High-Resolution Source Coding for Non-Difference Distortion Measures: The Rate-Distortion Function

Tamás Linder, *Member, IEEE*, and Ram Zamir, *Member, IEEE*

***Abstract*—The problem of asymptotic (i.e., low-distortion) behavior of the rate-distortion function of a random vector is investigated for a class of non-difference distortion measures. The main result is an asymptotically tight expression which parallels the Shannon lower bound for difference distortion measures. For example, for an *input-weighted* squared error distortion measure $d(x,y) = \|W(x)(y-x)\|^2$, $y, x \in \mathbb{R}^n$, the asymptotic expression for the rate-distortion function of $X \in \mathbb{R}^n$ at distortion level D equals**

$$h(X) - \frac{n}{2}\log(2\pi e D/n) + \boldsymbol{E}\log|\det W(X)|$$

where $h(X)$ is the differential entropy of X. Extensions to staionary sources and to high-resolution remote ("noisy") source coding are also given. In a companion paper in this issue these results are applied to develop a high-resolution quantization theory for non-difference distortion measures.

***Index Terms*—Asymptotic quantization theory, Shannon lower bound, non-difference distortion measures, rate-distortion function, remote source coding.**

I. Introduction and Summary of Results

A. Background

A fundamental component in the design and analysis of analog signal coding schemes is the choice of an appropriate fidelity criterion. The most commonly used fidelity criteria measure the distortion D of a coding scheme by the expected value of a nonnegative function ρ of the difference between the source $X \in \mathbb{R}^n$ and its reconstruction $Y \in \mathbb{R}^n$, i.e.,

$$D = \boldsymbol{E}\rho(Y - X). \tag{1}$$

Distortion measures of this type are called *difference* distortion measures. The single most popular distortion measure, the mean-squared error (MSE), is a difference distortion measure for which $\rho(y-x) = \|y-x\|^2$, where $\|\cdot\|$ denotes Euclidean norm. MSE and its variations, such as the frequency-weighted squared error [1], are widely used in speech, picture, and video compression. In general, difference distortion measures are popular because of their intuitive nature, and also because they are often mathematically tractable and simple to compute.

However, in many "real-life" applications the true distortion measure is not a function of the difference between the signal and its reconstruction. For example, a common perceptual criterion for quantizing the linear predictive coding (LPC) parameters in speech coding is the *log spectral distortion* (LSD) [2]

$$d(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(20\log_{10}|\mathcal{Y}(\omega)| - 20\log_{10}|\mathcal{X}(\omega)|)^2\,d\omega \tag{2}$$

where $\mathcal{X}(\omega)$ and $\mathcal{Y}(\omega)$ are the frequency responses of the corresponding linear prediction filters, i.e.,

$$\mathcal{X}(\omega) = 1 - \sum_{k=1}^{n} x_k e^{-j\omega k}$$

and

$$\mathcal{Y}(\omega) = 1 - \sum_{k=1}^{n} y_k e^{-j\omega k}.$$

Remote (or "noisy") source coding provides another example where the effective criterion for quantization may be a non-difference distortion measure even if the original distortion measure for the "clean" source is a difference distortion measure [3]. In contrast to the case of difference distortion measures, for non-difference distortion measures it is possible that the noisy source X and the reconstruction Y have different dimensions.

A large body of literature considers quantization under difference distortion measures. High-resolution quantization theory provides simple asymptotic expressions for the rate-distortion performance of fixed- and variable-rate quantizers, relative to difference distortion measures. For example, the minimum possible entropy of a scalar quantizer that encodes at MSE level D a source $X \in \mathbb{R}^1$ having a smooth density, is given for small D by [4]

$$H(Q_1(X)) \approx h(X) - \frac{1}{2}\log(12D) \tag{3}$$

where H and h denote regular and differential entropies, respectively, log denotes base 2 logarithm, and $\approx$ means that the difference between the corresponding quantities goes to zero as $D \to 0$. More generally, the entropy of a lattice quantizer Q_n that encodes a smooth source $X \in \mathbb{R}^n$ with

Manuscript received January 15, 1997; revised April 1, 1998. This work was supported in part by the Joint Program between the Hungarian Academy of Sciences and the Israeli Academy of Science and Humanities, and by the National Science Foundation and OTKA under Grant F 014174. This work was presented in part at ISIT-97, Ulm, Germany, June 1997.

T. Linder is with the Department of Mathematics and Statistics, Queen's University, Kingston, Ont. K7L 3N6 Canada (e-mail: linder@mast.queensu.ca).

R. Zamir is with the Department of Electrical Engineering–Systems, Tel-Aviv University, Ramat-Aviv, 69978, Israel (e-mail: zamir@eng.tau.ac.il).

Communicated by N. Merhav, Associate Editor for Source Coding.

Publisher Item Identifier S 0018-9448(99)01398-X.

MSE level D (so that the per-dimension distortion is D/n) is given for small D by [5]–[7]

$$H(Q_n(X)) \approx h(X) - \frac{n}{2}\log(D/(nG_n)) \tag{4}$$

where G_n denotes the normalized second moment of the lattice. Also, Shannon's rate-distortion function [3]

$$R(D) = \inf\{I(X,Y) : \boldsymbol{E}[d(X,Y)] \le D\} \tag{5}$$

characterizing the minimum achievable rate at distortion level D by any (multidimensional) quantizer, can be lower-bounded by the Shannon lower bound (SLB) [3] if d is a difference distortion measure. For the MSE criterion this lower bound states that

$$R(D) \ge h(X) - \frac{n}{2}\log(2\pi e D/n). \tag{6}$$

The SLB becomes tight in the limit as D goes to zero, i.e.,

$$R(D) \approx h(X) - \frac{n}{2}\log(2\pi e D/n) \tag{7}$$

provided $h(X)$ is finite and $E\|X\|^2 < \infty$ [8] (see also [9]). Properties (4) and (7) imply that the asymptotic rate redundancy of an entropy-coded lattice quantizer above the rate-distortion function is $\frac{1}{2}\log(2\pi e G_n)$ bit per dimension. (Note that $G_1 = 1/12$.)

The importance of the asymptotic expressions and bounds in (3)–(7) is in providing simple explicit formulas for quantities which are in general hard (or impossible) to compute analytically. In fact, the SLB is the only known tool for relating the high-resolution performance of entropy-coded quantizers to the rate-distortion function.

Although many other useful formulas and bounds for the high-resolution performance of fixed- and variable-rate coding schemes relative to specific difference distortion measures exist [10], [5], [11]–[13], there are significantly fewer results in the literature on the high-resolution performance for source coding under non-difference distortion measures. The first results extending bounds in [5] and [11] to locally quadratic non-difference distortion measures were given in [2]. The log spectral distortion and the Itakura–Saito distortion are examples of such measures. A more formal treatment of these bounds is given in [14], where a new lower bound on the variable-rate (i.e., entropy-coded) vector quantizer performance is developed using optimal point densities. It is also pointed out in [14] that some important "perceptual distortion measures" in image coding are locally quadratic.

In this work we take a rigorous approach to generalize some of the fundamental concepts of high-resolution source coding theory to locally quadratic non-difference distortion measures. In Section II, the small distortion behavior of the rate-distortion function is studied for a large class of non-difference distortion measures and sources. Theorem 1, the main result of the paper, gives an asymptotically tight formula for the rate-distortion function relative to a "input-weighted locally quadratic" distortion measure. In Section III, an application of this result to remote source coding is given and examples are provided. In the rest of this Introduction, an informal description of the main results and its corollaries is given. Exact statements and proofs are deferred until Sections II and III.

B. Main Result and Corollaries

Our main result is a generalization of the asymptotic formula for the rate-distortion function (7) to a fidelity criterion of the form

$$D = \boldsymbol{E}d(X,Y) \tag{8}$$

where d is a non-difference distortion measure satisfying certain regularity conditions. The basic requirement is that for a fixed x, the nonnegative function $d(x,y)$ is locally quadratic around $y = r(x) = \arg\min_y d(x,y)$, i.e.,

$$d(x,y) = d_{\min}(x) + (y - r(x))^t M(x)(y - r(x)) + O(\|y - r(x)\|^3)$$

where (by Taylor expansion) $M(x)$ is half the $n \times n$ matrix of second-order partial derivatives of $d(x,y)$ with respect to y at $y = r(x)$, and $d_{\min}(x) = d(x, r(x))$. Note that $M(x)$ must be nonnegative-definite. The fact that certain useful distortion measures can be expanded this way for the purpose of asymptotic analysis was first pointed out by Gardner and Rao [2], who considered the case $r(x) = x$.

The main result of this paper shows that under some regularity conditions (specified in detail in Section II), the rate-distortion function (5) is given as $D \to D_{\min}$ by

$$R(D) \approx h(r(X)) - \frac{n}{2}\log(2\pi e(D - D_{\min})/n) + \frac{1}{2}\boldsymbol{E}[\log\det M(X)] \tag{9}$$

where $D_{\min} = \boldsymbol{E}[d_{\min}(X)]$ and det denotes matrix determinant. The formal statement and the proof of (9) are given in Section II. Possible extensions of this result to more general $d(x,y)$ is discussed in Section IV.

For example, consider the $W(x)$-*weighted mean-squared error* (W-WMSE) criterion

$$d(x,y) = \|W(x)(y-x)\|^2 = (y-x)^t W^t(x) W(x)(y-x) \tag{10}$$

where $W(x)$ is some source-dependent weighting matrix, and $W^t(x)$ denotes the transpose of $W(x)$. In this case we have $r(x) = x$, $D_{\min} = 0$, and $M(x) = W^t(x)W(x)$, so that as $D \to 0$

$$R(D) \approx h(X) - \frac{n}{2}\log(2\pi e D/n) + \boldsymbol{E}[\log|\det W(X)|]. \tag{11}$$

Note that for $W(x) = I$ (the identity matrix), (11) coincides with the regular MSE case (7) as expected.

