Explanation of the Geometric Invariants

The modular diagonal quotient surface $Z_{N,e}$ is the quotient surface $Z_{N,e} = \Delta_e \backslash Y_N$ in which $Y_N = X(N) \times X(N)$ is the product of the modular curve $X(N)$ with itself and $\Delta_e \le G \times G$ is a certain "twisted diagonal" subgroup of $G = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. More precisely, let

$X(N) = \Gamma(N)\backslash\mathfrak{H}^*$	denote the modular curve of level N,
$G = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$,	viewed as a subgroup of the automorphism group of $X(N)$,
$\alpha_e \in \mathrm{Aut}(X(N))$	the automorphism of G defined by conjugation with Q_e;

to be precise, $\alpha_e(g) = Q_e g Q_e^{-1}$, where $Q_e = \left(\begin{smallmatrix} e & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$ and $e \in (\mathbb{Z}/N\mathbb{Z})^\times$. Then let

$Y_N = X(N) \times X(N)$	be the product surface of X by itself,
$\Delta_e = \{(g, \alpha_e(g)) : g \in G\}$	the "twisted diagonal" subgroup defined by α_e,
$Z_{N,e} = \Delta_e \backslash Y$	the (twisted) diagonal quotient surface defined by α_e.

Note that $Z_{N,e}$ has (isolated) singularities (because Δ_e has nontrivial stabilizers); we let

$\tilde{Z}_{N,e}$ denote its desingularization.

The following invariants of $Z_{N,e}$ and $\tilde{Z}_{N,e}$ are given in the tables:

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\begin{aligned}
m &= |G| = |\Delta_e| = |\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})|/2, \text{ the order of } G \text{ or of } \Delta_e \\
g &= g_{X(N)}, \text{ the genus of } X(N) \\
g_0 &= g_{\langle\sigma_0\rangle\backslash X(N)}, \text{ the genus of } \langle\sigma_0\rangle\backslash X(N) \text{ (see Note 1) below)} \\
g_1 &= g_{\langle\sigma_1\rangle\backslash X(N)}, \text{ the genus of } \langle\sigma_1\rangle\backslash X(N) \\
g_\infty &= \langle\sigma_\infty\rangle\backslash X(N), \text{ the genus of } \langle\sigma_\infty\rangle\backslash X(N) = X_1(N) \\
r_0 &= \#\text{ of singularities of } Z_{N,e} \text{ above } \bar{P}_0 \times \bar{P}_0 \\
r_1 &= \#\text{ of singularities of } Z_{N,e} \text{ above } \bar{P}_1 \times \bar{P}_1 \\
r_\infty &= \#\text{ of singularities of } Z_{N,e} \text{ above } \bar{P}_\infty \times \bar{P}_\infty \\
r &= r_0 + r_1 + r_\infty, \text{ the total } \# \text{ of singularities of } Z_{N,e} \text{ (not listed in tables)} \\
\mathbb{L} &= \#\text{ of irreducible components of the resolution curve} \\
b_2^s &= \dim H^2(Z_{N,e}), \text{ the 2nd Betti number of } Z_{N,e} \\
b_2 &= \dim H^2(\tilde{Z}_{N,e}), \text{ the 2nd Betti number of the desingularization } \tilde{Z}_{N,e} \\
h^{1,1} &= \dim H^1(\tilde{Z}_{N,e}, \Omega^1_{\tilde{Z}_{N,e}}), \text{ the (1,1)-Hodge number of } \tilde{Z}_{N,e} \\
p_g &= \text{the geometric genus of } \tilde{Z}_{N,e} \\
c_2 &= \text{the 2nd Chern number of } \tilde{Z}_{N,e} \\
K^2 &= \text{the self-intersection number of the canonical divisor } K \text{ of } \tilde{Z}_{N,e} \\
\kappa &= \text{the Kodaira dimension of } \tilde{Z}_{N,e} \ (\kappa = -1 \text{ means that } \tilde{Z}_{N,e} \text{ is rational})
\end{aligned}
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Notes: 1) If $k = 0, 1, \infty$, then $\langle\sigma_k\rangle$ denotes the stabilizer subgroup of G at a point above the three ramification points $\bar{P}_k \in X(1)$ of the quotient map $\pi : X(N) \to X(1) = G\backslash X(N) \simeq \mathbb{P}^1$. Thus $|\langle\sigma_k\rangle| = 2, 3, N$ for for $k = 0, 1, \infty$, respectively.

2) If $e' = f^2 e$ with $(f, N) = 1$, then $Z_{N,e'} \simeq Z_{N,e}$. Thus, the tables list only one representative e for each square class mod N.