}_{E,N} : H^{sm}_{E,N} \to U := \mathbb{P}^1 \setminus \pi_E(E[2])$ of $\delta_{E,N}$ to $H^{sm}_{E,N}$ is etale.*

(b) If $\mathrm{char}(K)\not| N!$, then $\delta^{sm}_{E,N}$ is also finite, and hence $\#\mathrm{Cov}^{(\mathrm{norm})}_{E,N,D_P} = \deg(\delta_{E,N})$, for all $P \in E(K) \setminus E[2]$.

Proof. (a) Let $f_{univ} : C_{univ} \to E_H$ denote the universal normalized genus 2 cover of degree N over $H = H_{E,N}$ (which exists by Theorem 11). Since $D := \mathrm{Disc}(f_{univ}) \in \mathrm{Div}(E_H/H)$ is an effective relative Cartier divisor, the set $H^{sm} \subset H$ where D is smooth over H is an open subset of H, and it is immediate that H^{sm} represents the functor $\mathcal{H}^{sm}_{E,N}$.

Since U represents the subfunctor of ${\mathrm{Div}^{(2)}_{E/K}}^{sym}$ of *smooth* divisors, the map $\delta^{sm}_{E,N}$ is etale if and only if the following deformation property holds (cf. [BLR], p. 288):

(*) *If A a local Artinian K-algebra with residue field $\overline{A}$ and if $D \in \mathrm{Div}^{(2)}_{E/K}(A)^{sym}$ is smooth symmetric divisor on E_A with image $\overline{D} \in \mathrm{Div}^{(2)}_{E/K}(\overline{A})$, then each normalized genus 2 cover $\overline{f} : \overline{C} \to E_{\overline{A}}$ with* $\mathrm{Disc}(\overline{f}) = \overline{D}$ *lifts uniquely to a normalized genus 2 cover $f : C \to E_A$ with* $\mathrm{Disc}(f) = D$.

To see that this property holds, we apply the deformation result of Fulton[Fu1], Theorem 4.8 (cf. also Wevers[Wev], Corollary 3.1.3) in the above situation to conclude that there exists a unique lift $f : C \to E_A$ of $\overline{f}$ with $\mathrm{Disc}(f) = D$. It remains to show that f is normalized.

We first show that f is minimal (in the sense of [Ka3], §7, p. 50). If not, then $\mathrm{Ker}(f^*)$ is a finite flat group scheme over A of rank $d > 1$ (because $f^* : J_{E_A} \to J_C$ is a homomorphism of abelian schemes), and so $\mathrm{Ker}(\overline{f}^*) = \mathrm{Ker}(f^*)_s$ also has rank d, where $\{s\} = \mathrm{Spec(A)}$. But this contradicts the fact that $\overline{f}$ is minimal.

Thus, f is minimal. By [Ka3], Theorem 3.2(c) it is thus enough to show that f is pseudo-normalized, i.e. that (the analogue of) (7) holds. Now since $f \circ \sigma_C$ and $[-1]_{E_A} \circ f$ both lift $\overline{f} \circ \sigma_{\overline{C}} = [-1]_A \circ \overline{f}$ and have discriminant $D = [-1]^*_{E_A} D$, there exists an isomorphism $\varphi : C \xrightarrow{\sim} C$ such that $\overline{\varphi} = id_{\overline{C}}$ and $f \circ \sigma_C = [-1]_{E_A} \circ f \circ \varphi$ (cf. [Fu1], Remark after Theorem 5.8). However, then $\varphi = id_C$ by [DM], Theorem (11.1), and so $f \circ \sigma_C = [-1]_{E_A} \circ f$, as desired.

(b) It is clear that the second assertion follows from the first together with part (a) (and Proposition 12). Now the first assertion could be proven by a method similar to that of the proof of Theorem 7.2 of Fulton[Fu1], but it seems simpler to deduce it from the results of the next section; cf. Theorem 26(e).

Corollary 14 *If $N \geq 3$, and D is an effective divisor of degree 2, then*

$$\#\mathrm{Cov}^{(\min)}_{E,N,D} \geq \frac{4}{\mu_D} \deg(\overline{\delta}_{E,N}) - (\mu_D - 1)(2g_{X(N)} - 2 + s_{E,N}), \tag{12}$$

where $\overline{\delta}_{E,N} : X(N) \to \mathbb{P}^1$ denotes the unique extension of $\delta_{E,N}$ to $X(N)$, $g_{X(N)}$ denotes the genus of $X(N)$, $s_{E,N} = \#(X(N) \setminus H_{E,N})$, and $\mu_D = 1$ if D is reduced and $\mu_D = 2$ otherwise. Moreover, equality holds in (12) if and only if $\mu_D = 1$ or if $\overline{\delta}_{E,N}$ is tamely ramified.

Proof. By Proposition 7(a),(b) we may assume that D is symmetric, so $D = D_P$ with $P \in E(K)$. Thus, by Propositions 7(c) and 12 we have

$$\#\mathrm{Cov}^{(\min)}_{E,N,D_P} = \sum_{i=0}^{3} \#\delta^{-1}_{E,N}(D_{T^*_{P_i}(P)}).$$

If $\mu_D = 1$, i.e. if $P \notin E[2]$, then also $T^*_{P_i}(P) \notin E[2]$ and so by Proposition 7 and Theorem 13(b) we have $\#\mathrm{Cov}^{(\min)}_{E,N,D_P} = 4\deg(\delta_{E,N}) = 4\deg(\overline{\delta}_{E,N})$. This proves that equality holds in (12) in this case.

Now suppose that $\mu_D = 2$, i.e. that $P \in E[2]$. Then by the above formula we have $\#\mathrm{Cov}^{(\min)}_{E,N,D_P} = \sum_{i=0}^{3} \#\delta^{-1}_{E,N}(D_{P_i})$. To calculate this, we shall apply the Riemann-Hurwitz formula to $\overline{\delta}$ to obtain

$$4\deg(\overline{\delta}_{E,N}) - \sum_{i=0}^{3} \#\overline{\delta}^{-1}_{E,N}(D_{P_i}) \le \deg(\mathrm{Diff}(\overline{\delta}_{E,N})) = (2g_{X(N)} - 2) - \deg(\overline{\delta}_{E,N})(-2).$$

From this (12) follows immediately after a short computation because by Theorem 13 we know that $\cup_i \delta^{-1}_{E,N}(D_{P_i}) \,\dot\cup\, S_{E,N} = \cup_i \overline{\delta}^{-1}_{E,N}(D_{P_i})$, where $S_{E,N} = X(N) \setminus H_{E,N}$.

Note since $\overline{\delta}_{E,N}$ is unramified outside the points $\pi_E(E[2]) = \{D_{P_0}, \ldots, D_{P_3}\}$ (cf. Theorem 13), we see that equality holds in the above equation (and hence in (12)) if and only if $\overline{\delta}_{E,N}$ is tamely ramified.

Remark. We shall see later (cf. Proposition 31) that $\overline{\delta}_{E,N}$ is always tamely ramified (provided that $\mathrm{char}(K) \nmid N!$).

4 Compactification of the universal cover

In the previous section we saw that the number $\#\mathrm{Cov}^{(\min)}_{E,N,D_P}$ is (mainly) determined by the degree of the discriminant map $\delta : H = H_{E,N} \to \mathbb{P}^1$. As we shall see below in §5, this degree is closely related to the bi-degree of the discriminant divisor of the universal cover $f_H : C_H \to H$. Since we want to use intersection theory to compute this degree, we first need to compactify the surface C_H as well as the cover f_H.

For this, recall first that by Theorem 11 we have a "universal cover" $f_H = f_{univ} : C_H \to E_H = E \times H$ over $H := H_{E,N} \subset X(N)$ with the property that all other normalized covers of degree N are obtained from this one by base-change. (Recall that we are always tacitly assuming that $N \ge 3$ and that $\mathrm{char}(K) \nmid 2N$.)

Let $p : \mathcal{C} = \mathcal{C}_{X(N)} \to X(N)$ denote the *minimal model* of the genus 2 curve C/F_N which is the generic fibre of C_H/H; here $F_N = \kappa(X(N))$ denotes the function field of $X(N)/K$. Thus, $\mathcal{C}$ is a projective, smooth surface over K, and $p : \mathcal{C} \to X(N)$ is a genus 2 fibration whose fibres do not contain any rational (-1)-curves and whose generic fibre is C/F_N. Note that $\mathcal{C}_H := p^{-1}(H) \simeq C_H$ because C_H/H is smooth and hence is the minimal model of C over H. Thus, $\mathcal{C}/X(N)$ is a natural compactification of C_H/H.

In order to study the geometry of $\mathcal{C}/X(N)$, it is useful to generalize the above situation slightly by allowing base-extensions of $X(N)$. Thus, let:

$\beta : X \to X(N)$ be a finite cover of degree $d = \deg(\beta)$, where
X is a smooth curve with function field $F = \kappa(X)$, and let
$C_F = C \otimes_{F_N} F$ be the base-change of the curve C/F_N,
$p = p_X : \mathcal{C} = \mathcal{C}_X \to X$ be the minimal model of C_F over X,

$J = J_F$ the Jacobian variety of C_F/F, and
$\mathcal{J}/X$ its Néron model.

We first note that the universal cover $f_H : \mathcal{C}_H \simeq C_H \to E_H$ naturally lifts and extends to a morphism $f : \mathcal{C} \to E_X$; this follows from the following general result.

Proposition 15 *Let $p_i : Y_i \to X$ be a two proper morphisms over a smooth curve X/K, and assume that Y_1 is a regular irreducible surface and that Y_2/X is either an abelian scheme or a smooth relative curve of genus $g \geq 1$. Then every X-rational map $f : Y_1 \dashrightarrow Y_2$ extends uniquely to an X-morphism $f : Y_1 \to Y_2$.*

Proof. Here we shall use the following fact:

(13) *If $\pi : Z \to Y_1$ is a birational proper morphism and if $C \subset Z$ is an irreducible curve such that $\pi(C)$ is a point, then C is a rational curve.*

[Indeed, if Z is also regular, then π is a sequence of blow-ups at smooth points (cf. [Ha], Corollary V.5.4), so in fact $C \simeq \mathbb{P}^1$ for any such C. In the general case, we can find by the resolution of singularities theorem (cf. [Art]) a regular surface Z' and a birational proper morphism $\pi' : Z' \to Z$. Then there is an irreducible curve $C' \subset Z'$ such that $\pi'(C') = C$. By what was just said, C' is rational, and hence so is C.]

