Principal Polarizations on Abelian Product Surfaces

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1 Introduction

Let A/K be an abelian variety over an algebraically closed field K. The question of determining the number N_A of isomorphism classes of principal polarizations lying on A has been considered by many authors.

This question was first considered by Hayashida [H1], who determined N_A in 1965 in the case that $A = E \times E'$, where $E \sim E'$ are isogeneous elliptic curves without complex multiplication, and he also determined N_A in 1968 in the case that $A = E \times E$, where E is an elliptic curve with complex multiplication (CM) by a maximal order; see Hayashida [H2]. In both cases he also determined the number N_A^* of isomorphism classes of smooth genus 2 curves lying on A. In 1986, Ibukiyama, Katsura and Oort [IKO] determined N_A and N_A^* in the case that $A = E \times E'$, where E and E' are supersingular elliptic curves. Moreover, Lange studied the question of determining N_A for certain higher dimensional abelian varieties in several papers; see [L1], [L2].

In the case of an abelian *surface* A, a new method for determining N_A^* was introduced in [K8]. Here we want to extend this method to compute the number N_A of all principal polarizations on A. This means that we want to compute the difference $N_A^{**} := N_A - N_A^*$, and for this we may assume that $A \simeq E \times E'$ is a product surface, for otherwise $N_A^{**} = 0$.

The key idea of this method is to study the sets $\mathcal{P}(A,q)$ and $\overline{\mathcal{P}}(A,q)$ which are attached to a given integral quadratic form q. Here $\mathcal{P}(A,q)$ consists of those principal polarizations $\theta \in \mathcal{P}(A)$ which are equivalent to the *refined Humbert invariant* $q_{(A,\theta)}$ (see [K3] and §2 below), and $\overline{\mathcal{P}}(A,q)$ denotes the set of orbits of $\mathcal{P}(A,\theta)$ under the action of the automorphism group of A. Thus,

$$\mathcal{P}(A,q) \;=\; \{\theta \in \mathcal{P}(A) : q_{(A,\theta)} \sim q\} \quad \text{and} \quad \overline{\mathcal{P}}(A,q) \;=\; \mathrm{Aut}(A)\backslash \mathcal{P}(A,q),$$

where $\mathcal{P}(A) \subset \mathrm{NS}(A)$ denotes the set of all principal polarizations on A and $\sim$ denotes the equivalence of quadratic forms.

The advantage of studying the set $\overline{\mathcal{P}}(A,q)$ is that there is an explicit formula for its cardinality in terms of the number of double cosets of certain subgroups of $\mathrm{Aut}(q_A)$, where $q_A(D) = \frac{1}{2}(D.D)$ denotes (one half of) the intersection form on A, which we view as an integral quadratic form on the Néron-Severi group $\mathrm{NS}(A)$. More precisely, if

$$G_A \;:=\; \{\alpha \in \mathrm{Aut}(q_A) : \alpha(\mathcal{P}(A)) = \mathcal{P}(A)\}$$

denotes the subgroup of $\mathrm{Aut}(q_A)$ consisting of those automorphisms of q_A which preserve the set $\mathcal{P}(A)$ of principal polarizations on A, then it is easy to see that $G_A \geq H_A$, where H_A is the image of $\mathrm{Aut}(A)$ in $\mathrm{Aut}(\mathrm{NS}(A))$ via its action on $\mathrm{NS}(A)$, and that $\overline{\mathcal{P}}(A,q) = H_A\backslash\mathcal{P}(A,q)$. By Theorem 1 of [K8] we have that

$$(1) \qquad |\overline{\mathcal{P}}(A,q)| \;=\; |H_A\backslash G_A/S_\theta|, \quad \text{for any } \theta \in \mathcal{P}(A,q),$$

where $S_\theta = \{\alpha \in G_A : \alpha(\theta) = \theta\}$ denotes the stabilizer subgroup of θ.

This formula immediately implies a *mass-formula* for the set $\overline{\mathcal{P}}(A,q)$. Indeed, for any $\theta \in \mathcal{P}(A)$, let us put $a(\theta) := |\mathrm{Aut}(\theta)|$, where $\mathrm{Aut}(\theta) := H_A \cap S_\theta$. Since the "weight" $a(\theta)$ is constant on H_A-orbits, we can write $a(\overline{\theta}) = a(\theta)$, if $\overline{\theta} = H_A\theta$. By Theorem 2 of [K8] we have the following mass formula:

$$(2) \qquad \mathbf{M}(\overline{\mathcal{P}}(A,q)) \;:=\; \sum_{\overline{\theta}\in\overline{\mathcal{P}}(A,q)} \frac{1}{a(\overline{\theta})} \;=\; \frac{[G_A : H_A]}{|\mathrm{Aut}(q)|}, \quad \text{provided that } \mathcal{P}(A,q) \neq \emptyset.$$

If the quadratic form q does not represent 1, then every $\theta \in \mathcal{P}(A,q)$ is equal to the theta-divisor θ_C of some smooth genus 2 curve C on A. In that case $a(\theta)$ only depends q, and $a(\theta)$ can be computed explicitly, as was explained in Theorem 3 of [K8]. However, when q does represent 1, then the situation is more complicated because in that case such a result does not always hold. Here we have:

Theorem 1 *If A/K is an abelian surface, and if $\theta \in \mathcal{P}(A,q)$, where q is a binary or ternary form which represents 1 but which is not equivalent to $x^2 + 4\kappa(y^2 + yz + z^2)$ nor to $x^2 + 4\kappa(y^2 + z^2)$, for any $\kappa > 1$, then $a(q) := a(\theta)$ only depends on q. More precisely, if $r_n^*(q) := |\{(x_1,\ldots,x_r) \in \mathbb{Z}^r : q(x_1,\ldots,x_r) = n, \gcd(x_1,\ldots,x_r) = 1\}|$ denotes the number of primitive representations of an integer n by q, then*

$$(3) \qquad a(q) \;=\; 2\max(1, r_4^*(q)),$$

and hence

$$(4) \qquad |\overline{\mathcal{P}}(A,q)| \;=\; [G_A : H_A]a(q)|\mathrm{Aut}(q)|^{-1}.$$

Note that the assertion of Theorem 1 may not be true for the exceptional cases when $q \sim x^2 + 4\kappa(y^2 + yz + z^2)$ or $q \sim x^2 + 4\kappa(y^2 + z^2)$, as Corollary 19 below shows.

By using formula (4) and the results of [K6], it not difficult to compute $N_A^{**} := N_A - N_A^*$ in most cases. (The exceptional cases can be computed by using (1).) Note that N_A^{**} equals the number of isomorphism classes of *reducible* principal polarizations (see §3), and that $N_A^{**} > 0$ if and only if A is a product surface, i.e. $A \simeq E \times E'$, for some elliptic curves E and E'; see Proposition 4. In the non-CM case we have:

Proposition 2 *Let* $A = E \times E'$, *where* $\mathrm{Hom}(E, E') = \mathbb{Z}h$ *with* $d := \deg(h) \geq 1$. *Then* $N_A^{**} = 1$, *if* $d = 1$, *and* $N_A^{**} = 2^{\omega(d)-1}$, *if* $d > 1$, *where* $\omega(d)$ *denotes the number of distinct prime divisors of* d.

The result for N_A^{**} is more complicated when A is a CM product surface, i.e., when $A \simeq E \times E'$, where $E \sim E'$ are isogeneous elliptic curves with complex multiplication.

Theorem 3 *Let* $A \simeq E \times E'$ *be a CM product surface.*

(a) *If* $A \simeq E'' \times E''$, *for some elliptic curve* E'', *and if* $\mathrm{End}(E'')$ *has discriminant* Δ, *then*

$$N_A^{**} = \frac{1}{2}(h(\Delta) + g(\Delta)), \tag{5}$$

where $h(\Delta)$ *denotes the class number and* $g(\Delta)$ *the number of genera of positive primitive integral binary quadratic forms of discriminant* Δ.

(b) *If* $A \not\simeq E'' \times E''$, *for any elliptic curve* E'', *then let* $\Delta = \mathrm{disc}(q_{E,E'})$ *be the discriminant of the degree form* $q_{E,E'}$ *on* $\mathrm{Hom}(E, E')$. *Furthermore, let* $\kappa = \mathrm{cont}(q_{E,E'})$ *denote the content of* $q_{E,E'}$ *and let* $u = |\,\mathrm{Aut}^+(q_{E,E'})|$ *denote the number of automorphisms of* $q_{E,E'}$ *of determinant* 1. *Then*

$$N_A^{**} = \frac{1}{u}(2^{\omega(\kappa)} + u - 2)h(\Delta). \tag{6}$$

Note that formula (5) generalizes the formula obtained by Hayashida [H2] in the case that $\mathrm{End}(E)$ is a maximal order and $K = \mathbb{C}$. (Hayashida's formula is reproved in [GHR].) Note also that the hypothesis of part (a) is equivalent to the hypothesis that $q_{E,E'}$ lies in the principal genus of forms of discriminant Δ; see Corollary 10.

2 The refined Humbert invariant

Let A/K be an abelian surface, and let $q_A : \mathrm{NS}(A) \to \mathbb{Z}$ be the integral quadratic form on $\mathrm{NS}(A)$ defined by (one-half of) the self-intersection pairing on the Néron-Severi group $\mathrm{NS}(A) = \mathrm{Div}(A)/\equiv$ of A. Its associated bilinear form β_A is therefore the intersection pairing, i.e.,

$$\beta_A(D, D') := q_A(D + D') - q_A(D) - q_A(D') = (D.D'). \tag{7}$$

Let $\mathcal{P}(A) \subset \mathrm{NS}(A)$ denote the set of *principal polarizations* of A. Thus,

$$\mathcal{P}(A) = \{\mathrm{cl}(D) : D \in \mathrm{Div}(A) \text{ is ample and } q_A(\mathrm{cl}(D)) = 1\},$$

where $\mathrm{cl}(D) \in \mathrm{NS}(A) = \mathrm{Div}(A)/\equiv$ denotes the class defined by the divisor $D \in \mathrm{Div}(A)$. In the sequel we will assume tacitly that $\mathcal{P}(A) \neq \emptyset$.