For the *log-spectral distortion* (2) we have $r(x) = x$, $D_{\min} = 0$, and the elements of $M(x)$ are (see the "sensitivity matrix" in [2])

$$m_{ik}(x) = \frac{(10/\ln(10))^2}{\pi}\int_{-\pi}^{\pi} \frac{e^{j\omega(i-k)}}{|\mathcal{X}(\omega)|^2}\,d\omega, \qquad 1 \le i,\ k \le n.$$

Thus the asymptotic expression for $R(D)$ can be calculated via (9) if the source distribution is known.

In Section II we also extend our asymptotic analysis to stationary sources and per-letter distortion measures. For example, if $d(x,y) = [w(x)(y-x)]^2$, then the rate-distortion function of a real stationary source $X_1, X_2, \cdots$ is given asymptotically as $D \to 0$ by

$$\bar{h} - \frac{1}{2}\log(2\pi e D) + \boldsymbol{E}\log|w(X_1)|$$

where $\bar{h} = \lim_n (1/n) h(X_1, \cdots, X_n)$ denotes the differential entropy rate of the source.

Two interesting corollaries follow from (9). Let $R(Z,d,D)$ denote the rate-distortion function of a vector source Z under the locally quadratic distortion measure d at distortion level D. Let $M(x)$ be the matrix of second derivatives associated with the distortion measure d, and let $r(x) = \arg\min_y d(x,y)$. Suppose that there exists $\tilde{M}(x)$ satisfying $M(x) = \tilde{M}(r(x))$ (this is possible, e.g., if r is invertible). Let $\tilde{W}(x)$ denote any $n \times n$ matrix for which $\tilde{W}^t(x)\tilde{W}(x) = \tilde{M}(x)$. Then, from (9) and (11) we have as $D \to D_{\min}$

$$R(X,d,D) \approx R(r(X), \tilde{W}\text{-WMSE}, D - D_{\min}) \tag{12}$$

where $\tilde{W}$-WMSE is the WMSE distortion measure defined in (10).

For the second corollary, suppose that $r(\cdot)$ is invertible, and there exists an invertible and continuously differentiable vector function $g : \mathbb{R}^n \to \mathbb{R}^n$ whose derivative matrix g' satisfies

$$g'(t) = W(r^{-1}(t)) \tag{13}$$

where $W^t(x)W(x) = M(x)$, that is,

$$[g'(t)]^t g'(t) = M(r^{-1}(t)).$$

For example, under the regularity condition given in Section II, such a function g always exists for $n = 1$ (scalar case). For the general case see [15]. Then, substituting $g(r(X))$ as the source in (7) and using the identity $h(g(Z)) = h(Z) + \boldsymbol{E}\log|\det g'(Z)|$, we have as $D \to D_{\min}$

$$R(X,d,D) \approx R(g[r(X)], \text{MSE}, D - D_{\min}). \tag{14}$$

Corollaries (12) and (14) are actually implied by the following stronger statement. At high resolution, optimal encoding of X with d-distortion level D results by optimally encoding $r(X)$ with $\tilde{W}$-WMSE-distortion level $D - D_{\min}$. Moreover, if the function g of (13) exists, then optimal encoding of X with d-distortion D results by optimally encoding $g(r(X))$ with MSE-distortion level $D - D_{\min}$ and then applying the function $g^{-1}(\cdot)$. This interpretation of our main result have important implications in the context of remote source coding, and for quantizing via a companding model, as discussed in the next two subsections.

C. A Wolf–Ziv-Type Encoder for Remote Source Coding

Consider the following indirect source-coding problem. An encoder observes a noisy version X of a "clean" source U. The encoder's objective is to encode X in such a way that the reconstruction Y satisfies the fidelity criterion

$$E\rho(U,Y) \le D \tag{15}$$

with respect to U. Clearly, it is only possible to achieve distortion levels that are not less than

$$D_{\min} = \boldsymbol{E}\rho[U, r(X)] \tag{16}$$

where

$$r(x) = \arg\min_y \boldsymbol{E}\{\rho(U,y) \mid X = x\} \tag{17}$$

is an optimal estimation function of U from X. Typically, the coding rate must go to infinity as $D \to D_{\min}$. It is a classical result [16], [3] that indirect coding of the remote source U under the distortion measure ρ is equivalent to direct coding of the "noisy" source X under the modified distortion measure

$$d(x,y) = \boldsymbol{E}\{\rho(U,y) \mid X = x\}. \tag{18}$$

In particular, the indirect rate-distortion function of U under ρ, characterizing the minimum possible rate in the remote coding scenario, is equal to the ordinary rate-distortion function of X under d of (18).

In the context of this paper, it is important to note that even if ρ is a difference distortion measure, the modified distortion measure d is in general not. Furthermore, the optimal reconstruction and the minimum distortion associated with d get here the "physical" meanings of optimal estimation function and minimum estimation error, respectively. Our formula (9) thus gives the asymptotic form of the indirect rate-distortion function, using the modified distortion measure d, provided that d satisfies the regularity conditions given in Section II.

In the special case when $\rho(y,u) = \|y-u\|^2$ is the squared error, Wolf and Ziv [17] showed that the optimal indirect encoder has the following intuitive structure; it first estimates U optimally from X, i.e.,

$$\hat{U} = r(X) = \boldsymbol{E}\{U \mid X\} \tag{19}$$

and then it encodes $\hat{U}$ with MSE-distortion $D - D_{\min}$ (where now $D_{\min}$ is the conditional covariance of U given X). However, this very efficient encoding structure does not apply to a general distortion measure ρ. (The separation theorem of [17] follows from the *orthogonality principle*, which applies specifically to second-order estimation.) Nevertheless, our results show that a "Wolf–Ziv-type" encoder is *always optimal at high-resolution conditions* provided that d satisfies the regularity conditions and $M(x)$ can be written as $M(x) = \tilde{M}(r(x))$, as discussed in (12). This fact follows from the structure of the "test channel" which realizes $R(X,d,D)$ asymptotically as $D \to D_{\min}$ (in the sense of achieving the minimum in (5)). This asymptotically optimal test channel has the form

$$X \to Y = r(X) + W^{-1}(X)N = r(X) + \tilde{W}^{-1}(r(X))N \tag{20}$$

where N is a white Gaussian vector (independent of X) with variance $(D - D_{\min})/n$ per component, and $W(x)$ is such that $M(x) = W^t(x)W(x)$ (see the proof of Proposition 1 in Section II). We conclude that asymptotically optimal indirect encoding of X with d-distortion level D can be obtained by first optimally estimating $\hat{U} = r(X)$, and then encoding $\hat{U}$ with $\tilde{W}$-WMSE-distortion level $D - D_{\min}$ using the encoder induced by the test channel of (20) (e.g., via random coding). A more efficient scheme using companding is briefly discussed in the next subsection.

Section III discusses in detail under what conditions the asymptotic expression for the rate-distortion function (9) applies to the remote-coding problem. Specific examples where these conditions are proven to be satisfied are also given.

D. A Variable-Rate Companding Model

In the case when d is a difference distortion measure, the test channel of (20) specializes to a regular additive noise channel (see [3] and [8])

$$X \to Y = X + N$$

where for the MSE case N is Gaussian with variance D. In this regard it was demonstrated in [6] and [18] that *entropy-coded randomized (dithered) uniform/lattice quantization* (ECDQ) simulates (in the rate-distortion sense) an additive noise test channel. In [7] it was shown that at high resolution, the randomization of ECDQ is not necessary, and its redundancy above the rate-distortion function is asymptotically $\frac{1}{2}\log(2\pi e G_n)$ bit per dimension.

For a non-difference distortion measure, the additive noise N in the the asymptotically optimal test channel (20) is multiplied by the source-dependent factor $W(X)$. In light of the analogy above, this motivates the application of a companding model [5], i.e., a combination of nonlinear mapping and uniform quantization, for efficient finite-dimensional variable-rate coding, under the non-difference distortion measure d.

To explain this idea, let us consider the scalar case, i.e., $n = 1$, and assume that $D_{\min} = 0$. In this case the test channel (20) becomes $Y = r(X) + N/\sqrt{m(X)}$, where N is Gaussian with variance D, and $m(x) = \frac{1}{2}\partial^2 d(x,y)/\partial y^2$ evaluated at $y = r(x)$. Further, the function g of (13) becomes $g(t) = \int^t \sqrt{m(r^{-1}(y))}\, dy$. The function g plays the role of the compressor mapping in our companding model. From high-resolution quantization theory (3) we know that if a uniform scalar quantizer Q_1, with step-size $\sqrt{12D}$, is applied to $g(r(X))$, its entropy is given for small D by

$$H(Q_1(g(r(X))) \approx h(g(r(X))) - \frac{1}{2}\log(12D). \tag{21}$$

Using the identity $h(g(Z)) = h(Z) + \boldsymbol{E}\log|g'(Z)|$, and comparing with (9), we obtain

$$\begin{aligned} H(Q_1(g(r(X))) &\approx h(r(X)) - \frac{1}{2}\log(12D) + \frac{1}{2}\boldsymbol{E}\log m(X) \\ &\approx R(X,d,D) + \frac{1}{2}\log(2\pi e/12). \end{aligned} \tag{22}$$

Namely, the entropy of a uniform quantizer exceeds the rate-distortion function of X relative to d by $\frac{1}{2}\log(2\pi e/12) \approx 0.254$ bit in the limit as $D \to 0$. Further derivation shows that the entropy-coded companding quantization scheme of the form

$$X \to r(\cdot) \to g(\cdot) \to Q_1(\cdot) \to \left|\begin{array}{c}\text{entropy}\\ \text{coding}\end{array}\right| \to g^{-1}(\cdot) \to \hat{X}$$

has asymptotic distortion D (with respect to the distortion measure d), and thus by (22) its rate-distortion performance is within $\frac{1}{2}\log(2\pi e/12)$ bit of the rate-distortion function of X.