Now let $U = \mathrm{def}(f)$ denote the largest open subset on which f is defined. Since p_2 is proper and f is an X-rational map, f is a strictly rational map in the sense of Iitaka[I], p. 134; cf. [I], Lemma 2.23. Thus, since Y_1 is normal, $B := Y_1 \setminus U$ has codimension ≥ 2 (cf. [I], Theorem 2.19), and hence is a finite set. Let $\Gamma_f \subset Y_1 \times_X Y_2 \subset Y_1 \times Y_2$ denote the graph of the rational map f. Then $p = (pr_1)_{|\Gamma_f} : \Gamma_f \to Y_1$ is proper and birational. Suppose $y \in B$. Then $\dim(q(p^{-1}(y))) > 0$ by Zariski's Main Theorem (cf. [I], Theorem 2.22), where $q = (pr_2)_{|\Gamma_f} : \Gamma_f \to Y_2$. Now every irreducible curve $C \subset p^{-1}(y)$ is rational by (13). In addition, since C lies in a closed fibre of $p_1 \circ p$, we see that $q(C)$ lies in a closed fibre of p_2. But the hypotheses on p_2 imply that its fibres cannot contain any rational curves, so $q(C)$ must be a point. Thus $p(q^{-1}(y))$ is 0-dimensional, contradiction, and hence $B = \emptyset$, i.e. $\mathrm{def}(f) = Y_1$, which means that f extends (uniquely) to a morphism $f : Y_1 \to Y_2$, as claimed.

Corollary 16 *Let $f_U : C_U \to E_U$ be the base-change of the above universal cover $f_H : C_H \to E_H$ with respect to $\beta_U = \beta_{|U} : U \to H$, where $U = \beta^{-1}(H)$. Then f_U extends uniquely to a proper, surjective morphism $f = f_X : \mathcal{C} \to E_X := E \times X$, and we have*

(14)
$$\beta_{E_X} \circ f_X = f_{X(N)} \circ \beta_{\mathcal{C}}$$

where $\beta_{E_X} = id_E \times \beta : E_X \to E_{X(N)}$ and $\beta_{\mathcal{C}} : \mathcal{C}_X \to \mathcal{C}_{X(N)}$ are the morphisms induced by base-change. Moreover, if $\sigma_{\mathcal{C}} \in \mathrm{Aut}_X(\mathcal{C})$ denotes the unique extension of the hyperelliptic involution σ_{C_F} to $\mathcal{C}$, then we have

(15)
$$f \circ \sigma_{\mathcal{C}} = [-1]_{E_X} \circ f.$$

Proof. The first assertion follows immediately from Proposition 15, for $E_X = E \times X$ is a smooth relative curve of genus 1 over X. The second assertion (14) is clear, for by construction the two morphisms agree on the dense open set $\mathcal{C}_U = \mathcal{C}_H \times_H U$. (Note that $\beta_{\mathcal{C}} = \beta_{(\mathcal{C})} \circ \nu$, where $\nu : \mathcal{C}_X \to \mathcal{C}_{X(N)} \times_{X(N)} X$ is birational and $\beta_{(\mathcal{C})} : \mathcal{C}_{X(N)} \times_{X(N)} X \to \mathcal{C}_{X(N)}$ is the base-change map.)

The existence of $\sigma_{\mathcal{C}}$ follows from the universal property of minimal models (cf. [Li]). Now since f_U is normalized by construction, the two morphisms $f \circ \sigma_{\mathcal{C}}$ and $[-1] \circ f$ agree on the dense open set $p^{-1}(U)$ (cf. [Ka3], Theorem 3.2(c)) and hence are equal on $\mathcal{C}$.

We now examine the geometry of $\mathcal{C}/X$ more closely. As a first step, let us compute its *modular height* $h_{\mathcal{C}/X}$. This important invariant, which was first introduced by Parshin[Pa1],[Pa2] and Arakelov[Ar], is defined for any (regular) semi-stable curve $\mathcal{C}/X$ of genus $g \geq 1$ by

$$h_{\mathcal{C}/X} := \deg_X(\wedge^g p_*(\omega^0_{\mathcal{C}/X})) = -\deg_X(\wedge^g R^1 p_* \mathcal{O}_{\mathcal{C}}).$$

where $\omega^0_{\mathcal{C}/X} = \omega_{\mathcal{C}/K} \otimes p^*(\omega_{X/K})^{-1}$ denotes the *relative dualizing sheaf* of $\mathcal{C}/X$ (cf. [BPV], p. 98 or [Kl]), and the equality follows from the relative duality isomorphism $p_*\omega^0_{\mathcal{C}/X} \simeq (R^1 p_* \mathcal{O}_{\mathcal{C}})^\vee$; cf. [BPV], p. 99 or [Kl], equation (1.1).

It turns out that the height of $\mathcal{C}/X$ is closely related to the modular height of $\mathcal{E}'/X$, where $\mathcal{E}' = \mathcal{E}'_X$ is the minimal model of the "universal elliptic curve" $\mathcal{E}'_F = \mathcal{E}'_{F_N} \otimes F$ over X. (More precisely, $\mathcal{E}'_{F_N}$ is the generic fibre of the universal elliptic curve $\mathcal{E}'_{X'(N)}$ over $X'(N) := X(N) \setminus X(N)_\infty$, where $X(N)_\infty = \{\text{cusps}\}$ denotes the set of cusps of $X(N)$; i.e. $\mathcal{E}'_{X'(N)}$ (together with its level-N-structure) is the universal object of the functor parameterizing elliptic curves with level-N-structure of fixed determinant.)

Theorem 17 *The relative curve $p : \mathcal{C} \to X$ is semi-stable and has modular height*

$$(16) \quad h_{\mathcal{C}/X} = h_{\mathcal{E}'/X} = \frac{N}{12} d \# X(N)_\infty = \frac{d}{12}\overline{sl}(N) := \frac{d}{12}\# \left(\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}\right).$$

Proof. The "basic construction" of [FK] or [Ka3], Corollary 5.19, shows that the Jacobian J_{F_N} of $C = C_{F_N}$ is isogenous to $E_{F_N} \times \mathcal{E}'_{F_N}$, and so we also have $J_F \sim E_F \times \mathcal{E}'_F$. Now $\mathcal{E}'_F$ has semi-stable (or semi-abelian) reduction over F, i.e. the minimal model $\mathcal{E}'/X$ of $\mathcal{E}'_F$ is semi-stable: this follows either from the work of [DR] or, more simply, from Raynaud's criterion (cf. [Gr2], Prop. 4.7) because the N-torsion points of $\mathcal{E}'_F$ are F-rational. Thus, since E_F has good reduction over F, we see that $E_F \times \mathcal{E}'_F$ and hence also J have semi-abelian reduction (use [BLR], 7.4/2 and 7.3/7). Thus, the Néron model $\mathcal{J}$ of J is semi-abelian, and so by [DM], Theorem (2.4), $\mathcal{C}/X$ is semi-stable.

To compute $h_{\mathcal{C}/X}$, we shall use the fact (due to Parshin[Pa2], §3, Proposition 1 and/or Arakelov[Ar], Lemma 1.4) that

$$h_{\mathcal{C}/X} = \deg_X(\lambda_{\mathcal{J}/X}), \quad \text{where } \lambda_{\mathcal{J}/X} = s^*\Omega^1_{\mathcal{J}/X},$$

in which $s : X \to \mathcal{J}$ denotes the zero-section of the Néron model $\mathcal{J}/X$ of the Jacobian J of C_F. Since here $J \sim E_F \times \mathcal{E}'_F$ (by an isogeny of degree N^2), we see that $\lambda_{\mathcal{J}/X} \simeq \lambda_{E_X/X} \otimes \lambda_{\mathcal{E}'/X} \simeq \lambda_{\mathcal{E}'/X}$ where the last isomorphism follows from the fact that E_X/X is constant (so $\lambda_{E_X/X} \simeq \mathcal{O}_X$).

Thus $h_{\mathcal{C}/X} = \deg(\lambda_{\mathcal{J}/X}) = h_{\mathcal{E}'/X}$, which proves the first equality of (16). To determine $h_{\mathcal{E}'/X}$, we use the fact that for any (regular) semi-stable elliptic curve $q : \mathcal{E}' \to X$ we have

(17) $$h_{\mathcal{E}'/X} = \tfrac{1}{12}\delta_{\mathcal{E}'/X} = \tfrac{1}{12}\deg(j_{\mathcal{E}'}),$$

where $\delta_{\mathcal{E}'/X}$ denotes the total number of singular points of all fibres of $\mathcal{E}'/X$ and $j_{\mathcal{E}'} : X \to \mathbb{P}^1$ denotes the morphism defined by the j-invariant $j_{\mathcal{E}'} \in \kappa(X)$. [Indeed, the first equation of (17) follows from Noether's formula (23) together with the fact that $(\omega^0_{\mathcal{E}'/X})^2 = 0$ (which in turn follows from the formula $\omega^0_{\mathcal{E}'/X} = q^*q_*\omega^0_{\mathcal{E}'/X}$; cf. [DR], Proposition II.1.6(ii)). To prove the second equation, we first note that $\mathcal{E}'/X$ has bad reduction at $x \in X$ if and only if $v_x(j_{\mathcal{E}'}) < 0$, i.e. if and only if $x \in j_{\mathcal{E}'}^{-1}(\infty)$, and that for such an x the fibre $\mathcal{E}'_x$ is a Néron polygon of length $-v_x(j_{\mathcal{E}'})$; cf. [Ta]. Thus $\delta_{\mathcal{E}'} = -\sum_x \min(0, v_x(j_{\mathcal{E}'})) = \deg(j_{\mathcal{E}'})$, which proves the second equality of (17).]

Now for $\mathcal{E}'/X$ as above, the map $j_{\mathcal{E}'}$ is given by $j_{\mathcal{E}'} = p_N \circ \beta$, where $p_N : X(N) \to X(1)$ is the standard cover which is Galois with group $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$. Thus, since each $x \in X(N)_\infty = p_N^{-1}(\infty)$ is ramified of degree N, we have $\deg(j_{\mathcal{E}'}) = d \cdot \overline{sl}(N) = dN\#X(N)_\infty$, which therefore proves the second and third equation of (16).