If $\theta \in \mathcal{P}(A)$, then put

(8) $$\tilde{q}_{(A,\theta)}(D) = \beta_A(D,\theta)^2 - 4q_A(D) = (D.\theta)^2 - 2(D.D), \quad \text{for } D \in \mathrm{NS}(A).$$

It is easy to see (see [K1]) that this defines a positive-definite quadratic form $q_{(A,\theta)}$ on the quotient space $\mathrm{NS}(A,\theta) := \mathrm{NS}(A)/\mathbb{Z}\theta$, so we have that $\tilde{q}_{(A,\theta)} = q_{(A,\theta)} \circ \pi_\theta$, where

$$\pi_\theta : \mathrm{NS}(A) \to \mathrm{NS}(A,\theta) := \mathrm{NS}(A)/\mathbb{Z}\theta$$

denotes the quotient map. The quadratic form $q_{(A,\theta)}$ or, more correctly, the quadratic module $(\mathrm{NS}(A,\theta), q_{(A,\theta)})$, is called the *refined Humbert invariant* of the principally polarized abelian surface (A,θ); cf. [K3]. Since $\mathrm{NS}(A,\theta) \simeq \mathbb{Z}^{\rho-1}$, where $\rho = \mathrm{rank}(\mathrm{NS})$ is the Picard number, we see that $q_{(A,\theta)}$ defines an equivalence class of integral, positive definite quadratic forms in $\rho - 1$ variables.

In [K8] it was shown that every isomorphism between two such quadratic modules $(\mathrm{NS}(A,\theta_i), q_{(A,\theta_i)})$ is induced by a suitable element of the automorphism group

$$\mathrm{Aut}(q_A) = \{\alpha \in \mathrm{Aut}(\mathrm{NS}(A)) : q_A \circ \alpha = q_A\}$$

of the quadratic form q_A. However, since $\mathrm{Aut}(q_A)$ does not act on $\mathcal{P}(A)$, it is useful to consider instead the subgroup

$$G_A := \{\alpha \in \mathrm{Aut}(q_A) : \alpha(\mathcal{P}(A)) = \mathcal{P}(A)\} \leq \mathrm{Aut}(q_A)$$

which does preserve the set of polarizations. Note that G_A has index 2 in $\mathrm{Aut}(q_A)$ because $\mathrm{Aut}(q_A) = \langle -1_{\mathrm{NS}(A)} \rangle \times G_A$ by Corollary 10 of [K8].

Let us now fix an integral quadratic form q in r variables, and consider the subset

$$\mathcal{P}(A,q) := \{\theta \in \mathcal{P}(A) : q_\theta \sim q\}$$

of $\mathcal{P}(A)$. Here $q_\theta = q_{(A,\theta)}$ is the refined Humbert invariant of (A,θ) and the condition $q_\theta \sim q$ means that we have an isomorphism $(\mathrm{NS}(A), q_{(A,\theta)}) \simeq (\mathbb{Z}^r, q)$ of quadratic modules. In the sequel we will tacitly assume that $r = \rho(A) - 1 = \mathrm{rank}(\mathrm{NS}(A)) - 1$, for otherwise $\mathcal{P}(A,q)$ is empty.

It is an immediate consequence of Theorem 9 of [K8] that the group G_A acts transitively on the set $\mathcal{P}(A,q)$. Thus, if $\theta \in \mathcal{P}(A,q)$, then the map $g \mapsto g(\theta)$ defines a bijection of G_A-sets

(9) $$G_A/S_\theta \overset{\sim}{\to} \mathcal{P}(A,q),$$

where $S_\theta := \{\alpha \in G_A : g(\theta) = \theta\}$ denotes the G_A-stabilizer of θ, and from this one deduces immediately formula (1) of the introduction; see Corollary 14 of [K8]. Moreover, by using elementary group theory the mass-formula (2) is easily deduced from (1), as was shown in the proof of Theorem 2 in [K8].

3 Reducible Polarizations

In order to determine the number $N_A^{**} = N_A - N_A^*$ of isomorphism classes of principal polarizations on an abelian surface A which do not come from a smooth genus 2 curve on A, we first study the set of such principal polarization. It turns out that this set is precisely the set of *reducible* polarizations, which are defined as follows.

Definition. A polarization $\theta \in \mathcal{P}(A)$ is called *reducible* (or *decomposable*) if $\theta = \mathrm{cl}(E_1 + E_2)$, for some elliptic curves E_1 and E_2 on A. Note that we can assume E_1 and E_2 to be elliptic subgroups of A. The set of reducible polarizations on A is denoted by $\mathcal{P}(A)^{\mathrm{red}}$.

The reducible polarizations can be classified as follows.

Proposition 4 *If $\theta = \mathrm{cl}(E_1 + E_2) \in \mathcal{P}(A)$, where $E_i \leq A$, $i = 1, 2$, are two elliptic subgroups of an abelian surface A, then there is an isomorphism*

$$f : E_1 \times E_2 \xrightarrow{\sim} A$$

such that $f(\theta_{E_i}) = E_i$, for $i = 1, 2$, where $\theta_{E_1} = E_1 \times \{0\}$ and $\theta_{E_2} = \{0\} \times E_2$, so in particular $f^*\theta = \theta_{E_1,E_2} := \mathrm{cl}(\theta_{E_1} + \theta_{E_2})$. Thus $\mathcal{P}(A)^{\mathrm{red}} \neq \emptyset$ if and only if A is an abelian product surface, i.e., $A \simeq E_1 \times E_2$ for some elliptic curves E_i, $i = 1, 2$. Furthermore, if $\theta \in \mathcal{P}(A)$ is any principal polarization, then

$$\theta \in \mathcal{P}(A)^{\mathrm{red}} \quad \Leftrightarrow \quad q_\theta(\bar{D}) = 1, \quad \text{for some } \bar{D} \in \mathrm{NS}(A, \theta). \tag{10}$$

Moreover, if $\theta \in \mathcal{P}(A)^* := \mathcal{P}(A) \setminus \mathcal{P}(A)^{\mathrm{red}}$, then $\theta = \theta_C$, for some smooth genus 2 curve on A.

Proof. The first assertion is a slight refinement of (one part of) Satz 2 of Weil [We]; see the proof of [K6], Proposition 8, for more details. The second assertion clearly follows from the first. The equivalence (10) is Proposition 6 of [K3], and the last assertion follows from another part of Satz 2 of Weil [We].

Remark 5 It follows from Proposition 4 that $\mathcal{P}(A)^* = \{\theta_C : C \in \mathfrak{C}(A)\}$, where $\mathfrak{C}(A) = \{C \subset A : g_C = 2\}$ denotes the set of smooth genus 2 curves on A. Thus, $N_A^* = |H_A \backslash \mathcal{P}(A)^*|$, and $N_A^{**} = |H_A \backslash \mathcal{P}(A)^{\mathrm{red}}|$.

We now relate the set $\mathcal{P}(A)^{\mathrm{red}}$ to the sets $\mathcal{P}(A, q)$, for suitable quadratic forms q. For this, we may assume that $A = E \times E'$ is an abelian product surface, for otherwise $\mathcal{P}(A)^{\mathrm{red}} = \emptyset$ by Proposition 4. For such a surface we have that

$$q_A \sim xy \perp (-q_{E,E'}), \tag{11}$$

where $q_{E,E'}$ denotes the degree form on $\mathrm{Hom}(E,E')$ which is defined by $q_{E,E'}(h) = \deg(h)$. Note that $q_{E,E'}$ is a positive quadratic form. To verify (11), recall from Proposition 23 of [K3] that we have an isomorphism

$$\mathbf{D} : \mathbb{Z} \times \mathbb{Z} \times \mathrm{Hom}(E,E') \quad \overset{\sim}{\to} \quad \mathrm{NS}(E \times E') \tag{12}$$

such that

$$q_A(\mathbf{D}(x,y,h)) \; = \; xy - \deg(h) \; = \; xy - q_{E,E'}(h), \tag{13}$$

and so (11) follows. Note that (11) implies that

$$\det(q_A) \; = \; (-1)^{\rho-1} \det(q_{E,E'}), \tag{14}$$

where $\det(q_A)$ denotes the determinant of a Gram matrix of the associated bilinear form $\beta_{q_A} = \beta_A$ with respect to some basis of $\mathrm{NS}(A)$, and $\det(q_{E,E'})$ is defined similarly. It thus follows from Proposition 9 of [K3] that for any $\theta \in \mathcal{P}(A)$ we have that

$$\det(q_{(A,\theta)}) \; = \; 2^{2\rho-3} \det(q_{E,E'}). \tag{15}$$

Consider the special case that $\theta = \theta_{E,E'} = \mathrm{cl}(\theta_E + \theta_E)$. Since $\theta_{E,E'} = \mathbf{D}(1,1,0)$, it follows from (13) and (8) that

$$q_{\theta_{E,E'}} \sim x^2 \perp 4q_{E,E'} \;\text{ because }\; \tilde{q}_{(E\times E',\theta_{E,E'})}(\mathbf{D}(x,y,h)) = (x-y)^2 + 4\deg(h). \tag{16}$$

In the sequel we denote by $\mathrm{gen}(q)$ the set of isomorphism classes of integral quadratic forms q' which are *genus-equivalent* to a given integral quadratic form q, i.e., those forms q' which are p-adically equivalent to q for all primes p (including the case $p = \infty$); cf. Jones [Jo], Chapter V.