A detailed treatment of high-resolution variable-rate companding for non-difference distortion measures is given in [15], where it is shown that g above is the optimal companding function for a uniform scalar quantizer, and where the analysis above is made rigorous and is extended to sources with memory, to lattice quantizers, and to vector distortion measures.

II. Main Result

A. Statement of the Main Result

Let X be an n-dimensional random vector. Given a distortion measure $d : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$, the rate-distortion function of X is defined for all $D > 0$ by

$$R(D) = \inf\{I(X;Y) : \boldsymbol{E}[d(X,Y)] \le D\}.$$

Here the infimum of the mutual information between X and the n-dimensional random vector Y is taken over all possible conditional distributions of Y given X such that $\boldsymbol{E}[d(X,Y)] \le D$. If no such Y exists, then $R(D) = \infty$ by definition. It is assumed that d is Borel measurable, that X can be represented with finite distortion, i.e.,

$$D_{\min} = \boldsymbol{E}\left[\min_y d(X,y)\right] < \infty,$$

and that X has an absolutely continuous distribution with n-dimensional density f.

Although our analysis of $R(D)$ will not rely on its operational meaning, we mention here that $R(D)$ is the minimum achievable rate in coding with distortion D a memoryless vector source $\{X_i, i = 1, 2, \cdots\}$, where the X_i are distributed as X.

In some important applications, $D_{\min} > 0$ is a natural assumption. However, the following argument [3] allows us to use the more convenient assumption $D_{\min} = 0$. Let $\hat{d}(x,y) = d(x,y) - \min_y d(x,y)$. Then $\min_y \hat{d}(x,y) = 0$ for all x, and

$$R(D) = \hat{R}(D - D_{\min})$$

for all $D > D_{\min}$, where $\hat{R}(D)$ is the rate-distortion function of X with respect to $\hat{d}$. The conditions imposed in the sequel on d will also hold for $\hat{d}$. Therefore, to determine the asymptotic behavior of $R(D)$ as D tends to $D_{\min}$ from above, it suffices to study $\hat{R}(D)$ as $D \to 0$. Thus we can assume without loss of generality that for all $x \in \mathbb{R}^n$

$$\min_y d(x,y) = 0$$

and that $D_{\min} = 0$.

We say that a mapping $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is piecewise-invertible if there exist a finite number of disjoint open sets $A_1, \cdots, A_k$ such that ϕ is one-to-one on each A_i, the union of the closures of the A_i's covers $\mathbb{R}^n$, and the boundary of each A_i has zero Lebesgue measure. If ϕ is also continuously differentiable on $\cup_i A_i$, then we say that ϕ is piecewise-invertible and continuously differentiable. Let $E(x) = \{e_{ij}(x)\}$ be an $n \times n$ matrix whose elements are real functions $e_{ij} : \mathbb{R}^n \to \mathbb{R}$. We say that $E(x)$ is continuously differentiable on an open set A, if each e_{ij} has a continuous derivative on A.

If E is an $n \times n$ matrix, $\|E\|$ will denote its norm induced by the Euclidean norm: $\|E\| = \max_{\|x\|=1} \|Ex\|$.

We assume that the distortion measure satisfies the following regularity conditions.

Conditions on $d(x,y)$:

a) For all fixed $x \in \mathbb{R}^n$, $d(x,y)$ is three times continuously differentiable in the variable y and the third-order partial derivatives

$$\frac{\partial^3 d(x,y)}{\partial y_i \partial y_j \partial y_k}, \qquad i,j,k \in \{1,\cdots,n\}$$

are uniformly bounded.

b) For all $x \in \mathbb{R}^n$, $d(x,y)$ has a unique minimum in y at $r(x) = \arg\min_y d(x,y)$. Thus

$$d(x,y) \geq 0 \quad \text{with equality if and only if } y = r(x).$$

We assume that $r : \mathbb{R}^n \to \mathbb{R}^n$ is piecewise-invertible and continuously differentiable (see the definition above).

c) For all $x \in \mathbb{R}^n$

$$\liminf_{\|y\|\to\infty} d(x,y) > 0.$$

d) Let $M(x) = \{m_{ij}(x)\}$ be the $n \times n$ matrix with entries $m_{ij}(x)$, where

$$m_{ij}(x) = \frac{1}{2} \left. \frac{\partial^2 d(x,y)}{\partial y_i \partial y_j} \right|_{y=r(x)}.$$

Then $M(x)$ is piecewise continuously differentiable with respect to x.

Note that $M(x)$ is symmetric for all x by condition a). Since $d(x,y)$ has a unique minimum at $y = r(x)$, it follows that

$$\frac{\partial d(x,y)}{\partial y}|_{y=r(x)} = 0$$

and thus $M(x)$ is also nonnegative-definite for all x. Therefore, we have $\det M(x) \geq 0$ for all x.

Next we state the conditions d and the distribution of X are assumed to satisfy jointly.

Conditions on X and $d(x,y)$:

e)

$$\boldsymbol{E}\,|\log\det M(X)| < \infty.$$

f) The random vector $r(X)$ has an n-dimensional density and a finite differential entropy $h(r(X))$ and

$$\boldsymbol{E}\|r(X)\|^2 < \infty.$$

g)

$$\boldsymbol{E}\,[(\mathrm{tr}\,\{M^{-1}(X)\})^{3/2}] < \infty$$

where $\mathrm{tr}\,\{M^{-1}(X)\}$ denotes the trace of the inverse of $M(X)$ (which exists almost surely by e)).

The following is the main result of the paper.

Theorem 1: Suppose $d(x,y)$ and X satisfy a)–g). Then

$$\lim_{D\to 0}\left[R(D) + \frac{n}{2}\log(2\pi e D/n)\right] = h(r(X)) + \frac{1}{2}\boldsymbol{E}\,[\log\det M(X)]$$

where r and M are defined in conditions b) and d) above.

The main theorem is a consequence of the following two results.

Proposition 1 (Achievability): Suppose $d(x,y)$ and X satisfy a)–g). Then

$$\limsup_{D\to 0}\left[R(D) + \frac{n}{2}\log(2\pi e D/n)\right] \leq h(r(X)) + \frac{1}{2}\boldsymbol{E}\,[\log\det M(X)].$$

Proposition 2 (Converse): Suppose $d(x,y)$ and X satisfy a)–f). Then

$$\liminf_{D\to 0}\left[R(D) + \frac{n}{2}\log(2\pi e D/n)\right] \geq h(r(X)) + \frac{1}{2}\boldsymbol{E}\,[\log\det M(X)].$$

The proofs of the two propositions are given in the next subsection. We now briefly discuss our hypotheses.

The first set of conditions a)–d) contain, for the most part, rather natural assumptions on the smoothness and regular behavior of the distortion measure. d is assumed to be regular in the sense that for a given input x there is a unique reproduction $y = r(x)$ minimizing the distortion (condition b)), and all other reproduction values y produce distortion which is bounded away from zero if y is bounded away from $r(x)$ (condition d)). As for smoothness, the assumption that $r(x)$ be piecewise-differentiable is quite mild. Also note that if $d(x,y)$ itself is three times continuously differentiable (as a function $d : \mathbb{R}^{2n} \to \mathbb{R}$), then condition d) holds. Two extra conditions here are hard to justify on intuitive grounds: the uniform boundedness of the third derivatives in a) and the requirement that $r(x)$ be piecewise-invertible in b). Both of these conditions are imposed for technical reasons and are not believed necessary for the validity of the main result.

Conditions e)–g) further specialize d. Most importantly, $\boldsymbol{E}\,|\log\det M(X)| < \infty$ implies that the locally quadratic behavior

$$d(x,y) \approx (y - r(x))^t M(x)(y - r(x))$$

dominates the higher order terms for all y in a neighborhood of $r(x)$. This follows since e) implies that $M(x)$ is positive-definite for all x except for a set of X-probability zero. Again, we were forced to introduce two technical conditions. The finiteness of $\boldsymbol{E}\,\|r(X)\|^2$ is required by our proof technique, while $\boldsymbol{E}\,[(\mathrm{tr}\,\{M^{-1}(X)\})^{3/2}] < \infty$ is an assumption we have not managed to eliminate.

It is instructive to observe what the conditions mean for the WMSE distortion measure

$$d(x,y) = \|W(x)(x-y)\|^2$$

where $W(x)$ is an $n \times n$ matrix that depends on the input x. In this case, $M(x) = W^t(x)W(x)$ and the third-order partial derivatives $\frac{\partial^3 d(x,y)}{\partial y_i \partial y_j \partial y_k}$ are all zero, so a) is satisfied. Conditions b) and c) obviously hold, and d) is also satisfied if the elements of $W(x)$ are (piecewise) continuously differentiable. We conclude that if X has a finite differential entropy and second moment, then $\boldsymbol{E}[\log|\det W(X)|] < \infty$ and $\boldsymbol{E}[(\mathrm{tr}\{W^{-1}(X)\})^3] < \infty$ are sufficient conditions for the validity of the main theorem. In particular, consider the magnitude weighted distortion measure (see [19])

$$d(x,y) = \frac{\|x-y\|^2}{\|x\|^2}.$$

We have $W(x) = \|x\|^{-1}I$. Suppose that $h(X)$ is finite. Then

$$\lim_{D\to 0}\Big[R(D) + \frac{n}{2}\log(2\pi e D/n)\Big] = h(X) - n\boldsymbol{E}[\log\|X\|]$$

provided $\boldsymbol{E}[\log\|X\|] > -\infty$ and $\boldsymbol{E}\|X\|^3 < \infty$ (conditions e) and g)) hold.