Remark 18 For $d = 1$ the genus of $X = X(N)$ is given by $2g_{X(N)} - 2 = \overline{sl}(N)(\frac{1}{6} - \frac{1}{N})$, and hence we can also write the above formula (16) in the form

$$h_{\mathcal{C}/X(N)} = h_{\mathcal{J}/X(N)} = \frac{1}{2}(2g_{X(N)} - 2 + \#X(N)_\infty).$$

Now since J has bad reduction precisely at $X(N)_\infty$ (cf. Proposition 19 below), this means that C and J have "maximal height", because by a theorem of Faltings[Fa] the right hand side of this equation is the maximum value that the height of a 2-dimensional abelian variety A/F can assume in characteristic 0 when A has good reduction outside of $X(N)_\infty$ (and A contains an elliptic curve defined over K).

In this connection it is interesting to note that in a recent preprint E. Viehweg and K. Zuo[VZ] give a beautiful structure theorem for abelian varieties (of arbitrary dimension) for which the Arakelov-Faltings inequality is an equality.

We next examine the *singular locus* of $\mathcal{C}/X$, i.e. the set $S(\mathcal{C}/X) \subset X$ of points $x \in X$ such that the fibre $\mathcal{C}_x$ is singular. Note that $S(\mathcal{C}/X)$ contains the set $S_0(\mathcal{C}/X) := \tilde{S}(\mathcal{J}/X)$ of points $x \in X$ where J has bad reduction, i.e. where the fibre $\mathcal{J}_x$ of the Néron model $\mathcal{J}$ is not proper (use [BLR], 9.4/4). Thus we can write

$$S(\mathcal{C}/X) = S_0(\mathcal{C}/X) \,\dot{\cup}\, S_1(\mathcal{C}/X), \quad \text{with } S_1(\mathcal{C}/X) := S(\mathcal{C}/X) \setminus S_0(\mathcal{C}/X).$$

Proposition 19 *(a) The set of points where the Jacobian J has bad reduction is the set $S_0(\mathcal{C}/X) = S(\mathcal{E}'/X) = X_\infty := \beta^{-1}(X(N)_\infty)$. If $x \in S_0(\mathcal{C}/X)$, then $\mathcal{C}_x$ contains a unique component $\mathcal{C}_{x,0}$ whose normalization has genus 1, and the induced morphism $f_{\mathcal{C}_{x,0}} : \mathcal{C}_{x,0} \to E_x = E$ is finite of degree N. Furthermore, there is at least one singular point $P_x \in \mathcal{C}_x$ such that $\mathcal{C}_x \setminus \{P_x\}$ is connected.*

(b) The set of points where the curve C has bad reduction (i.e. the singular locus of $\mathcal{C}/X$) is $S(\mathcal{C}/X) = X \setminus U$, where $U = \beta^{-1}(H)$, and so $S_1(\mathcal{C}/X) = X' \setminus U$, where $X' = X \setminus X_\infty$. If $x \in S_1(\mathcal{C}/X)$, then the fibre $\mathcal{C}_x$ of $\mathcal{C}$ at x is a chain $\mathcal{C}_x = \mathcal{C}_{x,0} \cup \ldots \cup \mathcal{C}_{x,r}$ of smooth curves $\mathcal{C}_{x,i}$. Furthermore, $\mathcal{C}_{x,0} = E_{x,1}$ and $\mathcal{C}_{x,r} = E_{x,2}$ are elliptic curves with self-intersection number $E_{x,i}^2 = -1$, whereas all others components are rational curves with self-intersection number $\mathcal{C}_{x,j}^2 = -2$. In addition, the induced maps $f_{x,i} : E_{x,i} \to E_x = E$ are finite of total degree N, i.e. $\deg(f_{x,1}) + \deg(f_{x,2}) = N$.

Proof. Many of these assertions follow from the discussion of [FK], pp. 161–166, at least when N is odd and the Weierstrass points are F-rational. However, since we do not make these assumptions here, it seems better to give a direct proof of these facts.

(a) Since $J \sim E_F \times \mathcal{E}'_F$, the connected component $\mathcal{J}_x^0$ of $\mathcal{J}_x$ is isogenous to $E \times (\mathcal{E}'_x)^0$ (cf. [BLR], 7.4/7), and so $S_0(\mathcal{C}/X) = S(\mathcal{E}'/X)$ (use [BLR], 7.4/2 and 7.3/7). Furthermore, $S(\mathcal{E}'/X) = X_\infty$, as was shown in the proof of Theorem 17.

Now if $x \in S(\mathcal{E}'/X)$, then $(\mathcal{E}'_x)^0 \simeq \mathbb{G}_m$, and hence $\mathcal{J}_x^0$ has a 1-dimensional torus part and a 1-dimensional abelian part. Thus, since $\mathrm{Pic}^0_{\mathcal{C}_x/K} \simeq \mathcal{J}_x^0$ (cf. [BLR], 9.5/4), it follows from [BLR], 9.2/8, that $\mathcal{C}_x$ has a unique component whose normalization has genus 1 and that the graph of components of $\mathcal{C}_x$ has a loop, i.e. there exists a point P_x with the asserted property.

To verify the assertion about $f_{\mathcal{C}_{x,0}}$, we will use the fact that since f is generically finite of degree N, we have $f_*f^*D = ND$, for every divisor D on E_X; cf. Fulton [Fu2], Example 8.1.7. Now for $D = E_x$ we have $f^*E_x = \mathcal{C}_x = \mathcal{C}_{x,0} + \mathcal{C}_{x,1} + \ldots + \mathcal{C}_{x,r}$, where the $\mathcal{C}_{x,i}$ are the components of $\mathcal{C}_x$. But since $\mathcal{C}_{x,i}$ is rational for $i \geq 1$ and since $E_x \simeq E$ is an elliptic curve, we see that $f(\mathcal{C}_{x,i})$ is a point and so $f_*(\mathcal{C}_{x,i}) = 0$ for $i \geq 0$. Thus $f_*\mathcal{C}_{x,0} = f_*f^*E_x = NE_x$, which means that $f_{\mathcal{C}_{x,0}} : \mathcal{C}_{x,0} \to E_x$ is finite and has degree N, as asserted.

(b) Since $\mathcal{C}_{X(N)}$ has semi-stable reduction, we have $S(\mathcal{C}/X) = \beta^{-1}S(\mathcal{C}_{X(N)}/X(N))$, and so it is enough to verify the first assertion for $X = X(N)$.

Since $\mathcal{C}_H/H$ is smooth, we see that $S := S(\mathcal{C}/X) \subset X(N) \setminus H$. To prove the opposite inclusion, we first recall some facts about the construction of $H = H_{E/K,N}$ from [Ka3], §5.3:

1) Since K is algebraically closed, we have $X_{E/K,K,-1} \simeq X'(N) = X(N) \setminus X(N)_\infty$ (cf. [Ka3], Theorem 4.1). Moreover, there is a principally polarized abelian scheme $(\mathcal{J}', \lambda')$ over $X' := X'(N)$ whose restriction to H is the Jacobian scheme J_{C_H}/H of the universal genus 2 curve C_H/H (use [Ka3], Corollary 5.11, Proposition 5.12 and Theorem 5.18).

2) The set H is the open subscheme of points $x \in X'(N)$ such that $(\mathcal{J}'_x, \lambda'_x)$ is *theta-smooth* (in the sense of [Ka3], §5.3); cf. [Ka3], Proposition 5.17.

Note that 1) shows that $\mathcal{J}'/X'$ is the Néron model of J_F over X' because $\mathcal{J}'/X'$ is an abelian scheme with generic fibre J_F (use [BLR], 1.2/8). Thus $\mathcal{J}' \simeq \mathcal{J}_{|X'}$ and λ' is the unique extention of the canonical polarization λ of the Jacobian J_{C_H}/H.

Now suppose that $x \in X(N) \setminus S$. Then $\mathcal{C}_x$ is smooth, and so $\mathcal{C}_T/T$ is a smooth relative curve over $T = \mathrm{Spec}(\mathcal{O}_{X,x})$. Then the principally polarized Jacobian $(J_{\mathcal{C}_T}, \lambda_T)$ of $\mathcal{C}_T$ is theta-smooth (cf. [Ka3], Proposition 5.14). But $J_{\mathcal{C}_T} \simeq \mathcal{J}'_{|T}$ because both are Néron models of J_F over T, and also $\lambda_T = \lambda'_{|J_{\mathcal{C}_T}}$, and so $\mathcal{J}'$ is theta-smooth at x, i.e. $x \in H$. Thus $X(N) \setminus S \subset H$ or $S \supset X(N) \setminus H$, and so $S = X(N) \setminus H$, as claimed.

Now let $x \in S_1 = S_1(\mathcal{C}/X)$, where X is again a general cover of $X(N)$. Then by definition $\mathcal{C}_x = \mathcal{C}_{x,0} \cup \ldots \cup \mathcal{C}_{x,r}$ is singular whereas $\mathcal{J}_x$ is an abelian variety. Thus, by [BLR], 9.3/12 (together with [BLR], 9.5/4) it follows that all components of $\mathcal{C}_x$ are smooth and the configuration of components is tree-like. Moreover, since $2 = \dim \mathcal{J}_x = \sum_i g(\mathcal{C}_{x,i})$ by [BLR], 9.2/8, we see that $\mathcal{C}_x$ has precisely two components of genus 1. These have to be in fact the two ends of the graph, for any rational end component would be a (-1)-curve on $\mathcal{C}_x$, which does not exist since $\mathcal{C}$ is minimal. Thus, $\mathcal{C}_x$ is a chain whose ends are the two elliptic curves.

The fact that the self-intersection numbers are as indicated is proved in Ogg[Ogg], p. 360; note that here we have type 13 in Ogg's list (and type IV is Parshin's list[Pa2]).

To prove the last assertion, we first observe that the same argument as in (a) shows that $N = \deg(f_{x,1}) + \deg(f_{x,2})$ (because all other components of $\mathcal{C}_x$ map to a point). Thus, at least one of the maps $f_{x,i}$ is finite, but it seems more difficult to show that *both* are finite. To prove this, it is clearly enough to show that $f(E_{x,i}) = E_x$, for $i = 1, 2$ and to verify this, we may assume (in view of (14) and the fact that β_{E_X} is finite) that F is sufficiently large. In that case this follows immediately from the following result which is also of independent interest.