Proposition 6 *If $A = E \times E'$ is an abelian product surface, then*

$$\mathcal{P}(A)^{\mathrm{red}} \; = \; \coprod_{q \in \mathrm{gen}(q_{E,E'})} \mathcal{P}(A, x^2 \perp 4q). \tag{17}$$

Proof. If $\theta \in \mathcal{P}(A, x^2 \perp 4q)$, then $q_\theta \sim f_q := x^2 \perp 4q$. Since f_q clearly represents 1, it follows from (10) that $\theta \in \mathcal{P}(A)^{\mathrm{red}}$, and so the right hand side of (17) is contained in the left hand side. Note also that the union is disjoint because by (an extension of) Proposition 54 of [K4] we have that $q \sim q' \Leftrightarrow x^2 \perp 4q \sim x^2 \perp 4q'$.

To prove the opposite inclusion, let $\theta = \mathrm{cl}(E_1 + E_2) \in \mathcal{P}(A)^{\mathrm{red}}$, where $E_i \leq A$ is an elliptic subgroups of A, for $i = 1, 2$. Then by Proposition 4 we have an isomorphism of principally polarized abelian surfaces:

$$f : (E_1 \times E_2, \theta_{E_1,E_2}) \quad \overset{\sim}{\to} \quad (A, \theta). \tag{18}$$

Thus, applying (11) to $A' = E_1 \times E_2$, we see that $q_{A'} \sim xy \perp (-q_{E_1,E_2})$. Since $A' \simeq A$, it follows that $q_{A'} \sim q_A$, and so $xy \perp (-q_{E_1,E_2}) \sim xy \perp (-q_{E,E'})$. From Corollary 26 of [K4] it thus follows that $q_{E_1,E_2} \in \mathrm{gen}(q_{E,E'})$.

Furthermore, by (18) and (16) we have that $q_\theta \sim q_{\theta_{E_1,E_2}} \sim x^2 \perp 4q_{E_1,E_2}$, and so $\theta \in \mathcal{P}(A, x^2 \perp 4q_{E_1,E_2})$. This shows that the left hand side of (17) is contained in the right hand side, and so (17) follows.

In the non-CM case the above result, together with (2) and results from [K6], yields the following result.

Corollary 7 *Let $A = E \times E'$, where $\mathrm{Hom}(E, E') = \mathbb{Z}h$ with $d := \deg(h) \geq 1$. Then $\mathcal{P}(A)^{\mathrm{red}} = \mathcal{P}(A, x^2 + 4dy^2)$, and hence its mass is $\mathbf{M}(\overline{\mathcal{P}}(A)^{\mathrm{red}}) = 2^{\omega(d)-2}$.*

Proof. Here $q_{E,E'}(xh) = dx^2$, so $q_{E,E'} \sim dx^2$ and hence $\mathrm{gen}(q_{E,E'}) = \{dx^2\}$. Thus, the first assertion follows immediately from (17). Moreover, since $|\mathrm{Aut}(x^2 + 4dy^2)| = 4$ and $\mathcal{P}(A, x^2 + 4dy^2) \neq \emptyset$ because $\theta_{E,E'} \in \mathcal{P}(A, x^2 + 4dy^2)$ by (16), and since $[G_A : H_A] = 2^{\omega(d)}$ by Theorem 1 of [K6], the formula for $\mathbf{M}(\overline{\mathcal{P}}(A)^{\mathrm{red}})$ follows from the mass-formula (2).

It is a bit more complicated to derive the mass formula for $\overline{\mathcal{P}}(A)^{\mathrm{red}}$ when A is a CM product surface. As a first step, we observe that all the sets $\mathcal{P}(A, x^2 \perp 4q)$ on the right hand side of (17) are indeed non-empty.

Lemma 8 *If $A = E \times E'$ is a CM abelian surface, and if $q \in \mathrm{gen}(q_{E,E'})$, then $\mathcal{P}(A, x^2 \perp 4q) \neq \emptyset$. More precisely, there exist $\theta \in \mathcal{P}(A, x^2 \perp 4q)$ and two elliptic subgroups $E_1, E_2 \leq A$ with $q_{E_1,E_2} \sim q$ such that $\theta = \mathrm{cl}(E_1 + E_2)$.*

Proof. Recall that $q_A \sim xy \perp (-q_{E,E'})$ by (11). Since $q \in \mathrm{gen}(q_{E,E'})$, we have by Remark 27 of [K4] that $xy \perp (-q) \sim xy \perp (-q_{E,E'}) \sim q_A$. This means that there exist $D_1, D_2 \in \mathrm{NS}(A)$ such that $(D_1.D_1) = (D_2.D_2) = 0$, $(D_1.D_2) = 1$, and such that $(q_A)_{|M^\perp} \sim -q$, where $M := \mathbb{Z}D_1 + \mathbb{Z}D_2$.

Put $D = D_1 + D_2$. Then $(D.D) = 2$, so by [K1], Corollary 2.2(b), either D or $-D$ is ample. By replacing D_1 and D_2 by their negatives, if necessary, we may assume that D is ample, so $\theta := D \in \mathcal{P}(A)$. Now since D_i is primitive in $\mathrm{NS}(A)$ (because $(D_1.D_2) = 1$), and since $(D_i.D) = 1$, it follows from [K1], Proposition 2.3, that $D_i = \mathrm{cl}(E_i)$, for some elliptic subgroups $E_i \leq A$, for $i = 1, 2$.

Thus, $\theta = \mathrm{cl}(E_1 + E_2) \in \mathcal{P}(A)^{\mathrm{red}}$ and so by Proposition 4 there exists an isomorphism $f : A' := E_1 \times E_2 \xrightarrow{\sim} A$ with $f(\theta_{E_i}) = E_i$, for $i = 1, 2$. Then f induces an isomorphism of the Néron-Severi groups of A' and A, and hence also an isomorphism

$$((\mathbb{Z}\overline{\theta}_{E_1} + \mathbb{Z}\overline{\theta}_{E_2})^\perp, (q_{A'})_{|(\mathbb{Z}\overline{\theta}_{E_1}+\mathbb{Z}\overline{\theta}_{E_2})^\perp}) \simeq (M^\perp, (q_A)_{|M^\perp}) = (M^\perp, -q),$$

where $\overline{\theta}_{E_i} = \mathrm{cl}(\theta_{E_i})$, for $i = 1, 2$. Since $(q_{A'})_{|(\mathbb{Z}\theta_{E_1}+\mathbb{Z}\theta_{E_2})^\perp} \sim -q_{E_1,E_2}$, it follows that $q_{E_1,E_2} \sim q$. Furthermore, we have as in (18) that f defines an isomorphism

$(A', \theta_{E_1,E_2}) \simeq (A, \theta)$ of principally polarized abelian surfaces, and so it follows by using (16) that $q_\theta \sim q_{\theta_{E_1,E_2}} \sim x^2 \perp 4q_{E_1,E_2} \sim x^2 \perp 4q$. Thus $\theta \in \mathcal{P}(A, x^2 \perp 4q)$, and hence $\mathcal{P}(A, x^2 \perp 4q) \neq \emptyset$, as claimed.

Remark 9 In the above proof we had used a non-trivial fact from the theory of quadratic forms (which was proved in Remark 27 of [K4]). It is possible to avoid using this fact, and instead derive the result by using the theory of [K2], particularly by using Proposition 65 of [K2]. If one does this, then one can obtain an independent proof of Remark 27 of [K4] for positive binary quadratic forms.

We observe that Lemma 8 implies the following interesting fact.

Corollary 10 *If $A = E \times E'$ is a CM product surface, then $A \simeq E'' \times E''$, for some elliptic curve E''/K if and only if $q_{E,E'}$ lies in the principal genus of (primitive) forms of discriminant $\Delta = \mathrm{disc}(q_{E,E'})$.*

Proof. ($\Rightarrow$) Recall from the proof of Proposition 6 that if $A \simeq E'' \times E''$, then $q_{E'',E''} \in \mathrm{gen}(q_{E,E'})$. Thus, $\mathrm{disc}(q_{E'',E''}) = \Delta$. Now since $1_{E''} \in \mathrm{End}(E'')$ and $q_{E'',E''}(1_{E''}) = \deg(1_{E''}) = 1$, it follows that $q_{E'',E''} \sim 1_\Delta$ is the principal form of discriminant Δ (and that $q_{E'',E''}$ is primitive). Thus, $q_{E,E'} \in \mathrm{gen}(1_\Delta)$ lies in the principal genus.

($\Leftarrow$) If $1_\Delta \in \mathrm{gen}(q_{E,E'})$, then by Lemma 8 (applied to $q = 1_\Delta$) we have that there exist two elliptic subgroups $E_1, E_2 \leq A$ such that $q_{E_1,E_2} \sim 1_\Delta$ and such that $\mathrm{cl}(E_1 + E_2) \in \mathcal{P}(A)$. The first property implies that there exists $h \in \mathrm{Hom}(E_1, E_2)$ such that $\deg(h) = q_{E_1,E_2}(h) = 1$, so $E_1 \simeq E_2$. The second property implies in view of Proposition 4 that $E_1 \times E_2 \simeq A$, so $A \simeq E_1 \times E_1$, as desired.

We also require the following two results from [K8], which are Proposition 16 and Lemma 33 of [K8], respectively.

