Explanation of the Basic Invariants

The modular diagonal quotient surface $Z_{N,e}$ is the quotient surface $Z_{N,e} = \Delta_e\backslash Y_N$ in which $Y_N = X(N)\times X(N)$ is the product of the modular curve $X(N)$ with itself and $\Delta_e \le G\times G$ is a certain "twisted diagonal" subgroup of $G = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. More precisely, let

$X(N) = \Gamma(N)\backslash\mathfrak{H}^*$	denote the modular curve of level N,
$G = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$,	viewed as a subgroup of the automorphism group of $X(N)$,
$\pi : X(N) \to X(1) = G\backslash X(N)$	the associated quotient map,
$\alpha_e \in \mathrm{Aut}(X(N))$	the automorphism of G defined by conjugation with Q_e;

to be precise, $\alpha_e(g) = Q_e g Q_e^{-1}$, where $Q_e = \left(\begin{smallmatrix} e & 0\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$ and $e \in (\mathbb{Z}/N\mathbb{Z})^\times$. Then let

$Y_N = X(N)\times X(N)$	be the product surface of X by itself,
$\Delta_e = \{(g,\alpha_e(g)) : g \in G\}$	the "twisted diagonal" subgroup defined by α_e,
$Z_{N,e} = \Delta_e\backslash Y$	the (twisted) diagonal quotient surface defined by α_e,
$\varphi : Y_N \to Z_{N,e} = \Delta_e\backslash Y_n$	the the associated quotient map.

Thus, the product map $\pi\times\pi : X(N)\times X(N) \to X(1)\times X(1)$ factors over φ:

$$X(N)\times X(N) \xrightarrow{\varphi} Z_{N,e} \xrightarrow{\psi} X(1)\times X(1).$$

Note that $Z_{N,e}$ has (isolated) singularities (because Δ_e has nontrivial stabilizers); we let

$$\tilde{Z}_{N,e} \quad \text{denote its desingularization.}$$

The geometric invariants of $Z_{N,e}$ and $\tilde{Z}_{N,e}$, such as the Betti and Hodge numbers etc., may be computed by simple expressions from the following list of basic invariants:

$$\begin{array}{rcl}
m &=& |G| = |\Delta_e| = |\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})|/2, \text{ the order of } G \text{ or of } \Delta_e\\
g &=& g_{X(N)}, \text{ the genus of } X(N)\\
r_0 &=& \#\text{ of singularities of } Z_{N,e} \text{ above } \bar{P}_0\times\bar{P}_0\\
r_1 &=& \#\text{ of singularities of } Z_{N,e} \text{ above } \bar{P}_1\times\bar{P}_1\\
s_{11} &=& \#\text{ of singularities of } Z_{N,e} \text{ above } \bar{P}_1\times\bar{P}_1 \text{ of type } (3,1)\\
r_\infty &=& \#\text{ of singularities of } Z_{N,e} \text{ above } \bar{P}_\infty\times\bar{P}_\infty\\
\mathbb{L}_\infty &=& \#\text{ of irreducible components of the resolution curves of the singularities above } \bar{P}_\infty\times\bar{P}_\infty\\
\mathbb{S}_\infty &=& \text{a certain sum of Dedekind sums (contribution at } \infty \text{ only)}\\
c_\infty &=& \tilde{C}^2_{\infty,1} = \tilde{C}^2_{\infty,2}, \text{ the self-intersection numbers of the two curves } \tilde{C}_{\infty,1}, \tilde{C}_{\infty,2}.
\end{array}$$

Here, the points $\bar{P}_0, \bar{P}_1$ and $\bar{P}_\infty \in X(1)$ are the three ramification points of the quotient map $\pi : X(N) \to X(1) = G\backslash X(N) \simeq \mathbb{P}^1$ (of orders $2, 3, N$, respectively).

From the above basic invariants, the other geometric invariants of the surface can be calculated readily, as is explained at the end of the tables.

Note: If $e' = f^2 e$ with $(f,N) = 1$, then $Z_{N,e'} \simeq Z_{N,e}$. Thus, the tables list only one representative e for each square class mod N.