B. Stationary Sources

Assume that $\{X_i\}$ is a real-valued stationary source and consider a single-letter distortion measure $d_1 : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ such that $d_1(x,x) = 0$ for all x. Then d_1 generates the family of distortion measures

$$d_n(x,y) = \frac{1}{n}\sum_{i=1}^{n} d_1(x_i, y_i), \qquad n = 1, 2, \cdots$$

between n blocks of the source and their reproductions. Let $X^n = (X_1, \cdots, X_n)$ and assume that the differential entropy rate of the source $\bar{h} = \lim_{n\to\infty}\frac{1}{n}h(X^n)$ is finite. Let $\bar{R}(D)$ be the rate-distortion function of $\{X_i\}$ relative to d_1, defined by $\bar{R}(D) = \lim_{n\to\infty} R_n(D)$, where

$$R_n(D) = \inf\{n^{-1}I(X^n;Y^n) : \boldsymbol{E}[d_n(X^n,Y^n)] \le D\}$$

is the nth-order rate-distortion function of $\{X_i\}$ [3]. Note that if $\{X_i\}$ is ergodic, $\bar{R}(D)$ is the minimum achievable rate in fixed-rate coding of $\{X_i\}$ with distortion D.

In what follows we show that if d_1 and X_1 satisfy the conditions of Theorem 1, then as $D \to 0$

$$\bar{R}(D) \approx \bar{h} - \frac{1}{2}\log(2\pi e D) + \frac{1}{2}\boldsymbol{E}[\log m(X_1)] \tag{23}$$

where $m(x) = \frac{1}{2}\frac{\partial^2 d_1(x,y)}{\partial y^2}|_{y=x}$.

To prove this claim first notice that if d_1 and X_1 satisfy the conditions of Theorem 1, then so do d_n and X^n for each $n \ge 1$. In particular, if $F_n(D)$ is defined by

$$F_n(D) = R_n(D) + \frac{1}{2}\log(2\pi e D)$$

then we have by Theorem 1 that

$$\lim_{D\to 0} F_n(D) = \frac{1}{n}h(X^n) + \frac{1}{2}\boldsymbol{E}[\log m(X_1)].$$

Since $\frac{1}{n}h(X^n) \to \bar{h}$ as $n \to \infty$, and

$$\lim_{n\to\infty} F_n(D) = \bar{R}(D) + \frac{1}{2}\log(2\pi e D)$$

(23) is equivalent to

$$\lim_{D\to 0}\lim_{n\to\infty} F_n(D) = \lim_{n\to\infty}\lim_{D\to 0} F_n(D).$$

This exchange of limits is legitimate if, as $n \to \infty$, $F_n(D)$ converges to its limit uniformly in D. But this uniform convergence holds if and only if $R_n(D)$ converges in n to $\bar{R}(D)$ uniformly in D, which is known to be true because the well-known lower bound of Wyner and Ziv [20], [3] implies that for all $D > 0$ and $n \ge 1$

$$\bar{R}(D) \le R_n(D) \le \bar{R}(D) + \frac{1}{n}h(X^n) - \bar{h}. \tag{24}$$

C. Proofs

In the proofs we will need the following simple fact. Suppose conditions a) and b) hold. Then there exists $0 \le C < \infty$ such that

$$d(x,y) = (y - r(x))^t M(x)(y - r(x)) + s(x,y) \tag{25}$$

where

$$|s(x,y)| \le C\|y - r(x)\|^3.$$

Consider the second-order Taylor expansion of $d(x,y)$ as a function of y around $y = r(x)$. Then (25) follows since $\frac{\partial d(x,y)}{\partial y}|_{y=r(x)} = 0$ and the remainder term is controlled by the uniform boundedness of the third derivatives.

Proof of Proposition 1: The asymptotic upper bound on $R(D)$ is proved by demonstrating the existence of an appropriate (not necessarily optimal) forward test channel. If $m \in \mathbb{R}^n$ and C is a symmetric nonnegative-definite $n \times n$ matrix, then $Z \sim N(m, C)$ denotes an n-dimensional random vector which has normal distribution with mean m and covariance C. For any $D > 0$, let $Z_D \sim N(0, (D/n)I)$ be independent of X, where I is the $n \times n$ identity matrix. Let $U = \{x : \det M(x) > 0\}$. Then $\boldsymbol{P}(X \in U) = 1$ by e). Let $W(x)$ denote any $n \times n$ matrix-valued (measurable) function of x such that $W^t(x)W(x) = M(x)$. For the sake of convenience, define $W^{-1}(x) = 0$ for all $x \notin U$. Set

$$Y_D = r(X) + W^{-1}(X)Z_D$$

and consider the test channel $X \to Y_D$. Then $W(X)(Y_D - r(X)) = Z_D$ almost surely, and

$$\begin{aligned}\boldsymbol{E}[(Y_D - r(X))^t M(X)(Y_D - r(X))] &= \boldsymbol{E}\|W(X)(Y_D - r(X))\|^2 \\ &= \boldsymbol{E}\|Z_D\|^2 = D.\end{aligned}$$

Thus

$$\boldsymbol{E}[d(X,Y_D)] = D + \boldsymbol{E}[s(X,Y_D)] \stackrel{\text{def}}{=} \delta(D) \tag{26}$$

where $s(x,y) = d(x,y) - (y - r(x))^t M(x)(y - r(x))$. In Appendix B it is shown that $\delta(D)$ is continuous at all $D > 0$, and $\boldsymbol{E}[s(X,Y_D)] = o(D)$ so that $\delta(D) = D + o(D)$ as $D \to 0$. Moreover,

$$\begin{aligned}I(X;Y_D) &= I(X; r(X) + W^{-1}(X)Z_D) \\ &= h(r(X) + W^{-1}(X)Z_D) - h(W^{-1}(X)Z_D \mid X)\end{aligned} \tag{27}$$

where the decomposition is possible because $h(W^{-1}(X)Z_D \mid X)$ is finite as shown below. Since $W^{-1}(x)Z_D \sim N(0,(D/n)M^{-1}(x))$ for all $x \in U$, and since X and Z_D are independent, we obtain

$$\begin{aligned} h(W^{-1}(X)Z_D \mid X=x) &= h(W^{-1}(x)Z_D) \\ &= \frac{n}{2}\log(2\pi eD/n) \\ &\quad + \frac{1}{2}\log\det M^{-1}(x), \qquad x \in U \end{aligned}$$

(see, e.g., [21, p. 230]). Thus

$$h(W^{-1}(X)Z_D \mid X) = \frac{n}{2}\log(2\pi eD/n) - \frac{1}{2}\boldsymbol{E}[\log\det M(X)] \tag{28}$$

where $\boldsymbol{E}[\log\det M(X)]$ is finite by condition e). Thus by the definition of the rate-distortion function and by (26)–(28)

$$\begin{aligned} R(\delta(D)) \le h(r(X) + W^{-1}(X)Z_D) - \frac{n}{2}\log(2\pi eD/n) \\ + \frac{1}{2}\boldsymbol{E}[\log\det M(X)]. \end{aligned}$$

In Appendix B it is proved that

$$\limsup_{D\to 0} h(r(X) + W^{-1}(X)Z_D) \le h(r(X))$$

and now $\delta(D) = D + o(D)$ implies

$$\begin{aligned} \limsup_{D\to 0}\Big[R(\delta(D)) + \frac{n}{2}\log(2\pi e\delta(D)/n)\Big] \\ \le h(r(X)) + \frac{1}{2}\boldsymbol{E}[\log\det M(X)]. \end{aligned}$$

Since $\delta(D)$ is continuous and $\delta(D) \to 0$ as $D \to 0$, the above gives

$$\begin{aligned} \limsup_{D\to 0}\Big[R(D) + \frac{n}{2}\log(2\pi eD/n)\Big] \\ \le h(r(X)) + \frac{1}{2}\boldsymbol{E}[\log\det M(X)]. \quad \square \end{aligned}$$

Before giving the somewhat involved proof of Proposition 2, we briefly go over the main idea.

Sketch of the proof of Proposition 2: To simplify things, consider the *scalar* WMSE distortion measure

$$d(x,y) = [w(x)(y-x)]^2$$

and assume that $w(x)$ has a continuous derivative $w'(x)$ and that i) $|w'(x)| \le a < \infty$; and ii) $w(x) \ge b > 0$. Note that i) and ii) are stronger versions of conditions d), e), and g) of Theorem 1. Let $\{Y_D : D > 0\}$ be an arbitrary collection of random variables, jointly distributed with the random variable X so that the (X, Y_D) pairs satisfy the distortion constraint $\boldsymbol{E}d(X,Y_D) = \boldsymbol{E}[w(X)^2(Y_D - X)^2] \le D$ for all $D > 0$. Assuming $h(X)$ is finite, we have

$$I(X;Y_D) = h(X) - h(X \mid Y_D) \tag{29}$$

whenever $I(X;Y_D)$ is finite. As in the proof of the Shannon lower bound (see, e.g., [22, Theorem 23]), we want to upper-bound $h(X \mid Y_D)$ using the distortion constraint. Let $\phi_y(x) = w(x)(y-x)$. By conditions i) and ii) above, for x in some small neighborhood of y (i.e., for $|x-y| \le \epsilon$ for some $\epsilon > 0$) the derivative of the function $\phi_y(x)$ with respect to x satisfies

$$\phi_y'(x) = w'(x)(y-x) - w(x) = -w(x) + O(\epsilon).$$

It follows that $\phi_y(x)$ is invertible in x in this neighborhood if ϵ is small enough. It is not hard to see that as $D \to 0$ the probability that $|Y_D - X| > \epsilon$ must go to zero for any $\epsilon > 0$. Thus by the identity $h(\phi(Z)) = h(Z) + E\log|\phi'(Z)|$, which holds if ϕ is invertible and continuously differentiable, we have

$$h(w(X)(Y_D - X) \mid Y_D) = h(X \mid Y_D) + E\log w(X) - o(1) \tag{30}$$

where $o(1) \to 0$ as $D \to 0$. The asymptotic equality (30) is made precise by conditioning on the event $|Y_D - X| > \epsilon$. Furthermore,

$$\begin{aligned} h(w(X)(Y_D - X) \mid Y_D) &\le h(w(X)(Y_D - X)) \\ &\le \frac{1}{2}\log(2\pi eD) \end{aligned} \tag{31}$$

where the first inequality follows since conditioning reduces entropy, and the second follows from the distortion constraint and the fact that $h(Z) \le \frac{1}{2}\log(2\pi e\boldsymbol{E}[Z^2])$. Combining (29)–(31) gives

$$I(X;Y_D) \ge h(X) - \frac{1}{2}\log(2\pi eD) + E\log w(X) - o(1)$$

for any collection of Y_D as above. Specifically the inequality holds for $R(D)$ as desired.