Proposition 11 *Let $A = E \times E'$, where $q_{E,E'}$ is a binary quadratic form of discriminant Δ and content κ. Put $\Delta' = \Delta/\kappa^2$. If q is a ternary quadratic form such that $\mathcal{P}(A, q) \neq \emptyset$, then*

$$\mathbf{M}(\overline{\mathcal{P}}(A, q))|\operatorname{Aut}^+(q)| \;=\; 2^{\omega(\kappa)} g(\Delta') \frac{h(\Delta)}{h(\Delta')}. \tag{19}$$

Lemma 12 *If q is a primitive positive binary quadratic form of discriminant Δ, then*

$$\sum_{q' \in \mathrm{gen}(q)} \frac{1}{|\operatorname{Aut}(q')|} \;=\; \frac{h(\Delta)}{2u(\Delta)g(\Delta)}, \tag{20}$$

where $u(\Delta) := |\operatorname{Aut}^+(q)|$ only depends on Δ.

We are now ready to derive the mass formula for $\overline{\mathcal{P}}(A)^{\mathrm{red}}$ for a CM product surface.

Proposition 13 *If $A = E \times E'$, where $q_{E,E'}$ is a binary quadratic form of discriminant Δ and content κ. Put $\Delta' = \Delta/\kappa^2$. Then the mass of $\overline{\mathcal{P}}(A)^{\mathrm{red}}$ is*

$$\mathbf{M}(\overline{\mathcal{P}}(A)^{\mathrm{red}}) \;:=\; \sum_{\overline{\theta}\in\overline{\mathcal{P}}(A)^{\mathrm{red}}} \frac{1}{a(\overline{\theta})} \;=\; 2^{\omega(\kappa)-1}\frac{h(\Delta)}{u(\Delta')}. \tag{21}$$

Proof. Write $f_q := x^2 \perp 4q$, where $q \in \mathrm{gen}(q_{E,E'})$. Since here q is a binary form, we have by [K4], Corollary 55, that

$$|\,\mathrm{Aut}^+(f_q)| \;=\; |\,\mathrm{Aut}(q)| \;=\; |\,\mathrm{Aut}(q/\kappa)|. \tag{22}$$

Since the map $q \mapsto q/\kappa$ induces a bijection between $\mathrm{gen}(q_{E,E'})$ and $\mathrm{gen}(q_{E,E'}/\kappa)$, we obtain from Proposition 6, Lemma 8, equations (19) and (22) that

$$\begin{aligned}\mathbf{M}(\overline{\mathcal{P}}(A)^{\mathrm{red}}) \;&=\; \sum_{q\in\mathrm{gen}(q_{E,E'})} \mathbf{M}(\overline{\mathcal{P}}(A, f_q)) \;=\; 2^{\omega(\kappa)}\frac{h(\Delta)}{h(\Delta')} \sum_{q\in\mathrm{gen}(q_{E,E'})} \frac{g(\Delta')}{|\,\mathrm{Aut}^+(f_q)|}\\ &=\; 2^{\omega(\kappa)}\frac{h(\Delta)}{h(\Delta')} \sum_{q'\in\mathrm{gen}(q_{E,E'}/\kappa)} \frac{g(\Delta')}{|\,\mathrm{Aut}(q'))|} \;=\; 2^{\omega(c)}\frac{h(\Delta)}{h(\Delta')}\frac{h(\Delta')}{2u(\Delta')},\end{aligned}$$

where the last equality follows from Lemma 12. This proves (21).

4 The computation of $a(\theta)$

We now study the weight $a(\theta) = |\,\mathrm{Aut}(\theta)|$ of a reducible principal polarization $\theta \in \mathcal{P}(A)^{\mathrm{red}}$. (Note that when $\theta \in \mathcal{P}(A)^*$ is irreducible, then the weight $a(\theta)$ was determined in [K8].)

For this, recall from the introduction and from [K8] that the group $\mathrm{Aut}(\theta)$ was defined as $\mathrm{Aut}(\theta) = S_\theta \cap H_A$, where S_θ is the G_A-stabilizer subgroup of θ, and where $H_A = \varphi_A(\mathrm{Aut}(A))$ is the image of $\mathrm{Aut}(A)$ in $G_A \le \mathrm{Aut}(q_A) \le \mathrm{Aut}(\mathrm{NS}(A))$ via the homomorphism $\varphi : \mathrm{Aut}(A) \to \mathrm{Aut}(\mathrm{NS}(A))$ induced by the action of $\mathrm{Aut}(A)$ on $\mathrm{NS}(A)$. As was mentioned in [K8], this group is closely related to the automorphism group $\mathrm{Aut}(A,\theta) = \{\alpha \in \mathrm{Aut}(A) : \alpha^*\theta = \theta\}$ of the principally polarized abelian surface (A,θ), for we have that

$$\mathrm{Aut}(\theta) \;=\; \varphi_A(\mathrm{Aut}(A,\theta)) \;\simeq\; \mathrm{Aut}(A,\theta)/\,\mathrm{Ker}(\varphi_A). \tag{23}$$

We first observe that $a(\theta)$ is invariant under isomorphisms in the following sense.

Lemma 14 *If $f : A \xrightarrow{\sim} A'$ is an isomorphism of abelian surfaces, then* $\mathrm{Ker}(\varphi_A) = f^{-1}\,\mathrm{Ker}(\varphi_{A'})f$, *and* $\mathrm{Aut}(A,\theta) = f^{-1}\,\mathrm{Aut}(A', f_*\theta)f$, *for all* $\theta \in \mathcal{P}(A)$. *In particular,* $a(f_*\theta) = a(\theta)$, *for all* $\theta \in \mathcal{P}(A)$.

Proof. If $g \in \mathrm{Aut}(A)$, then $\varphi_{A'}(f \circ g \circ f^{-1}) = f_* \circ \varphi_A(g) \circ (f_*)^{-1}$, where $f_* : \mathrm{NS}(A) \xrightarrow{\sim} \mathrm{NS}(A')$ is the induced isomorphism of the Néron-Severi groups. From this, the first two identities follow. In view of (23), these imply that $\mathrm{Aut}(f_*\theta) = f_*\,\mathrm{Aut}(\theta)(f_*)^{-1}$ because $\mathrm{Aut}(f_*\theta) = \varphi_{A'}(\mathrm{Aut}(A', f_*\theta)) = \varphi_{A'}(f\,\mathrm{Aut}(A,\theta)f^{-1}) = f_*\varphi_A(\mathrm{Aut}(A,\theta))(f_*)^{-1} = f_*\,\mathrm{Aut}(\theta)(f_*)^{-1}$, and so the last assertion follows.

It follows from this lemma and from Proposition 4 that in order to compute $a(\theta)$ when $\theta \in \mathcal{P}(A)^{\mathrm{red}}$, it suffices to consider the case that $\theta = \theta_{E_1,E_2}$ is the product polarization of $A = E_1 \times E_2$. As a first step, we relate the group $\mathrm{Aut}(\theta_{E_1,E_2})$ to the group $H_A \cap S_A(\theta_{E_1}, \theta_{E_2})$, which was studied in [K6].

Proposition 15 *Let $A = E_1 \times E_2$ be an abelian product surface and let* $\theta_{E_1,E_2} = \mathrm{cl}(\theta_{E_1} + \theta_{E_2})$. *Moreover, let* $\alpha \in \mathrm{Aut}(A)$. *If* $E_1 \not\simeq E_2$, *then* $\alpha \in \mathrm{Aut}(A, \theta_{E_1,E_2})$ *if and only if* $\alpha = \alpha_1 \times \alpha_2$, *for some* $\alpha_i \in \mathrm{End}(E_i)$, *for* $i = 1, 2$. *Thus*

$$\mathrm{Aut}(\theta_{E_1,E_2}) \;=\; H_A \cap S_A(\theta_{E_1}, \theta_{E_2}), \tag{24}$$

where $S_A(\theta_{E_1}, \theta_{E_2}) = \{g \in G_A : g(\mathrm{cl}(\theta_{E_i})) = \mathrm{cl}(\theta_{E_i}),$ *for* $i = 1, 2\}$.

On the other hand, if there exists an isomorphism $\tau : E_1 \xrightarrow{\sim} E_2$, *then* $\alpha \in \mathrm{Aut}(A, \theta_{E_1,E_2})$ *if and only if* $\alpha = \alpha_1 \times \alpha_2$, *or* $\alpha = \alpha_1 \times \alpha_2 \circ \tilde{\tau}$, *for some* $\alpha_i \in \mathrm{End}(E_i)$, *for* $i = 1, 2$, *where* $\tilde{\tau} \in \mathrm{Aut}(A)$ *is defined by* $\tilde{\tau}(x, y) = (\tau^{-1}(y), \tau(x))$. *Thus*

$$\mathrm{Aut}(\theta_{E_1,E_2}) \;=\; H_A \cap (S_A(\theta_{E_1}, \theta_{E_2}) \cup S_A(\theta_{E_1}, \theta_{E_2})\tilde{\tau}_*). \tag{25}$$

Proof. By definition, $\alpha \in \mathrm{Aut}(A, \theta_{E_1,E_2}) \Leftrightarrow \alpha^*(\theta_{E_1} + \theta_{E_2}) \equiv \theta_{E_1} + \theta_{E_2} \Leftrightarrow \theta := \theta_{E_1} + \theta_{E_2} \equiv \alpha(\theta_{E_1}) + \alpha(\theta_{E_2})$. Since θ and $\alpha(\theta_{E_1}) + \alpha(\theta_{E_2})$ are effective divisors and θ is a principal polarization, we see that $\alpha \in \mathrm{Aut}(A, \theta_{E_1,E_2}) \Leftrightarrow \alpha(\theta_{E_1}) + \alpha(\theta_{E_2}) = T_x(\theta)$, for some $x \in A(K)$. If this is the case, then $\{T_x(\theta_{E_1}), T_x(\theta_{E_2})\} = \{\alpha(\theta_{E_1}), \alpha(\theta_{E_2})\}$ is the set of irreducible components of $\alpha(\theta)$. But $\alpha(\theta_{E_1}) \cap \alpha(\theta_{E_2}) = \alpha(\theta_{E_1} \cap \theta_{E_2}) = \{0\}$, whereas $T_x(\theta_{E_1}) \cap T_x(\theta_{E_2}) = T_x(\theta_{E_1} \cap \theta_{E_2}) = T_x(\{0\}) = \{x\}$, so $x = 0$. Thus $\alpha(\theta) = \theta$. We thus have that $\alpha \in \mathrm{Aut}(A, \theta_{E_1,E_2})$ if and only if

$$\text{either } \alpha(\theta_{E_i}) = \theta_{E_i}, \text{for } i = 1, 2, \quad \text{or} \quad \alpha(\theta_{E_1}) = \theta_{E_2} \text{ and } \alpha(\theta_{E_2}) = \theta_{E_1}. \tag{26}$$

