Explanation of the Invariants of a MDQS

The modular diagonal quotient surface $Z_{N,e}$ is the quotient surface $Z_{N,e} = \Delta_e \backslash Y_N$ in which $Y_N = X(N) \times X(N)$ is the product of the modular curve $X(N)$ with itself and $\Delta_e \le G \times G$ is a certain "twisted diagonal" subgroup of $G = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$; its desingularization is denoted by $\tilde{Z}_{N,e}$. (See explgi for a more precise decription.)

The tables list the following information about the desingularization $\tilde{Z}_{N,e}$ of $Z_{N,e}$:

Basic numerical invariants:

- geometric (Betti, Hodge and Chern numbers): $p_g, h^{1,1}, b_2, sgn, c_2 = \chi_{top}, K^2$
- other: $m, g, r_0, g_0, r_1, g_1, s_{11}, r_\infty, g_\infty, h, \mathbb{L}_\infty, \mathbb{L}, \mathbb{S}_\infty, \mathbb{S}$

Singularities of the surface $Z_{N,e}$:

- singularities above $\bar{P}_0 \times \bar{P}_0$:
  - CM and anti-CM singularities (both of type (-2))
- singularities above $\bar{P}_1 \times \bar{P}_1$:
  - CM singularities (those of type (-3))
  - anti-CM singularities (those of type $(-2, -2)$)
- singularities above $\bar{P}_\infty \times \bar{P}_\infty$

See Description of Singularities for more explanations about the information given in the tables.

Intersection numbers of the basic curves:

- Table of the six non-exceptional basic curves $\tilde{C}_{0,1}, \tilde{C}_{0,2}, \tilde{C}_{1,1}, \tilde{C}_{1,2}, \tilde{C}_{\infty,1}, \tilde{C}_{\infty,2}$
  - table lists their genera and self-intersection numbers
- The intersection matrix of the six non-exceptional basic curves (with each other)
- The intersection matrix of the P_0-curves
  - these are the curves $\tilde{C}_{0,1}, \tilde{C}_{0,2}$ together with the r_0 exceptional curves arising from the resolution of the singularities over $\bar{P}_0 \times \bar{P}_0$
- The intersection matrix of the P_1-curves
  - these are the curves $\tilde{C}_{1,1}, \tilde{C}_{1,2}$ together with the $2r_1 - s_{11}$ exceptional curves arising from the resolution of the singularities over $\bar{P}_1 \times \bar{P}_1$
- The intersection matrix of the P_∞-curves
  - these are the curves $\tilde{C}_{\infty,1}, \tilde{C}_{\infty,2}$ together with the $\mathbb{L}_\infty$ exceptional curves arising from the resolution of the singularities over $\bar{P}_\infty \times \bar{P}_\infty$

Note: these tables give the entire intersection matrix of the basic curves because any other intersection number is 0. (The exceptional P_0-curves do not meet the exceptional P_1, P_∞-curves, etc.)

Intersection numbers of the Hecke curves:

- Properties of the Hecke curves $T_{n,k}$:
  - the table lists the degree $\deg(T)$, the arithmetic genus $p_a(T)$, the singularity degree $\delta(T)$ and the self-intersection number T^2 of each Hecke curve $T = T_{n,k}$
- The intersection numbers of the Hecke curves with the P_0-curves
  - for ease of identification, the 2nd to 4th columns of the table list n, k and $\deg(T_{n,k})$
- The intersection numbers of the Hecke curves with the P_1-curves
  - for ease of identification, the 2nd to 4th columns of the table list n, k and $\deg(T_{n,k})$
- The intersection numbers of the Hecke curves with the P_∞-curves
  - for ease of identification, the 2nd to 4th columns of the table list n, k and $\deg(T_{n,k})$
- The intersection numbers of the Hecke curves with each other
  - for ease of identification, the 2nd to 4th columns of the table list n, k and $\deg(T_{n,k})$

Notes: 1) For each pair (n, k) with $nk^2 \equiv e \pmod N$ there is a Hecke curve $T_{n,k}$ on $\tilde{Z}_{N,e}$ whose desingularization is the modular curve $X_0(n)$. (Only distinct curves are listed, using the fact that $T_{n,k_1} = T_{n,k_2}$ if $k_1 \equiv \pm k_2 \pmod N$.)

More precisely, the Hecke curve $T_{n,k}$ on $\tilde{Z}_{N,e}$ is the proper transform of the quotient curve $\Delta_e \backslash T'_{n,k} \subset Z_{N,e}$, where $T'_{n,k} = (\tau_k^{-1} \times id_{X(N)})(T_n)$ is the twist of the Hecke correspondence T_n on $X(N) \times X(N)$ by the diamond operator $\tau_k = \left(\begin{smallmatrix} k^{-1} & 0 \\ 0 & k \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

2) The degree of $T = T_{n,k}$ is the intersection number $\deg(T) = (T.F_i)$, where F_i is any fibre of the fibration $\psi_i : \tilde{Z}_{N,e} \to X(1) \simeq \mathbb{P}^1$, for $i = 1, 2$.

3) The singularity degree of $T = T_{n,k}$ is $\delta(T) = p_a(T) - g_{X_0(n)}$ because $X_0(n)$ is the desingularization of $T_{n,k}$.