The following lemma is proved in Appendix A.

Lemma 1: Let $W(x)$ be the square root of $M(x)$, i.e., the unique symmetric, nonnegative-definite $n \times n$ matrix for which $W(x)W(x) = M(x)$. If $M(x)$ is continuously differentiable on an open set A, then $W(x)$ is continuously differentiable on $A \cap \{x : \det M(x) > 0\}$.

Proof of Proposition 2: Let $\{Y_D : D > 0\}$ be an arbitrary collection of random n-vectors jointly distributed with X so that for all $D > 0$

$$\boldsymbol{E}[d(X,Y_D)] \le D \quad \text{and} \quad I(X;Y_D) < \infty.$$

Let $A_1, \cdots, A_k$ be the open sets on which r is one-to-one and continuously differentiable (see condition b)) and define the discrete random variable $C(X)$ by

$$C(X) = \begin{cases} i, & \text{if } X \in A_i, \qquad i = 1,\cdots,k \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\boldsymbol{P}(C(X) = 0) = 0$ and there exists a function $\omega(c,r)$ such that $\omega(C(X), r(X)) = X$ almost surely since X has a density. Thus we have

$$\begin{aligned} I(X;Y_D) &= I(C(X), r(X); Y_D) \\ &= I(C(X);Y_D) + I(r(X);Y_D \mid C(X)) \\ &= I(C(X);Y_D) + h(r(X) \mid C(X)) \\ &\quad - h(r(X) \mid Y_D, C(X)) \end{aligned} \tag{32}$$

where the above information quantities are finite since $h(r(X))$ is finite by condition f). Let

$$D_i = \boldsymbol{E}[d(X, Y_D) \mid C(X) = i]$$

and suppose we can prove that for all $i \geq 1$

$$\limsup_{D\to 0}\Big[h(r(X) \mid Y_D, C(X) = i) - \frac{n}{2}\log(2\pi e D_i/n)\Big] \leq -\frac{1}{2}\boldsymbol{E}[\log\det M(X) \mid C(X) = i] \quad (33)$$

and that

$$\liminf_{D\to 0} I(C(X); Y_D) \geq I(C(X); r(X)). \quad (34)$$

Then, on the one hand,

$$I(C(X); r(X)) + h(r(X) \mid C(X)) = h(r(X)). \quad (35)$$

On the other hand, by the convexity of the logarithm, we have

$$\begin{aligned}\sum_{i=1}^{k}\frac{n}{2}\log(2\pi e D_i/n)\boldsymbol{P}(C(X) = i) &\leq \frac{n}{2}\log(2\pi e\boldsymbol{E}[d(X, Y_D)]/n)\\ &\leq \frac{n}{2}\log(2\pi e D/n).\end{aligned}$$

Thus

$$\begin{aligned}\limsup_{D\to 0}&\Big[h(r(X) \mid Y_D, C(X)) - \frac{n}{2}\log(2\pi e D/n)\Big]\\ &\leq \limsup_{D\to 0}\Bigg[\sum_{i=1}^{k}\Big(h(r(X) \mid Y_D, C(X) = i)\\ &\quad - \frac{n}{2}\log(2\pi e D_i/n)\Big)\boldsymbol{P}(C(X) = i)\Bigg]\\ &\leq -\frac{1}{2}\boldsymbol{E}[\log\det M(X)]\end{aligned}$$

and by (32), (34), and (35)

$$\liminf_{D\to 0}\Big[I(X, Y_D) + \frac{n}{2}\log(2\pi e D/n)\Big] \geq h(r(X)) + \frac{1}{2}\boldsymbol{E}[\log\det M(X)]$$

which implies the statement of the proposition. Thus it suffices to prove (33) and (34). First consider (34). In Appendix C we prove that

$$\lim_{D\to 0}\boldsymbol{P}(\|r(X) - Y_D\| > \epsilon) = 0 \quad (36)$$

for all $\epsilon > 0$, i.e., $Y_D \to r(X)$ in probability. Then

$$(C(X), Y_D) \to (C(X), r(X))$$

in probability also, and therefore, (34) holds by the lower semicontinuity property of the information divergence (see, e.g., [23]).

To prove (33) we will assume without loss of generality that $\boldsymbol{P}(X \in A_i) = 1$, i.e., we replace X by a random variable which is distributed as the conditional distribution of X given A_i. This allows us to drop the conditioning on the event $\{C(X) = i\}$. Thus we have to prove that

$$\limsup_{D\to 0}\Big[h(r(X) \mid Y_D)) - \frac{n}{2}\log(2\pi e D/n)\Big] \leq -\frac{1}{2}\boldsymbol{E}[\log\det M(X)] \quad (37)$$

where $\boldsymbol{E}[d(X, Y_D)] \leq D$ for all $D > 0$, and where $r(x)$ is now invertible and continuously differentiable on the entire support A_i.

Let A be the open set such that $M(x)$ is continuously differentiable on A and $\mathbb{R}^n\backslash A$ has zero Lebesgue measure (see condition d)). Let r' denote the derivative of r on A_i, and let

$$B_i = \{x : |\det r'(x)| > 0\} \cap A_i.$$

Then B_i is open since r' is continuous on A_i, and by the assumption that $r(X)$ has an n-dimensional density, we have $\boldsymbol{P}(X \in B_i) = 1$. Setting

$$U = A \cap B_i \cap \{x : \det M(x) > 0\} \quad (38)$$

we have that U is open since $M(x)$ is continuous on A. Also, $M(x)$ and its square root $W(x)$ are positive-definite on U, and they are continuously differentiable there (see Lemma 1). Moreover, r is continuously differentiable and its derivative r' is nonsingular on U, and r has an inverse g with the same properties. Finally, $\boldsymbol{P}(X \in U) = 1$.

By a standard result of measure theory (see, e.g., [24]), there exist compact sets K_m, $m = 1, 2, \cdots$, such that

$$K_m \subset U \quad \text{and} \quad \lim_{m\to\infty}\boldsymbol{P}(X \in K_m) = 1.$$

Then by [24, Theorem 2.7], for each K_m there is an open set V_m with compact closure $\overline{V_m}$ such that

$$K_m \subset V_m \subset \overline{V_m} \subset U.$$

Fix $T > 0$ and define the binary random variable $B = B(D, T, m)$ by

$$B = \begin{cases}0, & \text{if } \|r(X) - Y_D\| < T \quad \text{and} \quad X \in V_m\\ 1, & \text{otherwise.}\end{cases}$$

The dependence of B on the parameters (D, T, m) will be hidden in the notation. Until the very last step in the proof, m will be fixed and the choice of T will depend only on V_m. Using B, we can upper-bound $h(r(X) \mid Y_D)$ as

$$\begin{aligned}h(r(X) \mid Y_D) &= h(r(X) \mid Y_D, B) + I(r(X); B \mid Y_D)\\ &\leq h(r(X) \mid Y_D, B = 1)\boldsymbol{P}(B = 1)\\ &\quad + h(r(X) \mid Y_D, B = 0)\boldsymbol{P}(B = 0) + H(B)\end{aligned} \quad (39)$$

where $H(B)$ is the Shannon entropy of B.

Bounding $H(B)$: We have

$$\lim_{D\to 0} \boldsymbol{P}(\|r(X)-Y_D\| \geq T, X \in V_m) = 0$$

for all $T, m > 0$ by (36). Thus

$$\lim_{D\to 0} \boldsymbol{P}(B=0) = \boldsymbol{P}(X \in V_m) \tag{40}$$

and

$$\lim_{D\to 0} H(B) = H_b(\boldsymbol{P}(X \in V_m)) \tag{41}$$

where H_b denotes the binary entropy function

$$H_b(\alpha) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha).$$

Bounding $h(r(X) \mid Y_D, B=1)\boldsymbol{P}(B=1)$: If the random vector Z has a density and $\boldsymbol{E}\|Z\|^2 < \infty$, then

$$h(Z) \leq \frac{n}{2}\log(n^{-1}2\pi e \boldsymbol{E}\|Z\|^2).$$

Since conditioning reduces differential entropy, this implies

$$\begin{aligned} h(r(X) \mid Y_D, B=1) &\leq h(r(X) \mid B=1) \\ &\leq \frac{n}{2}\log(n^{-1}2\pi e \boldsymbol{E}[\|r(X)\|^2 \mid B=1]) \\ &\leq \frac{n}{2}\log\left(n^{-1}2\pi e \frac{\boldsymbol{E}\|r(X)\|^2}{\boldsymbol{P}(B=1)}\right) \end{aligned} \tag{42}$$

and, therefore, by (40)

$$\begin{aligned} &\limsup_{D\to 0} h(r(X) \mid Y_D, B=1)\boldsymbol{P}(B=1) \\ &\quad \leq \frac{n}{2}\boldsymbol{P}(X \notin V_m)\log\left(n^{-1}2\pi e \frac{\boldsymbol{E}\|r(X)\|^2}{\boldsymbol{P}(X \notin V_m)}\right). \end{aligned} \tag{43}$$