If $E_1 \not\simeq E_2$, and if $\alpha \in \mathrm{Aut}(A, \theta_{E_1,E_2})$, then the second case of (26) cannot occur because $\theta_{E_i} \simeq E_i$ for $i = 1, 2$, so $\alpha(\theta_{E_i}) = \theta_{E_i}$, for $i = 1, 2$. Since $\theta_{E_i} = e_i(E_i)$, where $e_i : E_i \hookrightarrow A$ is the canonical inclusion, for $i = 1, 2$, the above condition is equivalent to the condition that $\alpha \circ e_i = e_i \circ \alpha_i$, for some $\alpha_i \in \mathrm{End}(E_i)$, for $i = 1, 2$, i.e., that $\alpha = \alpha_1 \times \alpha_2$. This proves the first assertion.

From this, equation (24) follows easily. Indeed, if $\alpha_* = \varphi_A(\alpha) \in \mathrm{Aut}(\theta_{E_1,E_2})$, then $\alpha \in \mathrm{Aut}(A, \theta_{E_1,E_2})$, and so by the above we have that $\alpha = \alpha_1 \times \alpha_2$ with $\alpha_i \in \mathrm{Aut}(E_i)$, for $i = 1, 2$. It is then clear that $\alpha_* \in H_A \cap S_A(\theta_{E_1}, \theta_{E_2})$. Conversely, if $\alpha_* \in H_A \cap S_A(\theta_{E_1}, \theta_{E_2})$, then $\alpha \in \mathrm{Aut}(A)$ satisfies $\alpha_* \,\mathrm{cl}(\theta_{E_i}) = \mathrm{cl}(\theta(E_i))$, i.e. $\alpha(\theta_{E_i}) \equiv \theta_{E_i}$, for $i = 1, 2$. Since $\alpha(\theta_{E_i})$ and θ_{E_i} are elliptic subgroups of A, it follows from Theorem 1.1 of [K1] that $\alpha(\theta_{E_i}) = \theta_{E_i}$, for $i = 1, 2$. Thus, by what was explained above, we have that $\alpha \in \mathrm{Aut}(A, \theta_{E_1,E_2})$, and so $\alpha_* \in \mathrm{Aut}(\theta_{E_1,E_2})$. This proves (24).

Now suppose that we have an isomorphism $\tau : E_1 \xrightarrow{\sim} E_2$, so $\tilde{\tau} \in \mathrm{Aut}(A)$. Clearly $\tilde{\tau}(\theta_{E_1}) = \theta_{E_2}$ and $\tilde{\tau}(\theta_{E_2}) = \theta_{E_1}$, so $\tilde{\tau} \in \mathrm{Aut}(A, \theta_{E_1,E_2})$ by the first part of the proof. It thus clear that $\alpha_1 \times \alpha_2$ and $\alpha_1 \times \alpha_2 \circ \tilde{\tau} \in \mathrm{Aut}(A, \theta_{E_1,E_2})$, for all $\alpha_i \in \mathrm{Aut}(E_i)$, $i = 1, 2$. Conversely, if $\alpha \in \mathrm{Aut}(A, \theta_{E_1,E_2})$, then (26) holds. If α satisfies the first condition of (26), then as before we conclude that $\alpha = \alpha_1 \times \alpha_2$ with $\alpha_i \in \mathrm{Aut}(E_i)$, for $i = 1, 2$. On the other hand, if α satisfies the second condition of (26), then $\alpha' := \alpha \circ \tilde{\tau}^{-1}$ satisfies the first condition of (26), and hence $\alpha' = \alpha_1 \times \alpha_2$, for some $\alpha_i \in \mathrm{Aut}(E_i)$, and so $\alpha = (\alpha_1 \times \alpha_2) \circ \tilde{\tau}$, as claimed. This proves the assertion about $\mathrm{Aut}(A, \theta_{E_1,E_2})$.

The proof of (25) is very similar to that of (24).

Corollary 16 *If $\theta = \mathrm{cl}(E_1 + E_2) \in \mathcal{P}(A)^{\mathrm{red}}$ is a reducible principal polarization, where the $E_i \le A$ are isogeneous elliptic subgroups, then*

$$a(\theta) \;=\; \varepsilon(E_1, E_2)|\varphi_A(\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2))|, \tag{27}$$

where $\varepsilon(E_1, E_2) = 2$, if $E_1 \simeq E_2$, and $\varepsilon(E_1, E_2) = 1$ otherwise. Moreover, if E_1 is not supersingular, then

$$a(\theta) \;=\; \varepsilon(E_1, E_2)\max(|\,\mathrm{Aut}(E_1)|, |\,\mathrm{Aut}(E_2)|). \tag{28}$$

Proof. By Proposition 4 we have an isomorphism $f : E_1 \times E_2 \xrightarrow{\sim} A$ such that $f_*\theta_{E_1,E_2} = \theta$, so $a(\theta) = a(\theta_{E_1,E_2})$ by Lemma 14.

It follows from Proposition 15 that $a(\theta_{E_1,E_2}) = \varepsilon(E_1, E_2)|H_A \cap S_A(\theta_{E_1}, \theta_{E_2})|$. Indeed, if $\varepsilon(E_1, E_2) = 1$, then this is clear by (24). If $\varepsilon(E_1, E_2) = 2$, then this follows from (25) because $|H_A \cap S_A(\theta_{E_1}, \theta_{E_2})| = |(H_A \cap S_A(\theta_{E_1}, \theta_{E_2}))\tilde{\tau}_*| = |H_A \cap S_A(\theta_{E_1}, \theta_{E_2})\tilde{\tau}_*|$ as $\tilde{\tau}_* \in H_A$, and because $S_A(\theta_{E_1}, \theta_{E_2})$ and $S_A(\theta_{E_1}, \theta_{E_2})\tilde{\tau}_*$ are disjoint since $\mathrm{cl}(\theta_{E_1}) \ne \mathrm{cl}(\theta_{E_2})$. (For the latter, note that $(\theta_{E_1}.\theta_{E_2}) = 1$ but $(\theta_{E_1}.\theta_{E_1}) = 0$, so $\mathrm{cl}(\theta_{E_1}) \ne \mathrm{cl}(\theta_{E_2})$.)

Thus $a(\theta) = a(\theta_{E_1,E_2}) = \varepsilon(E_1, E_2)|H_A \cap S_A(\theta_{E_1}, \theta_{E_2})|$, as claimed. Since $H_A \cap S_A(\theta_{E_1}, \theta_{E_2}) = \varphi_A(\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2))$ by [K6], Lemma 17, we see that (27) follows..

To prove (28), suppose first that $\mathrm{End}(E_i) = \mathbb{Z}$, for $i = 1, 2$. Then $|\,\mathrm{Aut}(E_i)| = |\,\mathrm{Ker}(\varphi_A)| = 2$ (see Theorem 1 of [K6]), and so $|\varphi_A(\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2))| = \frac{2\cdot 2}{2} = \max(|\,\mathrm{Aut}(E_1)|, |\,\mathrm{Aut}(E_2)|)$. Thus (28) follows from (27) in this case.

Now suppose that E_1 and E_2 are isogeneous CM elliptic curves. If $j_{E_i} \notin \{0, 1728\}$, for $i = 1, 2$, then $|\,\mathrm{Aut}(E_i)| = 2$ and also $|\,\mathrm{Ker}(\varphi_A)| = 2$ by Lemma 29 of [K6]. Thus, we see as before that (28) holds here.

Finally, suppose that $j_{E_1} \in \{0, 1728\}$ (and that E_1 is not supersingular, so E_1 is a CM curve). If $j_{E_2} \neq j_{E_1}$, then $|\operatorname{Aut}(E_2)| = |\operatorname{Ker}(\varphi_A)| = 2$ by [K6], Lemma 29 again, so $|\varphi_A(\operatorname{Aut}(E_1) \times \operatorname{Aut}(E_2)| = |\operatorname{Aut}(E_1)| = \max(|\operatorname{Aut}(E_1)|, \operatorname{Aut}(E_2)|)$. On the other hand, if $j_{E_2} = j_{E_1}$, then $|\operatorname{Aut}(E_2)| = |\operatorname{Ker}(\varphi_A)| = |\operatorname{Aut}(E_1)|$ by [K6], Lemma 29, and so (28) follows.

We now re-interpret the formula (28) for $a(\theta)$ in terms of quantities associated to the refined Humbert invariant q_θ. For this, we use the following notation which extends the notation given in the introduction.

Notation. If $q(x_1, \ldots, x_r)$ is a positive integral quadratic form in r variables, and if $n \geq 1$ is an integer, then let

$$R_n(q) := \{(x_1, \ldots, x_r) \in \mathbb{Z}^r : q(x_1, \ldots, x_r) = n\}$$

denote the set of representations of n by q, and let

$$R_n^*(q) \;:=\; \{(x_1, \ldots, x_r) \in R_n(q) : \gcd(x_1, \ldots, x_r) = 1\}$$

denote the subset of primitive representations of n by q. Moreover, put $r_n(q) = |R_n(q)|$ and $r_n^*(q) = |R_n^*(q)|$.