Proposition 20 *Suppose that either N is odd or that the Weierstrass points of C_F are F-rational. Then there exists a morphism $j' : \mathcal{C}' := p^{-1}(X') \to \mathcal{J}' := p_{\mathcal{J}}^{-1}(X') \subset \mathcal{J}$ such that*

(18) $$j' \circ \sigma_{\mathcal{C}'} = [-1]_{\mathcal{J}'} \circ j' \quad \text{and} \quad \lambda_{E_{X'}} \circ f_{|\mathcal{C}'} = f_* \circ j',$$

where $f_* : \mathcal{J}' \to J_{E_{X'}} = J_E \times X'$ is the homomorphism on the abelian schemes induced by f, and $\lambda_{E_{X'}} : E_{X'} \xrightarrow{\sim} J_{E_{X'}}$ is the canonical polarization. In addition, $\theta := j'(\mathcal{C}')$ is a theta-divisor of the principally polarized abelian scheme $(\mathcal{J}', \lambda_{\mathcal{J}'})$ over X', and $(f_*)_{|\theta} : \theta \to E_{X'}$ is finite. In particular, $f(E_{x,i}) = E_x$, for all $x \in S_1(\mathcal{C}/X) \subset X'$.

Proof. By [Ka3], Proposition 2.4, there exists a morphism $j_F : C_F \hookrightarrow J_F$ such that (18) holds for the generic fibres over F. Since $\mathcal{J}'/X'$ is an abelian scheme, it follows from Proposition 15 that j_F extends uniquely to a morphism $j' : \mathcal{C}' \to \mathcal{J}'$. Moreover, by the universal property of Néron models, $(f_F)*$ extends uniquely to a homomorphism $f_* : \mathcal{J}' \to J_{E_{X'}}$, and so (18) holds because it is true on the generic fibres.

Since j' is proper, $\theta = j'(\mathcal{C}')$ is the closure of $j_F(C_F)$ in $\mathcal{J}'$, and so is an irreducible divisor on the smooth 3-fold $\mathcal{J}'/K$. Thus θ cannot contain any component of a fibre of $\mathcal{J}'/X'$ and so is flat over X' (since X' is a smooth curve). Thus $\theta_x = \theta_{|\mathcal{J}'_x}$ is an effective divisor on $\mathcal{J}'_x$, for every $x \in X'$. Now since its generic fibre $\theta_F = \theta_{|J_F}$ is the theta-divisor of J_F by [Ka3], Corollary 2.5, we see that $\theta_x^2 = \theta_F^2 = 2$ for all $x \in X'$, and so each θ_x is the theta-divisor of a principal polarization on $\mathcal{J}'_x$, which means that $\lambda := \lambda_{\mathcal{L}(\theta)} : \mathcal{J}' \to \hat{\mathcal{J}}'$ defines a principal polarization. But since λ agrees with the canonical polarization $\lambda_{\mathcal{C}}$ on the generic fibre, we have $\lambda = \lambda_{\mathcal{J}'}$, and so θ is a theta-divisor associated to $\lambda_{\mathcal{J}'}$.

To show that $(f_*)_{|\theta}$ is finite, we first observe that we have an exact sequence

$$0 \to \mathcal{E}'_{X'} \xrightarrow{h'} \mathcal{J}' \xrightarrow{f_*} J_{E_{X'}} \to 0, \tag{19}$$

where $h' : \mathcal{E}'_{X'} \to \mathcal{J}'$ denotes the canonical extension (by the universal property of Néron models) of the map $h'_F : \mathcal{E}'_F \to J_F$ constructed in [Ka3], Proposition 2.7. Thus (19) is exact when restricted to the generic fibres over F. Now since $f_F^* : J_{E_F} \to J$ is an injection and $(f_*)_F \circ f_F^* = [N]$, the hypotheses of [BLR], 7.5/3(a) are satisfied and so (19) is an exact sequence because $\mathcal{J}'/X'$ is an abelian scheme.

Now let $x \in X'$. If θ_x is irreducible (hence also reduced), then $(f_*)_{|\theta_x} : \theta_x \to J_E$ is a surjective map f_x between integral proper curves and hence is finite. (Recall that f_x is surjective and hence also $(f_*)_{|\theta_x}$.) Thus assume that θ_x is reducible, i.e. $\theta_x = \theta_{x,1} + \theta_{x,2}$, where each $\theta_{x,i}$ is an elliptic curve on $\mathcal{J}'_x$; cf. [We], Satz 2. Now if $(f_*)_{|\theta_x}$ is not finite, then $f_*(\theta_{x,i})$ is a point for some i, and then a translate of $\theta_{x,i}$ lies in $E'_x := h'(\mathcal{E}'_x) = \mathrm{Ker}((f_*)_x)$. Since both are irreducible curves, we must have equality, and so $(\theta_x.E'_x) = (\theta_x.\theta_{x,i}) = 1$. But $(\theta_x.E'_x) = (\theta_F.E'_F) = \deg(f_F) = N$, contradiction. Thus $(f_*)_{|\theta_x}$ is finite for all x, and hence so is $(f_*)_\theta$.

Finally, if $x \in S_1$, then $j'(\mathcal{C}_x) = j'(E_{x,1}) \cup j'(E_{x,2}) = \theta_x$, and neither of the $j'(E_{x,i})$ can be a point for otherwise θ_x is an elliptic curve, which is impossible since $\theta_x^2 = 2$. Thus also $f(E_{x,i}) = f_*(j'(E_{x,i}))$ cannot be a point, and so $f(E_{x,i}) = E$, as claimed.

In the case that $X = X(N)$, the cardinality of the singular locus given by the following result.

Theorem 21 *If $X = X(N)$, then the number of points in $S_1 = S_1(\mathcal{C}/X)$ satisfies the inequality*

$$\#S_1 \le \frac{1}{12}(5N-6)\#X(N)_\infty = \frac{1}{12N}(5N-6)\overline{sl}(N). \tag{20}$$

Thus, the singular locus $S = S(\mathcal{C}/X)$ consists of at most $\frac{1}{12N}(5N+6)\overline{sl}(N)$ points.

Furthermore, equality holds in (20) if and only if either $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > N$*, i.e. we have*

$$\#S_1 = \frac{1}{12N}(5N-6)\overline{sl}(N) \;\Leftrightarrow\; \mathrm{char}(K)\not| N!. \tag{21}$$

Proof. By Proposition 19(b) we have $S_1 = X'(N)\backslash H = X_{E/K,N,-1}\backslash H_{E/K,N} = D_{E/K,N}$ (in the notation of [Ka3], §6). Since $\#X(N)_\infty = \frac{1}{N}\overline{sl}(N)$ (cf. (16)), we see that the assertions follow directly from [Ka3], Theorem 6.2 (which is actually a restatement of the results of [Ka1],[Ka2]).

In the case that $\mathrm{char}(K)\not| N!$, the precise structure of the fibres of $\mathcal{C}/X$ can be determined as follows.

Theorem 22 *If* $\mathrm{char}(K)\not| N!$ *and* $X = X(N)$*, then* $\mathcal{C}/X$ *is a stable curve which has precisely one singular point in each singular fibre* $\mathcal{C}_x$*, and so* $\delta_{\mathcal{C}/X} = \#S(\mathcal{C}/X) = \frac{1}{12N}(5N+6)\overline{sl}(N)$*. Thus, the structure of the singular fibres is as follows:*

(a) If $x \in S_1$*, then* $\mathcal{C}_x = E_{x,1} \cup E_{x,2}$ *is the union of two curves of genus* 1 *which meet transversely in a unique point* P_x*.*

(b) If $x \in S_0 = S_\infty$*, then the fibre* $\mathcal{C}_x$ *is an irreducible curve whose normalization is a curve of genus 1, and* $\mathcal{C}_x$ *has a unique singular point* $P_x \in \mathcal{C}_x$*.*

Proof. Let δ_1 (respectively δ_0) denote the number of singular points of the fibres of $\mathcal{C}/X$ which *disconnect* (respectively, which *do not disconnect*) the fibre; thus $\delta := \delta_{\mathcal{C}/X} = \delta_0 + \delta_1$ is the total number of singular points on all fibres of $\mathcal{C}/X$. By Proposition 19 we have the inequalities

$$\delta_0 \geq \#S_0 = \#X(N)_\infty \quad \text{and} \quad \delta_1 \geq \#S_1. \tag{22}$$

There are two important relations which connect the four invariants $h = h_{\mathcal{C}/X}$, $\omega^2 = (\omega^0_{\mathcal{C}/X})^2$, δ_0, and δ_1. The first of these is *Noether's formula*

$$12h = \omega^2 + \delta = \omega^2 + \delta_0 + \delta_1, \tag{23}$$

which is valid for any semi-stable curve of genus $g \geq 1$; cf. e.g. Szpiro[Sz], Lemma 1 (p. 48). For genus 2 curves, however, we have in addition *Mumford's formula*

$$\omega^2 = \frac{1}{5}\delta_0 + \frac{7}{5}\delta_1; \tag{24}$$

cf. [Mu], formula (8.5) on p. 317 (together with the remarks of [Sa] on p. 237), or Ueno[U], formula (2.4), together with Theorem 2.9 (specialized to the semi-stable case). (See also Remark 23(b) below.) Combining these two relations yields

$$12h = \frac{6}{5}\delta_0 + \frac{12}{5}\delta_1. \tag{25}$$

In our situation, this identity, together with the inequalities (22), yields

$$12h \geq \frac{6}{5}\#S_0 + \frac{12}{5}\#S_1 = \frac{6}{5}\#S_0 + \frac{1}{5}(5N-6)\#S_0 = N\#S_0, \tag{26}$$

where the second last equality follows from (21). (Here we used the hypothesis that $\mathrm{char}(K)\not| N!$.) But since $12h = N\#X(N)_\infty = N\#S_0$ by (16), it follows that all the inequalities in (26) and hence in (22) were equalities, i.e. we have

$$\delta_0 = \#S_0 = \frac{1}{N}\overline{sl}(N) \quad \text{and} \quad \delta_1 = \#S_1 = \frac{1}{12N}(5N-6)\overline{sl}(N). \tag{27}$$

We thus see that each singular fibre contains a unique singular point, as claimed. From this and Proposition 19, the assertions of parts (a) and (b) follow readily.

Remark 23 (a) Note that (16), (21) and (25) imply that the following converse of Theorem 22 holds: if each singular fibre $\mathcal{C}_x$ of $\mathcal{C}/X(N)$ contains a unique singular point, then $\mathrm{char}(K)\not| N!$.