Bounding $h(X \mid Y_D, B=0)\boldsymbol{P}(B=0)$: Let $G = r(U) = \{r(x) : x \in U\}$, where U is defined in (38), and let $g = r^{-1}$ be the inverse of r. Consider the mapping $\varphi : G \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\varphi(z,y) = W(g(z))(z-y)$$

where $W(x)$ the square root of $M(x)$. Then $g(z)$ is continuously differentiable and its derivative is nonsingular on its open domain $G = r(U)$ (G is open since r is an open mapping on U). Thus by Lemma 1, $W(g(z))$ is continuously differentiable, so $\varphi(z,y)$ is continuously differentiable with respect to z on G for all $y \in \mathbb{R}^n$. Its derivative Φ is

$$\Phi(z,y) \stackrel{\text{def}}{=} \frac{\partial}{\partial z}\varphi(z,y) = R(z,y) + W(g(z)) \tag{44}$$

where $R(z,y)$ is the $n \times n$ matrix with entries

$$r_{ik}(z,y) = \sum_{j=1}^{n}\left(\frac{\partial}{\partial z_k}w_{ij}(g(z))\right)(z_j - y_j) \tag{45}$$

where $z = (z_1, \cdots, z_n)$ and $y = (y_1, \cdots, y_n)$. Define $S_{m,T} \subset \mathbb{R}^n \times \mathbb{R}^n$ by

$$S_{m,T} = \{(z,y) : z \in r(V_m), \|z-y\| < T\} \tag{46}$$

and for any $y \in \mathbb{R}^n$ let $S_{m,T}(y) \subset \mathbb{R}^n$ be defined by

$$S_{m,T}(y) = \{z : (z,y) \in S_{m,T}\}.$$

It is proved in Appendix C that for all T small enough, and for a fixed y such that $S_{m,T}(y)$ is nonempty, $\varphi(z,y)$ is one-to-one and continuously differentiable in z on the open set $S_{m,T}(y)$. Recall now that by change of variables (see, e.g., [24, Theorem 7.26]), if $\phi(z)$ is invertible and continuously differentiable on an open set G, and if Z has a density and finite differential entropy, and $\boldsymbol{P}(Z \in G) = 1$, then

$$h(\phi(Z)) = h(Z) + \boldsymbol{E}[\log|\det \phi'(Z)|].$$

Since $\boldsymbol{P}(r(X) \in S_{m,T}(y) \mid B=0, Y_D = y) = 1$ for a.e. y conditioned on the event $\{B=0\}$, the above imply

$$\begin{aligned} &h(\varphi(r(X), Y_D) \mid B=0, Y_D=y) \\ &\quad = h(r(X) \mid B=0, Y_D=y) \\ &\qquad + \boldsymbol{E}[\log|\det \Phi(r(X), Y_D)| \mid B=0, Y_D=y] \end{aligned}$$

so that

$$\begin{aligned} &h(\varphi(r(X), Y_D) \mid B=0, Y_D) \\ &\quad = h(r(X) \mid B=0, Y_D) \\ &\qquad + \boldsymbol{E}[\log|\det\Phi(r(X), Y_D)| \mid B=0]. \end{aligned}$$

From this and the identity

$$\Phi(z,y) = W(g(z))[W^{-1}(g(z))R(z,y) + I]$$

valid for all $(z,y) \in S_{m,T}$, we obtain

$$\begin{aligned} &h(r(X) \mid Y_D, B=0) \\ &\quad = h(W(X)(r(X)-Y_D) \mid Y_D, B=0) \\ &\qquad - \frac{1}{2}\boldsymbol{E}[\log\det M(X) \mid B=0] \\ &\qquad - \boldsymbol{E}[\log|\det(W^{-1}(X)R(r(X),Y_D)+I)| \mid B=0]. \end{aligned} \tag{47}$$

Let χ_A denote the indicator function of the event A. Then by (25)

$$\begin{aligned} &\boldsymbol{E}\big[\|W(X)(r(X)-Y_D)\|^2 \chi_{\{B=0\}}\big] \\ &\qquad \leq D + \boldsymbol{E}\big[C\|r(X)-Y_D\|^3 \chi_{\{B=0\}}\big]. \end{aligned}$$

If $x \in V_m$ and $\|r(x) - y\| < T$, then

$$\begin{aligned} C\|r(x)-y\|^3 &\leq TC\|r(x)-y\|^2 \\ &\leq \beta(m,T)\|W(x)(r(x)-y)\|^2 \end{aligned}$$

where $\beta(m,T) = TC\sup_{x\in V_m}\|W^{-1}(x)\|^2 \to 0$ as $T \to 0$ for all fixed m. Thus

$$\boldsymbol{E}\big[\|W(X)(r(X)-Y_D)\|^2 \chi_{\{B=0\}}\big] \leq \frac{D}{1-\beta(m,T)} \tag{48}$$

where we have assumed that T in the definition of B is small enough to make $1 - \beta(m,T)$ positive. Using the same bound as in (42), we obtain

$$\begin{aligned} &h(W(X)(r(X)-Y_D) \mid Y_D, B=0) \\ &\qquad \leq \frac{n}{2}\log\left(n^{-1}2\pi e \frac{D}{1-\beta(m,T)}\frac{1}{\boldsymbol{P}(B=0)}\right) \end{aligned} \tag{49}$$

and, therefore, by (40)

$$\limsup_{D\to 0}\Big[h(W(X)(r(X)-Y_D)\mid Y_D, B=0)\times \boldsymbol{P}(B=0)-\frac{n}{2}\log(2\pi e D/n)\Big]$$
$$\le \limsup_{D\to 0}\Big[\Big(h(W(X)(r(X)-Y_D)\mid Y_D, B=0)-\frac{n}{2}\log(2\pi e D/n)\Big)\boldsymbol{P}(B=0)\Big]$$
$$\le -\frac{n}{2}\boldsymbol{P}(X\in V_m)\log((1-\beta(m,T))\boldsymbol{P}(X\in V_m)). \tag{50}$$

For the second term on the right-hand side of (47) we have

$$\lim_{D\to 0}\boldsymbol{E}\big[\log\det M(X)\chi_{\{B=0\}}\big] = \lim_{D\to 0}\boldsymbol{E}\left[\log\det M(X)\chi_{\{X\in V_m,\,\|r(X)-Y_D\|<T\}}\right] = \boldsymbol{E}\left[\log\det M(X)\chi_{\{X\in V_m\}}\right] \tag{51}$$

since $\log\det M(X)$ has a finite expectation and

$$\boldsymbol{P}(X\in V_m, \|r(X)-Y_D\|<T)\to\boldsymbol{P}(X\in V_m)$$

as $D\to 0$ for all $m,T>0$ by (40).

As for the last term on the right-hand side (47), the elements of $W^{-1}(g(z))R(z,y)$ tend to zero as $\|z-y\|\to 0$, uniformly on $S_{m,T}$. This follows from (45) and from the fact shown in Appendix C that $S_{m,T}$ has a compact closure which is contained in $G\times G$ for T small enough. Thus for such T, $\log|\det(W^{-1}(g(z))R(z,y)+I)|$ is bounded on $S_{m,T}$ and

$$\lim_{\|z-y\|\to 0}\log|\det(W^{-1}(g(z))R(z,y)+I)|=0$$

uniformly on $S_{m,T}$. It follows that $Y_D\to r(X)$ in probability implies

$$\lim_{D\to 0}\boldsymbol{E}\left[\log|\det(W^{-1}(X)R(r(X),Y_D)+I)|\chi_{\{B=0\}}\right]=0. \tag{52}$$

In summary, (47), (50), (51), and (52) show that

$$\limsup_{D\to 0}\Big[h(r(X)\mid Y_D, B=0)\boldsymbol{P}(B=0)-\frac{n}{2}\log(2\pi e D/n)\Big]$$
$$\le -\frac{1}{2}\boldsymbol{E}[\log\det M(X)\mid X\in V_m]\boldsymbol{P}(X\in V_m) - \frac{n}{2}\boldsymbol{P}(X\in V_m)\log((1-\beta(m,T))\boldsymbol{P}(X\in V_m)).$$

Combining this with (39) and (43) we obtain

$$\limsup_{D\to 0}\Big[h(r(X)\mid Y_D)-\frac{n}{2}\log(2\pi e D/n)\Big]$$
$$\le -\frac{1}{2}\boldsymbol{E}[\log\det M(X)\mid X\in V_m]\boldsymbol{P}(X\in V_m) + \frac{n}{2}\boldsymbol{P}(X\notin V_m)\log\left(n^{-1}2\pi e\frac{\boldsymbol{E}\|r(X)\|^2}{\boldsymbol{P}(X\notin V_m)}\right) - \frac{n}{2}\boldsymbol{P}(X\in V_m)\log((1-\beta(m,T))\boldsymbol{P}(X\in V_m)) + H_b(\boldsymbol{P}(X\in V_m)).$$

Note that the above inequality holds for all fixed m and $T=T(m)$ small enough. Since $\beta(m,T)\to 0$ as $T\to 0$ for all m, and since $\boldsymbol{P}(X\in V_m)\to 1$ as $m\to\infty$, by letting $T\to 0$ first and then $m\to\infty$ we finally obtain

$$\limsup_{D\to 0}\Big[h(r(X)\mid Y_D)-\frac{n}{2}\log(2\pi e D/n)\Big] \le -\frac{1}{2}\boldsymbol{E}[\log\det M(X)]. \quad \square$$