Theorem 17 *Let A be an abelian surface. If $\theta \in \mathcal{P}(A, q)$, where q is a binary or ternary quadratic form with $r_1(q) \neq 0$, then*

$$a(\theta) \;=\; 2\max(1, r_4^*(q)) \;=\; r_1^*(q)\max(1, r_4^*(q)), \tag{29}$$

except when $q \sim x^2 + 4\kappa(y^2 + yz + z^2)$ or $q \sim x^2 + 4\kappa(y^2 + z^2)$, for some integer $\kappa > 1$.

Proof. Since $r_1(q) \neq 0$, we have by (10) that $\theta = \operatorname{cl}(E_1 + E_2) \in \mathcal{P}(A)^{\mathrm{red}}$, and so $A \simeq E_1 \times E_2$ is an abelian product surface by Proposition 4. Thus, by the proof of Proposition 6 we know that $q \sim q_\theta \sim f_{q_1} = x^2 \perp 4q_1$, where $q_1 = q_{E_1,E_2}$. Thus, $r_1(q) = r_1^*(q) = 2$, and so the second equality of (29) follows.

To prove the first equality, we will use the formula (28). This is applicable because E_1 cannot be a supersingular curve since by hypothesis q is either a binary or ternary form and so $\operatorname{rank}(\operatorname{Hom}(E_1, E_2)) \leq 2$.

We want to relate the right hand side of (28) to $r_4^*(q)$. For this, we observe that $r_4^*(f_{q_1}) = r_1(q_1)$, and $r_1(q_1) = r_1(q_{E_1,E_2}) > 0$ if and only if there exists $h \in \operatorname{Hom}(E_1, E_2)$ with $\deg(h) = 1$, so $r_4^*(q) = r_4^*(f_{q_{E_1,E_2}}) > 0$ if and only if $E_1 \simeq E_2$.

Suppose first that $\operatorname{End}(E_1) = \mathbb{Z}$. Then $\operatorname{Hom}(E_1, E_2) = \mathbb{Z}h$, for some h, and $q_1 = q_{E_1,E_2} \sim dx^2$, where $d = \deg(h)$. If $d = 1$, then $E_1 \simeq E_2$ and $r_4^*(q) = r_1(q_1) = 2$. Thus, since $|\operatorname{Aut}(E_i)| = 2$, we see from (28) that $a(\theta) = 2 \cdot 2 = 2r_4^*(q) = 2\max(1, r_4^*(q))$, and so (29) holds in this case.

If $d > 1$, then $E_1 \not\simeq E_2$ and so $r_4^*(q) = 0$. Thus, by (28) we have that $a(\theta) = 1 \cdot 2 = 2\max(1, r_4^*(q))$, so (29) holds in this case as well.

Now suppose that $\mathrm{End}(E_1) \neq \mathbb{Z}$, so $q_1 = q_{E_1,E_2}$ is a binary form by our hypothesis. If $E_1 \simeq E_2$, then $\mathrm{Hom}(E_1, E_2) \simeq \mathrm{End}(E_1)$, and so $r_4^*(q) = r_1(q_1) = |\{h \in \mathrm{Hom}(E_1, E_2) : \deg(h) = 1\}| = |\mathrm{Aut}(E_1)|$. Thus, by (28) we obtain $a(\theta) = 2 \cdot |\mathrm{Aut}(E_1)| = 2r_4^*(q) = 2\max(1, r_4^*(q))$, and so (29) holds in this case.

Finally, suppose that $E_1 \not\simeq E_2$, so $r_4^*(q) = r_1(q_1) = 0$. Then the right hand side of (29) equals 2, so by formula (28) we see that (29) holds if and only if $\max(|\mathrm{Aut}(E_1)|, |\mathrm{Aut}(E_2)|) = 2$.

Suppose that $\max(|\mathrm{Aut}(E_1)|, |\mathrm{Aut}(E_2)|) > 2$. Then one of j_{E_i} is in $\{0, 1728\}$, say $j_{E_1} \in \{0, 1728\}$. Consider first the case that $j_{E_1} = 1728$, so $\mathrm{End}(E_1) \simeq \mathbb{Z}[i]$. Since $E_2 \not\simeq E_1$, we have that $\kappa := [\mathbb{Z}[i] : \mathrm{End}(E_2)] > 1$ because there is only one curve E (up to isomorphism) with $\mathrm{End}(E) \simeq \mathbb{Z}[i]$. Thus, by Proposition 40 of [K2] see that $q_{E_1,E_2} \sim \kappa q_2$, where q_2 is a binary form of discriminant -4. Thus $q_2 \sim x^2 + y^2$, and hence $q \sim f_{q_{E_1,E_2}} \sim x^2 + 4\kappa(y^2 + z^2)$, where $\kappa > 1$.

Similarly, if $j_{E_1} = 0$, then $\mathrm{End}(E_1) \simeq \mathbb{Z}[\zeta]$, where $\zeta = \frac{-1+\sqrt{-3}}{2}$, and by a similar argument we get that $q_{E_1,E_2} \sim \kappa(x^2 + xy + y)$ with $\kappa := [\mathbb{Z}[\zeta] : \mathrm{End}(E_2)]$, and so $q \sim x^2 + 4\kappa(y^2 + yz + z^2)$. Since the last two cases were excluded, this proves (29).

By using the results of [K8], we can generalize this theorem as follows.

Corollary 18 *Let A be an abelian surface. If $\theta \in \mathcal{P}(A, q)$, where q is a binary or ternary quadratic form, then*

(30) $$a(\theta) = a(q) := \max(1, r_1^*(q)) \max(1, r_4^*(q), 3r_4^*(q) - 12),$$

except when $q \sim x^2 + 4\kappa(y^2 + yz + z^2)$ or $q \sim x^2 + 4\kappa(y^2 + z^2)$, for some integer $\kappa > 1$.

Proof. Suppose first that $r_1^*(q) = 0$. Then by Proposition 4 we know that $(A, \theta) \simeq (J_C, \theta_C)$ is a Jacobian of a curve C/K, and so $q \sim q_{(A,\theta)} \sim q_C$ in the notation of [K8]. Now since $r_1^*(q) = 0$, we have that $r_4^*(q) = r_4(q)$, and so it follows from equation (33) of [K8] that $a(\theta) = \max(1, r_4^*(q), 3r_4^*(q) - 12)$. This proves (30) in this case.

Now suppose that $r_1^*(q) \neq 0$, so we are in the situation of Theorem 17. Then (30) follows from (29), provided we have that $3r_4^*(q) - 12 \leq r_4^*(q)$ or $r_4^*(q) \leq 6$.

To verify this, recall from the proof of Theorem 17 that $q \sim x^2 \perp 4q_1$, for some q_1, so $r_4^*(q) = r_1^*(q_1)$. If q is a binary form, then $q_1 \sim dx^2$, for some $d \geq 1$, so $r_1^*(q_1) \leq 2$. If q is a ternary form, then q_1 is a positive binary form. Thus, if $r_1^*(q_1) > 1$, then $q_1 \sim 1_\Delta$ is a principal form of some discriminant Δ, and hence $r_1^*(q_1) = 6, 4$ and 2, for $\Delta = -3, -4$ and $\Delta < -4$, respectively. Thus, $r_4^* \leq 6$ in all cases, and so formula (30) holds.

We now observe that formula (29) does not hold in general in the exceptional cases.

Corollary 19 *Suppose that $q \sim x^2 + 4\kappa(y^2 + z^2)$ or that $q \sim x^2 + 4\kappa(y^2 + yz + z^2)$, for some $\kappa > 1$, and that $\mathcal{P}(A, q) \neq \emptyset$. Then (29) does not hold for some $\theta \in \mathcal{P}(A, q)$. More precisely, if $\omega(\kappa) = 1$, then $a(\theta) = \frac{1}{2}|\operatorname{Aut}^+(q)| > 2$, for every $\theta \in \mathcal{P}(A, q)$, but if $\omega(\kappa) > 1$, then this formula holds for some but not all $\theta \in \mathcal{P}(A, q)$.*

Proof. Let $\theta \in \mathcal{P}(A, q)$. Since $r_1(q) > 0$, we have as in the proof of Theorem 17 that $\theta = \mathrm{cl}(E_1 + E_2)$, for some elliptic subgroups $E_i \leq A$, $i = 1, 2$, and that $q \sim f_{q_1}$, where $q_1 = q_{E_1,E_2}$.

Suppose first that $q \sim x^2 + 4\kappa(y^2 + z^2)$, so $q_1 \sim \kappa(x^2 + y^2)$ by [K4], Proposition 54. Thus $r_4^*(q) = r_1^*(q_1) = 0$ because $\kappa > 1$. In particular, $E_1 \not\simeq E_2$ because $r_1(q_{E_1,E_2}) = 0$. Since $\mathrm{disc}(x^2 + y^2) = -4$ is a fundamental discriminant, we see from [K2], Proposition 40, that $\mathrm{End}^0(E_i) \simeq \mathbb{Q}(i)$, and that $\gcd(f_1, f_2) = 1$, and $\mathrm{lcm}(f_1, f_2) = \kappa$, where $f_i = [\mathbb{Z}[i] : \mathrm{End}(E_i)]$, for $i = 1, 2$. We claim that

$$a(\theta) = \begin{cases} \frac{1}{2}|\operatorname{Aut}^+(q)|, & \text{if } \min(f_1, f_2) = 1, \\ 2, & \text{otherwise.} \end{cases} \tag{31}$$

Indeed, if $f_j > 1$, then $|\operatorname{Aut}(E_j)| = |\mathbb{Z}[f_j i]^\times| = 2$, so the second line of (31) follows immediately from (28) because in that case $f_1 > 1$ and $f_2 > 1$. On the other hand, if either $f_1 = 1$ (and hence $f_2 = \kappa$) or $f_2 = 1$ (and $f_1 = \kappa$), then either $\mathrm{End}(E_1) \simeq \mathbb{Z}[i]$ or $\mathrm{End}(E_2) \simeq \mathbb{Z}[i]$, and hence by (28) we see that $a(\theta) = 1 \cdot \max(|\operatorname{Aut}(E_1)|, |\operatorname{Aut}(E_2)|) = 4 = \frac{1}{2}|\operatorname{Aut}^+(f_{q_1})|$, the latter by Corollary 55 of [K4]. This proves (31).