(b) If we combine Noether's formula (23) and Mumford's formula (24) so as to eliminate δ_1, then we obtain the relation

$$h = \frac{1}{7}\omega^2 + \frac{1}{14}\delta_0, \tag{28}$$

which, via Noether's formula, is equivalent to Mumford's relation (24). Now in [FK], Corollary 4.2, a weak form of this equality was derived (in the arithmetic case) by studying the intersection numbers of the components of the Weierstrass divisor W. In the geometric case, this method can be refined to derive the above equality (28) directly.

Corollary 24 *If* $\mathrm{char}(K)\not| N!$*, then the self-intersection number of* $\omega^0_{\mathcal{C}/X(N)}$ *is given by*

$$(\omega^0_{\mathcal{C}/X(N)})^2 = \frac{1}{5}\#S_0 + \frac{7}{5}\#S_1 = \frac{1}{12N}(7N-6)\overline{sl}(N). \tag{29}$$

Proof. Substitute the values of δ_0, δ_1 and h which are given by (27) and (16) in Noether's formula (23).

Corollary 25 *If* $\mathrm{char}(K)\not| N!$ *and* $X = X(N)$*, then* $f : \mathcal{C} \to E_X$ *is finite and flat of degree* N*. Furthermore, we have* $f(P_x) \in E_x[2] = E[2]$*, for each singular point* $P_x \in \mathcal{C}_x$*.*

Proof. By Proposition 19 and Theorem 22 we know that f does not map any irreducible component of $\mathcal{C}_x$ to a point, and so f is quasi-finite. Since f is proper, this means that f is finite (cf. [Ha], Ex. III.11.2). Moreover, since $\mathcal{C}$ and E_X are smooth surfaces over K, it follows that f is also flat; cf. [Ha], Ex. III.9.3(a).

The last assertion is immediate: since P_x is the unique singular point in its fibre $\mathcal{C}_x$, it is fixed by the hyperelliptic involution $\sigma_{\mathcal{C}}$ and hence $f(P_x)$ is fixed by $[-1]_{E_X}$ by (15).

5 The different divisor

Let $D_F = \mathrm{Diff}(f_F)$ denote the different divisor of the generic cover $f_F : \mathcal{C}_F \to E_F$, and let $D_{\mathcal{C}/X}$ denote its closure in $\mathcal{C}$. Then there is a close connection between $D_{\mathcal{C}/X(N)}$, the discriminant map $\delta : H_{E,N} \to \mathbb{P}^1$ (defined in §3), the different divisor $\mathrm{Diff}(f)$ (cf. Appendix) of the compactified morphism $f : \mathcal{C} \to E_{X(N)}$, and the relative dualizing sheaf $\omega^0_{\mathcal{C}/X(N)}$, *provided that* $\mathrm{char}(K)\not| N!$, *which we assume from now on.*

Theorem 26 *Suppose $X = X(N)$, and write $D = D_{\mathcal{C}/X(N)}$.*

(a) We have $D = \mathrm{Diff}(f)$ and hence $\omega^0_{\mathcal{C}/X} \simeq \mathcal{L}(D)$.

(b) The divisor D is an irreducible curve on $\mathcal{C}$ which meets each singular fibre $\mathcal{C}_x$ precisely at the unique singular point P_x, and $f_{|D} : D \to f(D) \subset E \times X$ is birational.

(c) The morphism $\pi_D := pr_2 \circ f_{|D} : D \to X$ is a σ_D-invariant cover of X of degree 2, where $\sigma_D = (\sigma_{\mathcal{C}})_{|D}$ is the restriction of the hyperelliptic involution to D. In particular, $D/\langle\sigma_D\rangle \simeq X$.

(d) There exists a unique morphism $\delta_D : X \to \mathbb{P}^1$ such that

$$\pi_E \circ q = \delta_D \circ \pi_D, \tag{30}$$

where $q := pr_2 \circ f_{|D} : D \to E$ and $\pi_E : E \to E/\langle[-1]\rangle = \mathbb{P}^1$ is the usual double subcover of E. Furthermore, δ_D restricts to the discriminant map $\delta_{E,N} : H_{E,N} \to \mathbb{P}^1$ on $H_{E,N} \subset X$, and hence $\delta_D = \overline{\delta}_{E,N}$. In addition, we have

$$\deg(\overline{\delta}_{E,N}) = \deg(\delta_D) = \deg(q) = (\omega^0_{\mathcal{C}/X}.f^*(P \times X)). \tag{31}$$

(e) The restriction $\delta^{sm}_{E,N} : H^{sm}_{E,N} \to \mathbb{P}^1 \setminus \pi_E(E[2])$ of $\delta_{E,N}$ to $H^{sm}_{E,N}$ is finite and etale.

Proof. (a) Note first that $\mathrm{Diff}(f)$ exists by Corollary 35 of the Appendix. Now by Corollary 25 we know that for each component Γ of a fibre $\mathcal{C}_x$ of $\mathcal{C}/X$, the morphism $f_{|\Gamma} : \Gamma \to E_x = E$ is finite of degree $\deg(f_{|\Gamma}) \le N$. Since $\mathrm{char}(K)\not| N!$, it follows that $f_{|\Gamma}$ is automatically separable. Thus by Corollary 35(b) we see that $B(f) = 0$, and hence $\mathrm{Diff}(f) = D$. Now since E/K is an elliptic curve, we have $\omega_{E/K} \simeq \mathcal{O}_E$, and hence $\omega^0_{E_X/X} \simeq pr_1^*\omega_{E/K} \simeq \mathcal{O}_{E_X}$. Thus by (47) we obtain $\omega^0_{\mathcal{C}/X} \simeq f^*\omega^0_{E_X/X} \otimes \mathcal{L}(D) \simeq \mathcal{L}(D)$, as claimed.

(b) If $\Gamma \subset \mathcal{C}_x$ is an irreducible component of $\mathcal{C}_x$, where $x \in S = S(\mathcal{C}/X)$, then $\Gamma \setminus \{P_x\}$ is a smooth open subset of an elliptic curve, and so $\Gamma \setminus \{P_x\} \to E$ is etale (since separable). Thus $\mathrm{supp}(\mathrm{Diff}(f_x)) \subset \{P_x\}$, and so D can meet $\mathcal{C}_x$ only in P_x; cf. Corollary 35(b). This proves the second assertion.

The first assertion is a consequence of the second. For if $D = D_1 + D_2$ were reducible, then both components D_i must meet $\mathcal{C}_x$ in P_x. Since P_x is a singular point on $\mathcal{C}_x$, we have $(D_i.\mathcal{C}_x) \ge 2$, so $2 = (D.\mathcal{C}_x) = (D_1.\mathcal{C}_x) + (D_2.\mathcal{C}_x) \ge 4$, contradiction. Thus D is is irreducible.

To see that $f_{|D}$ is birational, we first observe that

$$f(D)^2 \geq D^2 > 0, \quad \text{and hence} \quad f(D) \not\sim A \times X, \text{ for any } A \in \mathrm{Div}(E). \tag{32}$$

Indeed, since $(A \times X)^2 = 0$, it is clear that the second follows assertion from the first. To prove the latter, write $f_*D = kf(D)$; note that $k = 1$ or 2 because $(f_*D.E \times x) = (D.\mathcal{C}_x) = 2$. Since $f^*f(D) = 2D + D'$, where D' is an effective divisor not containing D, we have $k(f(D))^2 = (f_*D.f(D)) = (D.(2D+D')) \geq 2D^2$. Since $D^2 = (\omega^0_{\mathcal{C}/X})^2 > 0$ by (a) and (29), we thus have $f(D)^2 \geq \frac{2}{k}D^2 \geq D^2 > 0$, as claimed.

We next observe that $\sigma^*_{\mathcal{C}}(D) = \mathrm{Diff}(f \circ \sigma_{\mathcal{C}}) = \mathrm{Diff}([-1] \circ f) = \mathrm{Diff}(f) = D$, i.e. $\sigma_{\mathcal{C}}$ maps D into itself, and hence $[-1]f(D) = f\sigma_{\mathcal{C}}(D) = f(D)$.

Suppose now that $D \to f(D)$ were not birational. Then $k = 2$ and $(f(D).E \times x) = 1$, for all $x \in X$, so $f(D)_F \in E(F)$. But since $[-1]f(D) = f(\sigma_{\mathcal{C}}(D)) = f(D)$, this forces $f(D)_F = P_i \in E[2]$. Thus $f(D) = P_i \times X$, which is impossible by (32), and hence $D \to f(D)$ is birational.

(c) Since $\pi_D^{-1} = D \cap \mathcal{C}_x$ and $(D.\mathcal{C}_x) = 2$, for all $x \in X$, we see that π_D is a double cover of X. Moreover, since $\sigma_{\mathcal{C}}$ maps D into itself (cf. proof of part (b)), $\sigma_D = (\sigma_{\mathcal{C}})_{|D}$ is an involution of D. In addition, we have $\pi_D \circ \sigma_D = pr_2 \circ f \circ (\sigma_C)_{|D} = pr_2 \circ [-1] \circ f_{|D} = \pi_D$, so $\pi_D \circ \sigma_D = \pi_D$, i.e. π_D is σ_D-invariant.

We claim that $\sigma_D \neq id_D$. Indeed, since $f(D) \neq A \times X$ by (32), there exists a point $(P, x) \in f(D)$ with $P \notin E[2]$. Then $f(\pi_D^{-1}(x)) = \{(P, x), (P', x)\}$, where $P' = [-1]P \neq P$, and so σ_D interchanges the two points of $\pi^{-1}(x)$; in particular, $\sigma_D \neq 1$. Thus the map $D/\langle\sigma_D\rangle \to X$ is birational and hence is an isomorphism as X is smooth.

(d) Since $f(D_F)$ is a $[-1]$-invariant effective divisor of degree 2 on E_F, there is a point $P \in \mathbb{P}^1_F(F)$ such that $\pi^*_E(P) = f(D_F)$. Thus, if Γ denotes the closure of P in $\mathbb{P}^1 \times X$, then we have $(\pi_E \times id_X)^*(\Gamma) = f(D)$. Now since $\deg(P) = 1$, we see that $(\Gamma.\mathbb{P}^1 \times x) = 1$, for all $x \in X$, and so $\Gamma = \Gamma^t_\delta \subset \mathbb{P}^1 \times X$ is the transpose of the graph of a unique morphism $\delta : X \to \mathbb{P}^1$. Note that the equation $(\pi_E \times id_X)^*\Gamma^t_\delta = f(D)$ implies that $\pi_E \circ (pr_2)_{|f(D)} = \delta \circ (pr_1)_{|f(D)}$, and from this (30) follows immediately.