III. High-Resolution Remote Source Coding

In the remote source-coding problem (also called noisy source coding) [16], [3], [25] the source U is corrupted by noise before the encoding operation. Thus the encoder has access only to the output X of a noisy channel whose input is U. In this context, U and X are usually called the clean and the noisy source, respectively. We assume that the decoder operates on the noise-free output of the encoder and that U and X are both n-dimensional random vectors. The channel connecting U to X is characterized by the conditional probability distribution $P_{X|U}$. For the memoryless case, the operational rate-distortion function is defined as follows. Let $(U_1,X_1),(U_2,X_2),\cdots$ be an independent and identically distributed (i.i.d.) sequence, each pair having the same joint distribution as (U,X). Let $(\hat{U}_1,\cdots,\hat{U}_k)=g(X_1,\cdots,X_k)$ be the output of the decoder for blocklength k. The fidelity of the reproduction is measured relative to the clean source as

$$D=\boldsymbol{E}\left[\frac{1}{k}\sum_{i=1}^{k}\rho(U_i,\hat{U}_i)\right]$$

where $\rho:\mathbb{R}^n\times\mathbb{R}^n\to[0,\infty)$ is a distortion measure between n-vectors. The rate of this encoding scheme is given by the logarithm of the number of output values of g normalized by the blocklength k. Let $R(D,\rho,U,P_{X|U})$ be the OPTA of this scheme defined in the usual sense of lossy source coding, i.e., $R(D,\rho,U,P_{X|U})$ is the minimum rate asymptotically (as $k\to\infty$) achievable at distortion level $D>0$. Thus $R(D,\rho,U,P_{X|U})$ depends on the distribution of U, on the channel connecting X with U, and on the original distortion measure ρ. It is a classical result of rate-distortion theory [3] that

$$R(D,\rho,U,P_{X|U})=R(D,d,X) \tag{53}$$

where $R(D,d,X)$ is the ordinary rate-distortion function of the source X with respect to a modified distortion measure d defined by the conditional expectation

$$d(x,y)=\boldsymbol{E}[\rho(U,y)\mid X=x]. \tag{54}$$

The modified distortion measure d is in general not a difference distortion measure even if ρ is, so that our Theorem 1 is a natural tool to evaluate $R(D,\rho,U,P_{X|U})$ for high resolution. Note that $\boldsymbol{E}[\inf_y d(X,y)]=D_{\min}>0$ in all nontrivial cases. Let $a(D)\approx b(D)$ mean $\lim_{D\to 0}(a(D)-b(D))=0$. Then by Theorem 1 and (53) we can formally write

$$R(D+D_{\min},\rho,U,P_{X|U}) \approx h(r(X))+\frac{1}{2}\boldsymbol{E}[\log\det M(X)]-\frac{n}{2}\log(2\pi e D/n) \tag{55}$$

where

$$r(x) = \arg\min_y \boldsymbol{E}[\rho(U,y) \mid X = x],$$
$$D_{\min} = \boldsymbol{E}[\rho(U, r(X))],$$

and

$$M(x) = \frac{\partial^2}{\partial y^2}\boldsymbol{E}[\rho(U,y) \mid X = x]|_{y=r(x)}$$

provided the above quantities are well defined and satisfy conditions a)–g).

As it turns out, the condition that r is piecewise-invertible and $r(X)$ has a finite differential entropy is the hardest to check. Indeed, if ρ itself satisfies conditions a) and c), then so does d, as one can see using standard arguments for exchanging the order of differentiation and integration. If, furthermore, $\rho(u,y)$ is assumed to be strictly convex in y for all u, then $d(x,y)$ has a unique minimum in y for all x by [26, Lemma A], and thus $r(x)$ is well defined. Also, if the conditional density $f_{U|X}(u \mid x)$ exists and satisfies appropriate differentiability and boundedness conditions, then $r(x)$ and $M(x)$ are easily seen to be continuously differentiable. The integrability conditions e)–g) can be taken care of by the convexity assumption and by assuming that the distribution of X has a light enough tail. Unfortunately, these quite natural conditions do not imply that r is piecewise-invertible. In what follows we show a specific class of original distortion measures ρ and clean-noisy source pairs for which we can check exactly all conditions of Theorem 1. To keep the discussion simple, we consider the scalar case $n = 1$.

Let the scalar WMSE distortion measure (10) be defined by

$$\rho(u,y) = \hat{m}(u)(u-y)^2$$

where $\hat{m}$ is a positive function,

One can easily show that in this case (see [27]) that

$$d(x,y) = d_{\min}(x) + m(x)(y - r(x))^2 \tag{56}$$

where

$$m(x) = \boldsymbol{E}[\hat{m}(U) \mid X = x],$$
$$r(x) = \frac{1}{m(x)}\boldsymbol{E}[U\hat{m}(U) \mid X = x], \tag{57}$$

and

$$d_{\min}(x) = \boldsymbol{E}[\hat{m}(U)(U - r(X))^2 \mid X = x]$$

assuming all these quantities are well defined. If $\hat{m}(u) = 1$, we get back the well-known decomposition result [28], [17] for remote coding with mean-square original distortion

$$d(x,y) = d_{\min}(x) + (y - r(x))^2$$

where $r(x)$ is the regression function $\boldsymbol{E}[U \mid X = x]$, which leads to the "Wolf–Ziv encoder" discussed in the Introduction. If U is Gaussian and $P_{X|U}$ is an additive Gaussian channel, we obtain the following.

Theorem 2: Let $U \sim N(0, \sigma_U^2)$ and let $X = U + Z$, where $Z \sim N(0, \sigma_Z^2)$ is independent of U. Assume that

$$\rho(u,y) = \hat{m}(u)(u-y)^2$$

where $\hat{m}(u) > 0$ is bounded, symmetric to zero, monotone decreasing in $[0, \infty)$

$$\lim_{u\to\infty} \frac{\hat{m}(u+h)}{\hat{m}(u)} = 1, \qquad \text{for all } h > 0 \tag{58}$$

and

$$\limsup_{u\to\infty} \frac{\hat{m}(u/2)}{\hat{m}(u)} < \infty. \tag{59}$$

Then

$$R(D + D_{\min}, \rho, U, P_{X|U}) \approx h(r(X)) + \frac{1}{2}\boldsymbol{E}[\log m(X)] - \frac{1}{2}\log(2\pi e D)$$

where m and r are defined in (57) above, and

$$D_{\min} = \boldsymbol{E}[\hat{m}(U)(U - r(X))^2].$$

Remark: Note that (58) and (59) are satisfied, for example, if there exists $p > 0$ such that $\hat{m}(u)u^p$ tends to a finite limit as $u \to \infty$. Thus the above result holds for

$$\hat{m}(u) = \frac{1}{c + |u|^p}$$

if $c, p > 0$. In general, (58) and (59) hold if $\hat{m}$ is of regular variation, i.e., for some finite positive function $\alpha(t)$, we have $\hat{m}(tu)/\hat{m}(u) \to \alpha(t)$ as $u \to \infty$ for all $t > 0$. In this case, there exists $\beta > 0$ such that $\alpha(t) = t^{-\beta}$ (see [29]) and thus (58) and (59) are easy to obtain.

Proof of Theorem 2: Let $\varphi_\sigma(u) = (\sqrt{2\pi}\sigma)^{-1}e^{-u^2/(2\sigma^2)}$. Then the conditional density of U given $X = x$ is $\varphi(u - ax)$, where $\sigma = (1/\sigma_Z^2 + 1/\sigma_U^2)^{-1}$ and $a = (1 + \sigma_Z^2/\sigma_U^2)^{-1}$. Since $\hat{m}$ is bounded

$$m(x) = \int \hat{m}(u)\varphi(u - ax)\,du$$

and

$$r(x) = \frac{1}{m(x)}\int u\hat{m}(u)\varphi(u - ax)\,du$$

are finite for all x. By the form (56) of $d(x,y)$, conditions a) and c) are satisfied. Clearly, $m(x)$ corresponds to $M(x)$ of Theorem 1. Since $\hat{m}$ is uniformly bounded, an application of the dominated convergence theorem shows that m is continuously differentiable, and thus d) holds. To see that $\boldsymbol{E}|\log m(X)|$ is finite, we note that (58) readily implies that there exists $\alpha > 0$ such that $\hat{m}(u) > e^{-\alpha|u|}$ for $|u| > 0$ large enough. This implies

$$m(x) = \int \hat{m}(v + ax)\varphi_\sigma(v)\,dv > ce^{-\alpha a|x|}$$

for some $c > 0$ if $|x|$ is large enough. Since X is Gaussian, this implies

$$\boldsymbol{E}|\log m(X)| < \infty$$

and

$$\boldsymbol{E}\left(\frac{1}{m(X)^{3/2}}\right) < \infty$$

hence e) and g) are satisfied. It remains to prove that r is piecewise-invertible and continuously differentiable and that f) holds. In Appendix D we show that $r'(x)$ is continuous, positive for all x, and

$$\lim_{|x|\to\infty} r'(x) = a.$$

Thus r is invertible. Since r' is continuous, it is bounded and bounded away from zero. Therefore, $\boldsymbol{E}\,|\log r'(X)|$ is finite and thus $h(r(X))$ is finite. Since $\boldsymbol{E}[r(X)^2] < \infty$ trivially follows, condition f) holds and the proof is complete. □

IV. Concluding Remarks

We have studied the asymptotic (low-distortion) behavior of the rate-distortion function for a class of non-difference distortion measures. Our main result (Theorem 1) presents an asymptotically tight expression which can play the same very useful role in high-resolution source-coding theory for non-difference distortion measures as does the Shannon lower bound for difference distortion measures. Some applications of this result are dealt with informally in Section I. These include the study of structure of the optimal forward channel realizing the rate-distortion function at high resolution, the computation of the (high-resolution) rate-distortion function in the remote source-coding problem, and high-resolution quantization theory for non-difference distortion measures. Some aspects of these implications for remote source coding are explored rigorously in Section III. A full and rigorous treatment of the application of the main result to high-resolution vector quantization is given in [15].