From (31) the assertion of the corollary follows immediately (in the case that $q \sim x^2 + 4\kappa(y^2 + z^2)$). Indeed, if $\omega(\kappa) = 1$, then only the first case of (31) is possible, and so $a(\theta) = \frac{1}{2}|\operatorname{Aut}^+(q)|$, for all $\theta \in \mathcal{P}(A, q)$. On the other hand, if $\omega(\kappa) > 1$, then the second case also occurs (here we need to use Proposition 40 of [K2] again), and so both cases occur. Moreover, since $\frac{1}{2}|\operatorname{Aut}^+(q)| = 4 > 2 = 2\max(1, r_4^*(q))$, we see that for any $\kappa > 1$ there exists a $\theta \in \mathcal{P}(A, q)$ such that (29) does not hold.

If $q \sim x^2 + 4\kappa(y^2 + yz + z^2)$, then a similar proof shows that (31) also holds. In this case we have that $a(\theta) = 6 = \frac{1}{2}|\operatorname{Aut}^+(q)|$ when $\min(f_1, f_2) = 1$. Thus, a similar reasoning as before shows that the assertion of the corollary holds here as well.

We now use the above results to determine $N_A^{**} = |\overline{\mathcal{P}}(A)^{\mathrm{red}}|$. If E does not have CM, then the result was formulated as Proposition 2 in the introduction.

Proof of Proposition 2. By Corollary 7 we have that $\mathcal{P}(A)^{\mathrm{red}} = \mathcal{P}(A, q)$, where $q = x^2 + 4dy^2$, and by Theorem 17 we have that $a(\theta) = 2\max(1, r_4^*(q))$, for all $\theta \in \mathcal{P}(A)^{\mathrm{red}}$. Thus, $N_A^{**} = \mathbf{M}(\overline{\mathcal{P}}(A)^{\mathrm{red}})(2\max(1, r_4^*(q)) = 2^{\omega(d)-1}\max(1, r_4^*(q))$, the latter by Corollary 7. Now if $d = 1$, then $E_1 \simeq E_2$ and $r_4^*(q) = r_4^*(x^2 + 4y^2) = 2$, so $N_A^{**} = 2^{\omega(1)} = 1$. If $d > 1$, then $E_1 \not\simeq E_2$ and $r_4^*(q) = r_4^*(x^2 + 4dy^2) = 0$, and thus $|\overline{\mathcal{P}}(A)^{\mathrm{red}}| = 2^{\omega(d)-1}$.

Similarly, part (a) of Theorem 3 is easily deduced from the above results.

Proof of Theorem 3(a). Since $A \simeq E'' \times E''$, we see that $q_1 := q_{E'',E''}$ represents 1 and so $q_1 \sim 1_\Delta$ is equivalent to the principal form 1_Δ of discriminant Δ. Thus, if, as before, $f_q = x^2 \perp 4q$, then by (17) and by Theorem 17 we obtain that

$$N_A^{**} = \sum_{q \in \mathrm{gen}(q_1)} |\overline{\mathcal{P}}(A, f_q)| = \sum_{q \in \mathrm{gen}(q_1)} \mathbf{M}(\overline{\mathcal{P}}(A, f_q)) 2 \max(1, r_4^*(f_q)). \tag{32}$$

because $N_A^{**} = |\overline{\mathcal{P}}(A)^{\mathrm{red}}|$ and because none of the f_q's can be one of the exceptional forms since every $q \in \mathrm{gen}(q_1)$ is primitive.

Now by (19) and Lemma 8 we have for all $q \in \mathrm{gen}(q_1)$ that $\mathbf{M}(A, f_q)|\,\mathrm{Aut}^+(f_q)| = 2^{\omega(\kappa)} g(\Delta) \frac{h(\Delta)}{h(\Delta'} = g(\Delta)$ because here $\kappa = 1$ and $\Delta' = \Delta$. Thus, since $|\,\mathrm{Aut}^+(f_q)| = |\,\mathrm{Aut}(q)|$ by (22), we obtain from (32) that

$$N_A^{**} = \frac{2g(\Delta) r_4^*(f_q)}{\mathrm{Aut}(q_1)} + \sum_{\substack{q \,\in\, \mathrm{gen}(q_1) \\ q \not\sim q_1}} \frac{2g(\Delta)}{|\,\mathrm{Aut}(q)|} = g(\Delta) + \sum_{\substack{q \,\in\, \mathrm{gen}(q_1) \\ q \not\sim q_1}} \frac{2g(\Delta)}{|\,\mathrm{Aut}(q)|} \tag{33}$$

because $2r_4^*(f_{q_1}) = 2r_1(q_1) = 2|\,\mathrm{Aut}(E'')| = 2u(\Delta) = |\,\mathrm{Aut}(q_1)|$.

Suppose first that $\Delta \in \{-3, -4\}$. Then $h(\Delta) = g(\Delta) = 1$, and the sum in (33) is empty, so $|\overline{\mathcal{P}}(A)^{\mathrm{red}}| = g(\Delta) = 1 = \frac{1}{2}(h(\Delta) + g(\Delta))$. Thus (5) holds in this case.

Now suppose that $\Delta \notin \{-3, -4\}$. Then $u(\Delta) = 2$, so $|\,\mathrm{Aut}(q_1)| = 2u(\Delta) = 4$. Then the right hand side of (33) becomes $\frac{g(\Delta)}{2} + \sum_{q \in \mathrm{gen}(q_1)} \frac{2g(\Delta)}{|\,\mathrm{Aut}(q)|} = \frac{g(\Delta)}{2} + \frac{h(\Delta)}{2}$ by (20), which proves (5) in all cases.

As we will now see, a similar (but easier) proof works for part (b) of Theorem 3 as long as $\Delta' \notin \{-3, -4\}$.

Proof of Theorem 3(b) when $\Delta' \notin \{-3, -4\}$. Put $q_1 = q_{E,E'}$. The hypothesis that $A \not\simeq E'' \times E''$, for any E'', means that $r_1(q) = 0$, for all $q \in \mathrm{gen}(q_1)$. Indeed, if $r_1(q) \neq 0$, then by Lemma 8 there exists $\theta = \mathrm{cl}(E_1 + E_2) \in \mathcal{P}(A)^{\mathrm{red}}$ with $q_{E_1,E_2} \sim q$, and then $E_1 \simeq E_2$ because $r_1(q_{E_1,E_2}) > 0$. But then by Proposition 4 we have that $A \simeq E_1 \times E_2 \simeq E_1 \times E_1$, contrary to the hypothesis.

Since $\Delta' \notin \{-3, -4\}$, we have that $u(\Delta') = 2$ (see [BV], p. 29) and that if $q \in \mathrm{gen}(q_1)$, then $q \not\sim \kappa(x^2 + xy + y^2)$ and $q \not\sim \kappa(x^2 + y^2)$, for any $\kappa \geq 1$. It therefore follows from Proposition 54 of [K4] that $f_q \not\sim x^2 + c(y^2 + yz + z^2)$ and $f_q \not\sim x^2 + c(y^2 + z^2)$, for any $q \in \mathrm{gen}(q_1)$ and any $\kappa \geq 1$. Thus, by Theorem 17 (and Proposition 6) we have that $a(\theta) = 2$, for all $\theta \in \mathcal{P}(A)^{\mathrm{red}}$, and so we obtain from this and Proposition 13 that

$$N_A^{**} = |\overline{\mathcal{P}}(A)^{\mathrm{red}}| = 2\mathbf{M}(\overline{\mathcal{P}}(A)^{\mathrm{red}}) = 2\left(2^{\omega(\kappa)-1}\frac{h(\Delta)}{u(\Delta')}\right) = 2^{\omega(\kappa)-1} h(\Delta).$$

This proves (6) in this case because here $u = |\,\mathrm{Aut}^+(q_1)| = u(\Delta') = 2$.

Unfortunately, the above method will not work when $\Delta' \in \{-3,-4\}$ because in that case $a(\theta)$ does not have the same value for all $\theta \in \mathcal{P}(A, f_q)$, as Corollary 19 shows (when $\omega(\kappa) > 1$). Thus, here we need to compute $|\overline{\mathcal{P}}(A, f_q)|$ directly by using formula (1). We will prove:

Proposition 20 *Let $A = E_1 \times E_2$ be a CM product surface and suppose that $q := q_{E_1,E_2}$ has discriminant Δ and content $\kappa > 1$. Put $\Delta' = \Delta/\kappa^2$. Then*

$$|\overline{\mathcal{P}}(A, f_q)| \;=\; (2^{\omega(\kappa)} + u(\Delta') - 2)\frac{2g(\Delta')h(\Delta)}{|\operatorname{Aut}(q)|h(\Delta')}. \tag{34}$$

Before proving this, let us see why this result allows us to complete the proof of Theorem 3.