Since $\deg(\pi_D) = \deg(\pi_E) = 2$, the second equality of (31) follows directly from (30), and the third is clear because we have by definition that $\deg(q) = \deg(q^*(P)) = \deg((f^*pr_1^*(P)_{|D}) = (D.f^*(P \times X))$.

It thus remains to show that $(\delta_D)_{|H} = \delta_{E,N}$. For this, we first note that by definition (cf. §3) we have:

$$\delta_{E,N}(x) = \overline{P} \Leftrightarrow \mathrm{Disc}(f_x) = \pi^*_E(\overline{P}) \Leftrightarrow \mathrm{supp}(\mathrm{Disc}(f_x)) \subset \pi_E^{-1}(\overline{P}),$$

for all $x \in H_{E,N}(K)$ and $\overline{P} \in \mathbb{P}^1(K)$, where the latter equivalence follows from the fact that $\mathrm{supp}(\mathrm{Disc}(f_x))$ is $[-1]_E$-invariant (and $\pi_E^{-1}(\overline{P})$ contain no proper $[-1]_E$-invariant subset. On the other hand, for any $x \in X(K)$ we have $\mathrm{supp}(\mathrm{Disc}(f_x)) = pr_1(f(D) \cap (E \times x)) = q(D \cap \mathcal{C}_x) = q(\pi_D^{-1}) \subset \pi_E^{-1}(\delta_D(x))$, where the inclusion follows from (30) because π_D is surjective. Thus we see that $\delta_{E,N}(x) = \delta_D(x), \forall x \in H$.

(e) By Theorem 13(a) we know that $\delta^{sm}_{E,N}$ is etale, hence quasi-finite, so it is enough to show that $\delta^{sm}_{E,N}$ is proper. Moreover, since $\delta_D : X \to \mathbb{P}^1$ is proper, it is enough to show (in view of part (d)) that $H^{sm}_{E,N} = \delta_D^{-1}(\mathbb{P}^1 \setminus \pi_E(E[2]))$. Now if $x \in \delta_D^{-1}(\mathbb{P}^1 \setminus \pi_E(E[2]))$, then $\mathcal{C}_x$ is smooth by part (b), and so $x \in H^{sm}_{E,N}$ by definition (and (30)). Thus $H^{sm}_{E,N} = \delta_D^{-1}(\mathbb{P}^1 \setminus \pi_E(E[2]))$ since the opposite inclusion is trivial.

Remark 27 From the above theorem we see that the curve $f(D) = f_*(\mathrm{Diff}(f))$ is the *discriminant* of f. Moreover, formula (31) shows that $(f(D).P \times X) = \deg(q)$, so $f(D) \subset E \times X$ has bi-degree $(\deg(q), 2)$.

6 The Weierstrass divisor W

In order to compute the degree of q, we shall exploit the properties of the *Weierstrass divisor* $W_{\mathcal{C}/X}$ on $\mathcal{C}$. By definition, $W_{\mathcal{C}/X}$ is the closure in $\mathcal{C}$ of the (usual) Weierstrass divisor W_F on C_F; recall that the latter is the locus of fixed points of the hyperelliptic involution σ_F on C_F, and hence is an effective divisor of degree $\deg(W_F) = 6$.

As in [Pa2] and [FK], the discussion of Weierstrass divisors becomes much easier when all the Weierstrass points of C_F are F-rational. We therefore study $W_{\mathcal{C}/X}$ first in the case that $X = X(2N)$ and then relate it to that of $X(N)$.

Proposition 28 *Let $X = X(2N)$. Then we have:*

(a) The Weierstrass points of C_F are F-rational, and so $W_{\mathcal{C}/X} = W_1 + \ldots + W_6$, where the W_i's are disjoint *sections of $\mathcal{C}/X$.*

(b) For each $x \in S_1(\mathcal{C}/X)$, the base-change map $(\beta_{\mathcal{C}})_x : \mathcal{C}_x \to \mathcal{C}_{\beta(x)}$ is an isomorphism.

(c) For each $x \in S_0(\mathcal{C}/X) = X_\infty$, the fibre $\mathcal{C}_x = C_x \cup B_x$ is the union of an elliptic curve C_x and a rational curve $B_x \simeq \mathbb{P}^1$, and we have $C_x^2 = B_x^2 = -(B_x.C_x) = -2$. Furthermore, $(\beta_{\mathcal{C}})_x(B_x) = P_{\beta(x)}$, the unique singular point of $\mathcal{C}_{\beta(x)}$.

(d) If $D \in \mathrm{Div}(\mathcal{C}_{X(N)})$ is the closure of an effective divisor D_{F_N} on C, and if $D^* \in \mathrm{Div}(\mathcal{C})$ denotes the closure of the divisor $(\beta_{C_F})^*(D_{F_N})$ on C_F, then we have

(33) $$\beta_{\mathcal{C}}^*(D) = D^* + \sum_{x \in X_\infty} n_x B_x, \quad \text{where} \quad n_x = \tfrac{1}{2}(D^*.B_x).$$

In particular, we have $\beta_{\mathcal{C}}^*(W_{\mathcal{C}/X(N)}) = W_{\mathcal{C}/X} + B$, where $B := \sum_{x \in X_\infty} B_x$.

(e) We have $\omega^0_{\mathcal{C}/X} \simeq \beta_{\mathcal{C}}^*(\omega^0_{\mathcal{C}/X(N)})$, and so

(34) $$(\omega^0_{\mathcal{C}/X(N)}.W_{\mathcal{C}/X(N)}) + W^2_{\mathcal{C}/X(N)} = \#X(N)_\infty = \frac{1}{N}\overline{sl}(N).$$

Proof. (a) The first assertion follows from the fact that the 2-torsion points of J_F are F-rational. To verify the latter claim, recall that $J_F = A_F/H$, where $A_F = E_F \times \mathcal{E}'_F$ and $H \le A_F[N]$ is a suitable subgroup (of order N^2); cf. [Ka3], Corollary 5.19. Since the $2N$-torsion points $\mathcal{E}'_F$ are F-rational, it follows that the same is true for $A_F = E_F \times \mathcal{E}'_F$, and so the 2-torsion points of $J_F = A_F/H$ are F-rational, as desired.

Thus we see that $W_{\mathcal{C}/X} = W_1 + \ldots + W_6$ is the sum of six sections. Now by the same argument as in [FK], these meet each fibre in different places (because for each $x \in X$ the kernel of the reduction maps $J_C \to (\mathcal{J})_x$ does not contain any non-trivial 2-torsion points), and so it follows that these sections are disjoint.

(b) This is clear because $\beta : X(2N) \to X(N)$ is unramified outside of $X(N)_\infty$.

(c) Since $e_x(\beta) = 2N/N = 2$, the singularity $P_{\beta(x)}$ of $\mathcal{C}_{\beta(x)}$ (cf. Theorem 22) is resolved by a line $B_x \simeq \mathbb{P}^1$ which meets the elliptic curve C_x in two distinct points. Thus we have case II of Parshin's list[Pa2], and so the intersection numbers are as indicated.

(d) By parts (b) and (c) we see that $\beta^*_{\mathcal{C}}(D) = D^* + \sum_{x\in X_\infty} n_x B_x$ for some integers $n_x \in \mathbb{Z}$. Since $(\beta_{\mathcal{C}})_* B_x = 0$, we obtain from the projection formula that $0 = (\beta^*_{\mathcal{C}}(D).B_x) = (D^*.B_x) + n_x B_x^2$, and so $n_x = \frac{1}{2}(D^*.B_x)$, which proves (33).

By Parshin[Pa2], p. 80, we know that $(W_{\mathcal{C}/X}.B_x) = 2$, for all $x \in X_\infty$, and so the formula for $\beta^*_{\mathcal{C}}(W_{\mathcal{C}/X(N)})$ follows from (33).

(e) The first assertion is a general fact about relative dualizing sheaves; cf. [Ar], Lemma 3.4 or [Sz], Lemma 3(b), p. 50.

To prove (34), write $W = W_{\mathcal{C}/X(N)}$, $\omega = \omega^0_{\mathcal{C}/X(N)}$ and $\tilde{W} = W_{\mathcal{C}/X(2N)}$, $\tilde{\omega} = \omega^0_{\mathcal{C}/X(2N)}$. In addition, let $d = \deg(\beta) = \overline{sl}(2N)/\overline{sl}(N)$. (Note that $d = 8$ if N is even and $d = 6$ if N is odd.) Then $(\beta_{\mathcal{C}})_*(\tilde{W}) = dW$, and so by the projection formula and part (d) we have

$$d((\omega + W).W) = (\beta^*_{\mathcal{C}}(\omega + W).\tilde{W}) = ((\tilde{\omega} + \tilde{W} + B).\tilde{W}).$$

Now by the adjunction formula we have for any section W_i of $\mathcal{C}/X$ that $(\tilde{\omega}.W_i) = -W_i^2$, and so by part (a) we have $((\tilde{\omega}+\tilde{W}).\tilde{W}) = 0$ because $\tilde{W} = W_1+\ldots+W_6$ consists of 6 *disjoint* sections (so $\tilde{W}^2 = W_1^2 + \ldots + W_6^2$). Moreover, since $((\tilde{W} + B).B) = (\beta^*_{\mathcal{C}}(W).B) = (W.(\beta_{\mathcal{C}})_*(B)) = 0$, we have $(\tilde{W}.B) = -B^2 = 2\#X_\infty$, the latter by part (c). Now since $X_\infty = \beta^{-1}(X(N)_\infty)$, and each $x \in X_\infty$ is ramified of degree $e_x(\beta) = 2$, we see that $\#X_\infty = \frac{d}{2}\#X(N)_\infty$, and so $(\tilde{W}.B) = 2\#X_\infty = d\#X(N)_\infty$. Thus $d((\omega + W).W) = ((\tilde{\omega} + \tilde{W}).\tilde{W}) + (B.\tilde{W}) = 0 + d\#X(N)_\infty$, which proves (34).

Proposition 29 *Suppose that $X = X(N)$ and write $D = D_{\mathcal{C}/X}$ and $W = W_{\mathcal{C}/X}$. Then we have $6D \sim 2W + p^*A$, for some divisor $A \in \mathrm{Div}(X)$ of degree*

$$\deg(A) = \#X(N)_\infty - \frac{4}{3}W^2 = \frac{1}{6}(9(\omega^0_{\mathcal{C}/X})^2 - W^2) = \frac{1}{N}(N-1)\overline{sl}(N). \tag{35}$$

Proof. We first note that

$$(W.E_{x,1}) = (W.E_{x,2}) = 3, \quad \text{for all } x \in S_1. \tag{36}$$

Indeed, by [Pa2], p. 81, this is true after base-change to $X(2N)$, with W replaced by $W_{\mathcal{C}/X(2N)}$, and so (36) follows from this together with Proposition 28(b),(d).