Proof of Theorem 3 when $\Delta' \in \{-3,-4\}$. The hypothesis implies that $h(\Delta') = 1$, so $q := q_{E,E'} \sim \kappa(1_{\Delta'})$, i.e., $q \sim \kappa(x^2 + xy + y^2)$ or $q \sim \kappa(x^2 + y^2)$. Moreover, gen(q) consists only of one class. Recall from above that the hypothesis that $A \not\simeq E'' \times E''$ implies $1_\Delta \notin \operatorname{gen}(q)$, and so we must have that $\kappa > 1$. Thus, by Proposition 6 and Proposition 20 we obtain

$$N_A^{**} \;=\; |\overline{\mathcal{P}}(A, f_q)| \;=\; (2^{\omega(\kappa)} + u(\Delta') - 2)\frac{2g(\Delta')h(\Delta)}{2u(\Delta')h(\Delta')} \;=\; (2^{\omega(\kappa)} + u - 2)\frac{h(\Delta)}{u}$$

because $|\operatorname{Aut}(q)| = 2u(\Delta') = 2u$ and $h(\Delta') = g(\Delta') = 1$. This proves (6) in all cases.

We now turn to the proof of Proposition 20. This follows easily from the results proven in [K6] which concern the computation of the (H_A, S)-double coset decomposition of G_A, where $A = E_1 \times E_2$ and $S = S_A(\theta_{E_1}, \theta_{E_2})$ is is the joint stabilizer subgroup as in Proposition 15.

Proposition 21 *If $A = E_1 \times E_2$ is a CM abelian product surface, and if q, Δ, Δ' and κ are as in* Proposition 20*, then*

$$|H_A \backslash G_A / S_A(\theta_{E_1}, \theta_{E_2})| \;=\; s(\Delta, \kappa)\frac{2g(\Delta')h(\Delta)}{|\operatorname{Aut}(q)|h(\Delta')}, \tag{35}$$

where $s(\Delta, \kappa) := 2(2^{\omega(\kappa)} + u(\Delta/\kappa^2) - 2)$.

Proof. This follows immediately from Corollary 27 of [K6], together with Proposition 14 of [K6].

To deduce Proposition 20 from Proposition 21 we observe the following.

Lemma 22 *Let $\theta = \operatorname{cl}(\theta_{E_1} + \theta_{E_2})$ be the product polarization on $A = E_1 \times E_2$. Then there exists $\tau \in S_\theta$ such that $S_\theta \;=\; S_A(\theta_{E_1}, \theta_{E_2}) \,\dot\cup\, \tau S_A(\theta_{E_1}, \theta_{E_2})$.*

Proof. Define $\tau \in \mathrm{End}(\mathrm{NS}(A))$ by $\tau(\mathbf{D}(x,y,h)) = \mathbf{D}(y,x,h)$, where $\mathbf{D}$ is the isomorphism of (12). From (13) we see that $\tau \in \mathrm{Aut}(q_A)$. Moreover, since $\theta = \mathbf{D}(1,1,0)$, it follows that $\tau(\theta) = \theta$, so $\tau \in S_\theta$. Since clearly $S_A(\theta_{E_1}, \theta_{E_2}) \le S_\theta$, we thus have that $\langle S_A(\theta_{E_1}, \theta_{E_2}), \tau\rangle \le S_\theta$. Moreover, since $\tau \notin S_A(\theta_{E_1}, \theta_{E_2})$ (because $\mathrm{cl}(\theta_{E_1}) \ne \mathrm{cl}(\theta_{E_2})$), this shows that

$$(36) \qquad |S_\theta| \;\ge\; |\langle S_A(\theta_{E_1}, \theta_{E_2}), \tau\rangle| \;\ge\; 2|S_A(\theta_{E_1}, \theta_{E_2})| \;=\; 2|\mathrm{Aut}(q_{E_1,E_2})|,$$

the latter by [K6], Proposition 16. On the other hand, since $q_\theta \sim f_{q_{E_1,E_2}} = x^2 \perp 4q_{E_1,E_2}$, we have by Corollary 10 of [K8] and equation (46) of [K4] that $|S_\theta| = |\mathrm{Aut}(q_\theta)| = |\mathrm{Aut}(f_{q_{E_1,E_2}})| = 2|\mathrm{Aut}(q_{E_1,E_2})|$, and so we see from (36) that $|S_\theta| = 2|S_A(\theta_{E_1}, \theta_{E_2})|$. Thus, the assertion follows.

To analyze the (H_A, S_θ)-double cosets of G_A, we recall some facts about the $(H_A, S_{\theta_{E_1},\theta_{E_2}})$-double cosets of G_A which were established in [K6]. For this, recall the following notation from [K6].

If E is an elliptic curve, then let $\mathcal{S}(A,E) = \{E' \le A : E' \simeq E\}$ denote the set of elliptic subgroups of A which are isomorphic to E. Moreover, if E_1', E_2' are two elliptic curves, then let

$$G_A(E_1', E_2') = \{g \in G_A : g(\mathrm{cl}(\theta_{E_i})) = \mathrm{cl}(E_i''),\ \text{for some } E_i'' \in \mathcal{S}(A, E_i'),\ \text{for } i = 1,2\}.$$

Lemma 23 *If $g \in G_A(E_1', E_2')$, then $G_A(E_1', E_2') = H_A g S_{\theta_{E_1},\theta_{E_2}}$. Moreover, in this case we have that $G_A(E_1', E_2') = G_A(E_1'', E_2'')$ if and only if $E_1' \simeq E_1''$ and $E_2' \simeq E_2''$. In addition, if $g \in G_A$, then there exist elliptic curves E_1', E_2' such that $g \in G_A(E_1', E_2')$.*

Proof. The first assertion is Proposition 11 of [K6], and the second and third follow from Remark 13 of [K6].

Lemma 24 *If $g \in G_A(E_1', E_2')$, then $g\tau \in G_A(E_2', E_1')$, where τ is as in* Lemma 22. *Thus*

$$(37) \qquad H_A g S_\theta \;=\; H_A g S_A(\theta_{E_1}, \theta_{E_2}) \;\cup\; H_A g\tau S_A(\theta_{E_1}, \theta_{E_2}),$$

and the union is disjoint if and only if $E_1' \not\simeq E_2'$.

Proof. Since $\tau(\mathrm{cl}(\theta_{E_1})) = \mathrm{cl}(\theta_{E_2})$ and $\tau(\mathrm{cl}(\theta_{E_2})) = \mathrm{cl}(\theta_{E_1})$, we see that $g\tau(\mathrm{cl}(\theta_{E_1})) = g(\mathrm{cl}(\theta_{E_2})) = \mathrm{cl}(E_2'')$, with $E_2'' \in \mathcal{S}(A, E_2')$, and similarly $g\tau(\mathrm{cl}(\theta_{E_2})) = \mathrm{cl}(E_1'')$, with $E_1'' \in \mathcal{S}(A, E_1')$. Thus $g\tau \in G_A(E_2', E_1')$, which proves the first assertion.

The formula (37) clearly follows from Lemma 22. Moreover, from Lemma 23 and the first assertion we see that the union is disjoint if and only if $E_1' \not\simeq E_2'$ (or $E_2' \not\simeq E_1'$), and so the last assertion follows.

We can now establish the following relation between the number of (H_A, S_θ)-double cosets of G_A and the number of $(H_A, S_{\theta_{E_1},\theta_{E_2}})$-cosets of G_A.

Proposition 25 *Let $A = E_1 \times E_2$ be a CM abelian product surface and let $\theta = \mathrm{cl}(\theta_{E_1} + \theta_{E_2})$. If $E_1 \not\simeq E_2$, then $|H_A \backslash G_A / S_\theta| = \frac{1}{2}|H_A \backslash G_A / S_A(\theta_{E_1}, \theta_{E_2})|$.*

Proof. By Lemma 23 we know that G_A is a disjoint union of sets (double cosets) of the form $G_A(E_1', E_2')$. Now by Proposition 14 of [K6] we have that

$$G_A(E_1', E_2') \neq \emptyset \quad \Leftrightarrow \quad A \simeq E_1' \times E_2' \quad \text{and} \quad q_{E_1,E_2} \sim q_{E_1',E_2'}. \tag{38}$$

Since $E_1 \not\simeq E_2$, we know that $q_{E_1,E_2} \not\sim 1_\Delta$. If $G_A(E_1', E_2') \neq \emptyset$, then by (38) we have that $q_{E_1',E_2'} \not\sim 1_\Delta$, and hence $E_1' \not\simeq E_2'$. Thus, by Lemma 24 we see that each double coset $H_A g S_\theta$ decomposes into exactly two disjoint $(H_A, S_A(\theta_{E_1}, \theta_{E_2}))$-double cosets, and so $|H_A \backslash G_A / S_A(\theta_{E_1}, \theta_{E_2})| = 2|H_A \backslash G_A / S_\theta|$, which proves the assertion.

Proof of Proposition 20. Let $\theta = \mathrm{cl}(\theta_{E_1} + \theta_{E_2})$. Then $q_\theta \sim f_q := x^2 \perp 4q$, where $q = q_{E_1,E_2}$, so $\theta \in \mathcal{P}(A, f_q)$. Then by formula (1) we have that $|\overline{\mathcal{P}}(A, f_q)| = |H_A \backslash G_A / S_\theta|$.

Since $\kappa > 1$, it is clear that $q_{E_1,E_2} \not\sim 1_\Delta$, so $E_1 \not\simeq E_2$. We thus obtain from Proposition 25 and (35) that

$$|\overline{\mathcal{P}}(A, f_q)| = \frac{1}{2}|H_A \backslash G_A / S_A(\theta_{E_1}, \theta_{E_2})| = \frac{2(2^{\omega(\kappa)} + u(\Delta/\kappa^2) - 2)}{2} \frac{2g(\Delta')h(\Delta)}{|\operatorname{Aut}(q)|h(\Delta')},$$

which proves formula (34).

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