Now since $6D_F$ and $2W_F$ are both 6-canonical divisors, we have $6D_F \sim 2W_F$. Moreover, since the fibres over S_1 are the only reducible fibres of $\mathcal{C}/X$ (cf. Theorem 22) and since for $x \in S_1$, both $6D$ and $2W$ meet each component $E_{x,i}$ of $\mathcal{C}_x$ with multiplicity 6 (by Theorem 26(b) and by (36), respectively), we can conclude that

$$6D \sim 2W + p^*(A), \quad \text{for some } A \in \mathrm{Div}(X). \tag{37}$$

Here we have used the following general fact (which is valid for any regular proper curve $\mathcal{C}/X$ without multiple fibres):

Fact. *Suppose that $D_1, D_2 \in \mathrm{Div}(\mathcal{C})$ are two divisors such that their restrictions $(D_i)_F$ to the generic fibre are linearly equivalent, i.e. $(D_1)_F \sim (D_2)_F$ and which have the property that $(D_1.\Gamma) = (D_2.\Gamma)$, for all components Γ of reducible fibres $\mathcal{C}_x$ of $\mathcal{C}/X$. Then $D_1 \sim D_2 + p^*A$, for some divisor $A \in \mathrm{Div}(X)$.*

[Indeed, the first hypothesis yields that $D_1 \sim D_2 + B$, for some divisor B consisting entirely of fibre components, and the second shows that $(B.\Gamma) = 0$, for all fibre components Γ, and so $B = p^*A$, for some A by Zariski's Lemma (cf. [BPV], p. 90).]

We now compute the degree of A. For this, recall first from Theorem 26(a) that $\omega := \omega^0_{\mathcal{C}/X} \simeq \mathcal{L}(D)$. Thus, by (34) and (37) we have $\#X(N)_\infty - W^2 = (\omega.W) = \frac{1}{6}((2W + p^*(A)).W) = \frac{1}{3}W^2 + \deg(A)$, and so $\deg(A) = \#X(N)_\infty - \frac{4}{3}W^2$, which proves the first equality of (35).

Next, by (37) we have $36\omega^2 = (2W + p^*A)^2 = 4W^2 + 4(W.p^*A) + (p^*A)^2 = 4W^2 + 24\deg(A) + 0$, so $\deg(A) = \frac{1}{6}(9\omega^2 - W^2)$, which proves the second equality of (35).

Thus we have $6\deg(A) = (9\omega^2 - W^2) = 6\#X(N)_\infty - 8W^2$, and so

$$W^2 = \frac{6}{7}\#X(N)_\infty - \frac{9}{7}\omega^2 = -\frac{3}{4N}(N-2)\overline{sl}(N), \tag{38}$$

the latter by (29). From this, the last equation of (35) is immediate.

Corollary 30 *The degree of q is given by*

$$\deg(q) = \frac{N}{6}\deg(A) = \frac{1}{6}(N-1)\overline{sl}(N). \tag{39}$$

Proof. Since f is normalized, we have $f_*(W) = \sum_{i=0}^{3} k_i(P_i \times X)$ with suitable multiplicities $k_i \geq 0$ (cf. equation (6)). Now since $2P_i \sim 2P_0$, we thus have $2f_*W \sim 12(P_0 \times X)$, and so, since $\omega := \omega^0_{\mathcal{C}/X} \sim D$ by Theorem 26(a), we obtain

$$6f_*\omega \sim 6f_*D \sim 2f_*W + f_*(p^*A) \sim 12(P_0 \times X) + N(E \times A).$$

Thus, by (31) and the projection formula we have $6\deg(q) = 6(\omega.f^*(P_0 \times X)) = ((12(P_0 \times X) + N(E \times A)).(P_0 \times X)) = N\deg(A)$, which proves the first equality of (39). The second follows immediately from (34).

By the above corollary we have computed the degree $\deg(\delta_{E,N})$ and hence also the right hand side of formula (12). Thus, the desired formula for $\mathrm{Cov}^{(\min)}_{E,N,D}$ of Theorem 3 follows once we know that $\overline{\delta}_{E,N}$ is tamely ramified.

Proposition 31 *If* $\mathrm{char}(K) \nmid N!$, *then the map* $\overline{\delta}_{E,N} : X(N) \to \mathbb{P}^1$ *is tamely ramified.*

Proof. We shall use the criterion given in Corollary 14. For this, fix a point $P \in E(K)$. Since $\deg(\overline{\delta}_{E,N}) = \deg(q) = \frac{1}{6}(N-1)\overline{sl}(N)$ by (31) and (39) and since $(2g_{X(N)} - 2 + s_{E,N}) = \frac{N-6}{6N}\overline{sl}(N) + \frac{5N+6}{12N}\overline{sl}(N) = \frac{7N-6}{12N}\overline{sl}(N)$ by Theorem 21, we see that (12) reduces to the inequality

$$(40) \qquad \#\mathrm{Cov}^{(\min)}_{E,N,2P} \geq \left(\frac{N-1}{3} - \frac{7N-6}{12N}\right)\overline{sl}(N) = \frac{(4N-3)(N-2)}{12N}\overline{sl}(N),$$

and that equality holds if and only if $\overline{\delta}_{E,N}$ is tamely ramified. In particular, we see that equality holds if $\mathrm{char}(K) = 0$.

We now derive another formula for $\#\mathrm{Cov}^{(\min)}_{E,N,2P}$ by using the *tame fundamental group* $\pi_1^t(E_P, x)$, where $E_P = E \setminus \{P\}$, and $x \in E_P(K)$ is a fixed base point. For this, let $p = \mathrm{char}(K)$, if $\mathrm{char}(K) > 0$, and otherwise let p be any prime with $p > N$. Now the hypothesis $\mathrm{char}(K) \nmid N!$ guarantees that each $f \in \mathrm{Cov}^{(\min)}_{E,N,2P}$ is tamely ramified and that the degree of its Galois closure $\tilde{f} : \tilde{C} \to E$ is not divisible by p. Thus, $\mathrm{Gal}(\tilde{f})$ is a quotient of the group $\pi_1^t(E_P, x)^{(p')}$, the prime-to-p fundamental group, and so we have a natural bijection

$$(41) \qquad \mathrm{Cov}^{(\min)}_{E,N,2P} \xrightarrow{\sim} \mathrm{Hom}'(\pi_1^t(E_P, x)^{(p')}, S_N)/S_N$$

where the symmetric group S_N acts by conjugation on $\mathrm{Hom}'(\ldots)$ and the prime denotes the subset of those homomorphisms $h : \pi_1^t(E_P, x)^{(p')} \to S_N$ satisfying the following conditions: 1) $Im(h)$ is a transitive and primitive subgroup of S_N; 2) the ramification generator $t = t_P \in \mathrm{Hom}'(\pi_1^t(E_P, x)^{(p')}$ at P is mapped either to a 3-cycle or to a $(2,2)$-cycle.

Now by a fundamental result of Grothendieck (cf. e.g. [Wev], Theorem 4.3.1), $\pi_1^t(E_P, x)^{(p')} \simeq \hat{\Gamma}_{1,1}^{(p')}$, the prime-to-$p$ quotient of the group $\hat{\Gamma}_{1,1}$ which is the pro-finite completion of the discrete group $\Gamma_{1,1} = \langle \sigma_1, \sigma_2, \tau : \sigma_1\sigma_2\sigma_1\sigma_2\tau = 1\rangle$. Thus, by applying the above results to an elliptic curve E_0/K_0 with $\mathrm{char}(K_0) = 0$, we see from (41) and (40) that

$$\#\mathrm{Hom}'(\Gamma_{1,1}^{(p')}, S_N)/S_N = \frac{(4N-3)(N-2)}{12N}\overline{sl}(N), \quad \text{for any } p > N;$$

here we have used the fact that $\overline{\delta}_{E_0,N}$ is tamely ramified because $\mathrm{char}(K_0) = 0$. Thus, if we substitute this formula in (41), we see that equality holds in (40), and hence $\overline{\delta}_{E,N}$ is tamely ramified.

Proof of Theorem 3. We may assume $N \geq 3$ because the case $N = 2$ is settled by Proposition 6. Since $\overline{\delta}_{E,N}$ is tamely ramified by Proposition 31, we have by Corollary 14 that equality holds in (12). This proves (3) because $\deg(\overline{\delta}_{E,N}) = \frac{1}{6}(N-1)\overline{sl}(N)$ and $(2g_{X(N)} - 2 + s_{E,N}) = \frac{7N-6}{12N}\overline{sl}(N)$; cf. the proof of Proposition 31.

7 Appendix: The Kähler different divisor

In this appendix we gather together some basic facts concerning the Kähler different divisor which were used above but which are difficult to find explicitly in the literature.

If $f : X \to Y$ is any morphism of finite type, then the *Kähler different ideal sheaf* $\mathcal{D}_f \subset \mathcal{O}_X$ is defined as the 0-th Fitting ideal sheaf $\mathcal{D}_f = F_0(\Omega^1_{X/Y})$ of the sheaf $\Omega^1_{X/Y}$ of relative differentials; cf. Kunz[Ku], p. 159 or [LK], p. 102. If $\mathcal{D}_f$ is *invertible*, then there is a unique effective Cartier divisor $\mathrm{Diff}(f)$ such that $\mathcal{L}(\mathrm{Diff}(f)) = \mathcal{D}_f^{-1}$, which we call the *Kähler different divisor*. We observe:

Remark 32 (a) If the Kähler different divisor $\mathrm{Diff}(f)$ exists, then by [Ku], Theorem 10.7, its support is the *ramification locus* of f:

$$\mathrm{supp}(\mathrm{Diff}(f)) = \mathrm{Ram}(f) := \mathrm{supp}(\Omega^1_{X/Y}). \tag{42}$$

(b) If $g : \tilde{Y} \to Y$ is any morphism of schemes such that $g_X(\tilde{X}') \not\subseteq \mathrm{Ram}(f)$, for all irreducible components $\tilde{X}'$ of $\tilde{X} := X \times_Y \tilde{Y}$, then the Kähler different divisor of $\tilde{f} = f_{\tilde{Y}} : \tilde{X} \to \tilde{Y}$ also exists and is given by

$$\mathrm{Diff}(\tilde{f}) = g_X^* \mathrm{Diff}(f). \tag{43}$$

[Indeed, the hypothesis on g_X implies in view of (42) that the pullback Cartier divisor $g_X^*\mathrm{Diff}(f)$ is defined and corresponds to the invertible ideal sheaf $g_X^* F_0(\Omega^1_{X/Y}) = F_0(\Omega^1_{X/Y}) \cdot \mathcal{O}_{\tilde{X}}$. Since $F_0(\Omega^1_{X/Y}) \cdot \mathcal{O}_{\tilde{X}} = F_0(\Omega^1_{\tilde{X}/\tilde{Y}}) = \mathcal{D}_{\tilde{f}}$ by [Ku], Rule 10.3a), we see that $\mathcal{D}_{\tilde{f}}$ is invertible, and so $\mathrm{Diff}(\tilde{f})$ exists and is given by $g_X^*\mathrm{Diff}(f)$.]

In general, $\mathrm{Diff}(f)$ need not exist, but we do have the following result which suffices for our purposes (and which generalizes the discussion of [I], p. 202):

Proposition 33 *If $f : X \to Y$ is a generically etale morphism between smooth varieties over a field K, then the Kähler different ideal sheaf $\mathcal{D}_f$ is invertible, and* *hence the Kähler different divisor* $\mathrm{Diff}(f) \in \mathrm{Div}(X)$ *exists. Moreover, we have the Riemann-Hurwitz relation*

$$\omega_{X/K} \simeq f^*\omega_{Y/K} \otimes \mathcal{L}(\mathrm{Diff}(f)), \tag{44}$$

where $\omega_{X/K}$ *and* $\omega_{Y/K}$ *denote the canonical sheaves of* X *and* Y*.*

Proof. Since f is generically etale, we see that $\Omega^1_{X/Y}$ is a torsion $\mathcal{O}_X$-module (use [Ku], Corollary 10.5). Moreover, since $\Omega^1_{Y/K}$ is locally free and f is generically etale, we have the exact sequence

$$0 \to f^*\Omega^1_{Y/K} \xrightarrow{u} \Omega^1_{X/K} \to \Omega^1_{X/Y} \to 0. \tag{45}$$

(Indeed, to see that u is injective, note that the local freeness of $\Omega^1_{Y/K}$ implies that it is naturally a subsheaf of the sheaf $\mathcal{M}_Y(\Omega^1_{Y/K}) = \mathcal{M}_Y \otimes_{\mathcal{O}_Y} \Omega^1_{Y/K}$ of meromorphic differential forms, and that hence we have $f^*\Omega^1_{Y/K} \subset f^*\mathcal{M}_Y(\Omega^1_{Y/K}) \xrightarrow{\sim} \mathcal{M}_X(\Omega^1_{X/K})$. Thus, the map $f^*\Omega^1_{Y/K} \to \Omega^1_{X/K}$ is induced by the map $f^*\mathcal{M}_Y(\Omega^1_{Y/K}) \to \mathcal{M}_X(\Omega^1_{X/K})$, which is an isomorphism since f is generically etale.)

Since $u : f^*\Omega^1_{Y/K} \to \Omega^1_{X/K}$ is an injective map of locally free sheaves of rank n, the map $\wedge(u) : f^*\omega_{Y/K} \simeq \wedge^n f^*\Omega^1_{Y/K} \to \omega_{X/K} = \wedge^n\Omega^1 X/K$ is also injective, and so $f^*\omega_{Y/K} \simeq \mathrm{Im}(u) = \mathcal{D}_f\omega_{X/K}$, where the latter equality follows from the definition of the Fitting ideal (and that of $\wedge(u)$). Thus $\mathcal{D}_f$ is invertible and we have (44).

Corollary 34 *Let* S *be an arbitrary scheme, and let* X/S *and* Y/S *be two smooth relative varieties of dimension* n *over* S*, i.e.* X/S *and* Y/S *are smooth of finite presentation over* S *with geometrically integral fibres of dimension* n*. If* $f : X \to Y$ *is any* S*-morphism such that* $f_s : X_s \to Y_s$ *is generically etale, for all* $s \in S$*, then* $\mathcal{D}_f$ *is invertible and the Kähler different divisor* $\mathrm{Diff}(f) \in \mathrm{Div}(X/S)$ *is a relative Cartier divisor of* X/S *which satisfies the Riemann-Hurwitz relation, i.e.* $\omega_{X/S} \simeq f^*\omega_{Y/S} \otimes \mathcal{L}(\mathrm{Diff}(f))$*. Furthermore, the formation of* $\mathrm{Diff}(f)$ *commutes with arbitrary base-change in the sense that for any* $g : S' \to S$ *we have* $\mathrm{Diff}(f_{(S')}) = g_X^*(\mathrm{Diff}(f))$*.*

Proof. We first show that the sequence

$$0 \to f^*\Omega^1_{Y/S} \xrightarrow{u} \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0. \tag{46}$$

is exact. As usual, this sequence is right-exact. To show that u is injective, we follow the method of [LK], p. 106. For this we first observe that $\Omega^1_{X/S}$ is S-flat because $\Omega^1_{X/S}$ is a locally free $\mathcal{O}_X$-module (of rank n) and X/S is flat. Moreover, by the proof of Proposition 33 we know that $u_s = u \otimes \kappa(s) : f_s^*\Omega^1_{Y_s/\kappa(s)} \to \Omega^1_{X_s/\kappa(s)}$ is injective, for all $s \in S$. Thus, by Corollary A.2 of [LK], it follows that u is (universally) injective. (Note that this corollary is applicable because we may reduce to case that S is noetherian by using [EGA] (IV,8.9.1) since X/S and Y/S are of finite presentation.)

From the exact sequence (46) we conclude as before that $f^*\omega_{Y/S} \simeq \mathrm{Im}(u) = \mathcal{D}_f\omega_{X/S}$, and that hence $\mathcal{D}_f$ is invertible and that the Riemann-Hurwitz relation holds. Furthermore, $\mathcal{D}_f$ is then S-flat, and so $\mathrm{Diff}(f) \in \mathrm{Div}(X/S)$.

Finally, the formation of $\mathrm{Diff}(f)$ commutes with base-change because the hypothesis of Remark 32(b) is satisfied by [EGA], (IV, 21.15.9).

Corollary 35 *Let Y/X and Z/X be two regular relative curves over a smooth base curve X/K, where K is a perfect field, and let $f : Y \to Z$ be a generically etale X-morphism. In addition, let D denote the closure in Y of the different divisor $\mathrm{Diff}(f_F) \in \mathrm{Div}(Y_F)$ of the generic cover $f_F : Y_F \to Z_F$, where $F = \kappa(X)$. Then there is an effective Cartier divisor $B = B(f) \geq 0$ consisting entirely of components of fibres of Y/X with following properties:*

(a) The divisor $\mathrm{Diff}(f) := D + B \in \mathrm{Div}(Y)$ is the different divisor of f and hence satisfies (42). Moreover, we have the relative Riemann-Hurwitz relation:

$$\omega^0_{Y/X} \simeq f^*\omega^0_{Z/X} \otimes \mathcal{L}(\mathrm{Diff}(f)). \tag{47}$$

(b) Suppose that the fibre Y_x of Y at $x \in X$ is reduced. Then a component Γ of Y_x does not appear in B if and only if the induced map $f_{|\Gamma} : \Gamma \to f(\Gamma) \subset Z_x$ is finite and separable. If this condition holds for all components of Y_x, then $D_x = D_{|Y_x}$ is the different divisor of $f_x : Y_x \to Z_x$, i.e. $D_x = \mathrm{Diff}(f_x)$.

(c) If X is complete, then for any a fibre component Γ of Y/X we have

$$(B.\Gamma) = 2p_a(\Gamma) - 2 - \Gamma^2 - (D.\Gamma) - (\omega^0_{Z/X}.f_*\Gamma). \tag{48}$$

Proof. By hypothesis, Y and Z are smooth irreducible surfaces over K, and so $\mathrm{Diff}(f)$ exists by Proposition 33. We have $\mathrm{Diff}(f)_{|Y_F} = \mathrm{Diff}(f_F)$ (apply Remark 32(b) with $\tilde{Y} = Y_F$), and so $\mathrm{Diff}(f) = D + B$, where B consists entirely of fibre components. This proves the existence of B.

(a) The first assertion is clear from the construction. Now if we multiply both sides of (44) by $p_Y^*\omega^{-1}_{X/K} = f^*p_Z^*\omega^{-1}_{X/K}$, where $p_Y : Y \to X$ and $p_Z : Z \to X$ denote the structure maps, then we obtain (47) because $\omega^0_{Y/X} = \omega_{Y/K} \otimes p_Y^*\omega^{-1}_{X/K}$ and $f^*\omega^0_{Z/X} = f^*(\omega_{Z/K} \otimes p_Z^*\omega^{-1}_{X/K}) = f^*\omega_{Z/K} \otimes p_Y^*\omega^{-1}_{X/K}$.

(b) Since Y_x is reduced, we have $\mathfrak{m}_{Y,\Gamma} = \mathfrak{m}_{X,x}\mathcal{O}_{Y,\Gamma} = \mathfrak{m}_{Z,f(\Gamma)}\mathcal{O}_{Y,\Gamma}$, and so $\Gamma \subset \mathrm{Ram}(f) \Leftrightarrow \kappa(\Gamma)/\kappa(f(\Gamma))$ is a finite separable field extension; cf. [BLR], 2.2/2. In view of (42), this proves the first assertion.

To prove the second, let $g : Z_x \hookrightarrow Z$ denote the canonical closed immersion. If no components Γ of appears in B, then its base-change $g_Y : Y_x \to Y$ satisfies the hypothesis of Remark 32(b) 33, and so the assertion follows.

(c) Since Γ is a component of a fibre of Y/X, we have $(\omega^0_{Y/X}.\Gamma) = (\omega_{Y/K}.\Gamma) = 2p_a(\Gamma)-2-\Gamma^2$, the latter by the adjunction formula. Now by (47) we have $(\omega^0_{Y/X}.\Gamma) = ((D+B).\Gamma)+(f^*\omega^0_{Z/X}.\Gamma) = ((D+B).\Gamma)+(\omega^0_{Z/X}.f_*\Gamma)$ by the projection formula, and so (48) follows.

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