Further research can be done for relaxing the conditions on the class of distortion measures considered here. Namely, analogous expressions are likely to hold for distortion measures for which $r(x) = \arg\min_y d(x,y)$ is not piecewise-invertible. Furthermore, the case when x and y have different dimensions (as can be the case for remote source coding) needs also study. Another nontrivial problem is to consider distortion measures for which the quadratic behavior $d(x,y) \approx d_{\min}(x) + (y - r(x))^t M(x)(y - r(x))$ is not dominant for "almost all" x, i.e., when $M(X)$ is singular with positive probability. For example, consider the relatively simple scalar case where $d(x,y) = 0$ iff $x = y$ and for some $r > 0$, the limit

$$\lim_{y\to x} \frac{d(x,y)}{|x-y|^r} = m(x)$$

is finite and positive for all x. Then, assuming some technical conditions hold, using a modification of the proof of the main result it is not hard to show that

$$R(D) \approx h(X) + \frac{1}{r}\boldsymbol{E}\left[\log m(X)\right] - \frac{1}{r}\log(reD) - \log(2\Gamma(1+1/r))$$

where Γ is the gamma function. This formula generalizes Gray *et al.*'s expression [11] for the SLB for the rth-power distortion measure.

An interesting question is to find a usable *lower* bound on the rate-distortion function for non-difference distortion measures. Note that we did not claim

$$\frac{n}{2}\log(2\pi eD/n) + h(r(X)) + \frac{1}{2}\boldsymbol{E}\left[\log\det M(X)\right]$$

to be a lower bound on $R(D)$ for any $D > 0$. In this direction, it is not hard to prove (following the proof of Proposition 2) that for the scalar case and WMSE distortion measure with a "nice" weighting function $w(x)$, the difference between the asymptotic expression and $R(D)$ is upper-bounded by $O(\sqrt{D})$ as $D \to 0$.

Appendix A

Proof of Lemma 1: Let $U = A \cap \{x : |\det M(x)| > 0\}$. Then U is open since $M(x)$ is continuous on A. Consider the power series expansion

$$\sqrt{z} = \sqrt{1+(z-1)} = \sum_{i=0}^{\infty} a_i (z-1)^i, \qquad |z-1| < 1. \tag{A.1}$$

Recall that $M(x)$ is positive definite for all $x \in U$. Fix an arbitrary $x_0 \in U$ and let $c > 0$ be small enough such that the largest eigenvalue of $cM(x_0)$ is less than 1, so that $(I - cM(x_0))$ is positive-definite and $\|I - cM(x)\| < 1$ in an open ball $V \subset U$ centered at x_0. Then for all $x \in V$

$$(cM(x))^{1/2} = (I + (cM(x) - I))^{1/2} = \sum_{i=0}^{\infty} a_i (cM(x) - I)^i \tag{A.2}$$

where the a_i are the same as in (A.1), since the power series in (A.1) converges absolutely for all $|z-1| < 1$. Moreover, the convergence in (A.2) is uniform in any closed ball $K \subset V$ centered at x_0. We will prove that the elements of the matrix-valued function defined by the above power series have continuous partial derivatives. Without loss of generality we can assume that $c = 1$. For any matrix-valued function $A(x) = \{a_{ij}(x)\}$ with differentiable entries let

$$\frac{\partial}{\partial x_k}A(x) = \left\{\frac{\partial}{\partial x_k}a_{ij}(x)\right\}.$$

The elements of $(M(x) - I)^i$ are differentiable by assumption and it is easy to see that

$$\left\|\frac{\partial}{\partial x_k}(M(x)-I)^i\right\| \le i\left\|\frac{\partial}{\partial x_k}M(x)\right\| \cdot \|M(x)-I\|^{i-1}.$$

It follows that $\sum_{i=0}^{\infty} a_i \frac{\partial}{\partial x_k}(M(x)-I)^i$ converges absolutely in V and uniformly in any closed ball $K \subset V$ centered x_0. Thus the same argument which proves the term-by-term differentiability of real power series applies and we obtain

$$\frac{\partial}{\partial x_k}W(x) = \sum_{i=0}^{\infty} a_i \frac{\partial}{\partial x_k}(M(x)-I)^i$$

at $x = x_0$, for all $1 \le k \le n$. The continuity of $\frac{\partial}{\partial x_k}W(x)$ at x_0 is obvious since it is the limit of a uniformly convergent series of continuous functions. □

APPENDIX B

Proof that $\limsup_{D\to 0} h(r(X)+W^{-1}(X)Z_D) \le h(r(X))$*:* Recall that we have defined $W^{-1}(x) = M^{-1}(x) = 0$ on $\{x : \det M(x) = 0\}$. Define

$$\Delta(D) = \boldsymbol{E}\|r(X) + W^{-1}(X)Z_D\|^2 \text{ and } \Delta(0) = \boldsymbol{E}\|r(X)\|^2.$$

Then, by the independence of X and Z_D

$$\Delta(D) = \boldsymbol{E}\|r(X)\|^2 + \frac{D}{n}\boldsymbol{E}[\mathrm{tr}\,\{M^{-1}(X)\}]$$

where $\mathrm{tr}\,\{A\}$ denotes the trace of the matrix A. $\Delta(D)$ is finite for all D by conditions f) and g). Thus $\Delta(D)$ is continuous at 0. Let $Z_{\Delta(D)} \sim N(0, n^{-1}\Delta(D)I)$. Then $\boldsymbol{E}\|Z_{\Delta(D)}\|^2 = \Delta(D)$. If U and V are n-dimensional random vectors let $\boldsymbol{D}(U\|V)$ denote their information divergence (relative entropy). By a well-known identity for normal distributions and information divergence (see, e.g., [21]), for all $D \ge 0$, we have

$$\begin{aligned} h(r(X) + W^{-1}(X)Z_D) - h\big(Z_{\Delta(D)}\big) \\ = -\boldsymbol{D}\big(r(X) + W^{-1}(X)Z_D\|Z_{\Delta(D)}\big). \end{aligned}$$

Therefore,

$$\begin{aligned} h(r(X) + W^{-1}(X)Z_D) - h(r(X)) \\ = \boldsymbol{D}\big(r(X)\|Z_{\Delta(0)}\big) - \boldsymbol{D}(r(X) + W^{-1}(X)Z_D\|Z_{\Delta(D)}) \\ + h\big(Z_{\Delta(D)}\big) - h\big(Z_{\Delta(0)}\big). \end{aligned} \tag{B.1}$$

Now $r(X) + W^{-1}(X)Z_D \to r(X)$ in distribution since $\boldsymbol{E}\|W^{-1}(X)Z_D\|^2 \to 0$ as $D \to 0$. Also, we have by the continuity of $\Delta(D)$ at zero that $Z_{\Delta(D)} \to Z_{\Delta(0)}$ in distribution. Thus the lower semicontinuity property of the divergence [23] implies that

$$\liminf_{D\to 0} \boldsymbol{D}\big(r(X) + W^{-1}(X)Z_D\|Z_{\Delta(D)}\big) \ge \boldsymbol{D}\big(r(X)\|Z_{\Delta(0)}\big).$$

Since $h(Z_{\Delta(D)}) \to h(Z_{\Delta(0)})$, we obtain

$$\limsup_{D\to 0} h(r(X) + W^{-1}(X)Z_D) \le h(r(X))$$

which was to be proved. □

Proof That $\delta(D)=D+o(D)$*:* Since we assume $d(x, r(x)) = 0$, by (25)

$$|d(x,y) - \|W(x)(r(x) - y)\|^2| \le C\|r(x) - y\|^3.$$

Let $Z \sim N(0, n^{-1}I)$ be independent of X. Then

$$\begin{aligned} |\delta(D) - D| &= |\boldsymbol{E}d(X, Y_D) - \boldsymbol{E}\|W(X)(r(X) - Y_D)\|^2| \\ &\le C\boldsymbol{E}\|r(X) - Y_D\|^3 \\ &= C\boldsymbol{E}\|W^{-1}(X)Z_D\|^3 \\ &= CD^{3/2}\boldsymbol{E}\|W^{-1}(X)Z\|^3 \\ &\le CD^{3/2}\boldsymbol{E}[\|W^{-1}(X)\|^3\|Z\|^3] \\ &= CD^{3/2}\boldsymbol{E}\|W^{-1}(X)\|^3\boldsymbol{E}\|Z\|^3. \end{aligned}$$

Let $\hat{w}_{ij}(x)$ denote the elements of $W^{-1}(x)$. By the well-known bound on the matrix norm

$$\|W^{-1}(x)\| \le \left(\sum_{i,j}\hat{w}_{ij}(x)^2\right)^{1/2} = \left(\sum_i \hat{m}_{ii}(x)\right)^{1/2}$$

where $M^{-1}(x) = \{\hat{m}_{ij}(x)\}$. Thus

$$|\delta(D) - D| \le CD^{3/2}\boldsymbol{E}\,[(\mathrm{tr}\,\{M^{-1}(X)\})^{3/2}]\boldsymbol{E}\|Z\|^3$$

which proves the statement since $\boldsymbol{E}\,[(\mathrm{tr}\,\{M^{-1}(X)\})^{3/2}]$ is finite by assumption.

Proof That $\delta(D)$ *Is Continuous:* Fix $D_0 > 0$. Then $Y_D = r(X) + \sqrt{D}W^{-1}(X)Z \to Y_{D_0}$ almost surely as $D \to D_0$. Since $d(x,y)$ is continuous in y, $d(X, Y_D) \to d(X, Y_{D_0})$ a.s. The continuity of $\boldsymbol{E}d(X, Y_D)$ now follows from the dominated convergence theorem since

$$d(X, Y_D) \le D\|Z\|^2 + CD^{3/2}(\mathrm{tr}\,\{M^{-1}(X)\})^{3/2}\|Z\|^3$$

almost surely, where the right-hand side has a finite expectation. □

APPENDIX C

Proof That $Y_D \to r(X)$ *in Probability as* $D \to 0$*:* For any $\epsilon > 0$ define

$$g_\epsilon(x) = \inf_{y:\|y - r(x)\| > \epsilon} d(x,y).$$

Since $d(x,y)$ is continuous in y by condition a), it follows from conditions b) and c) that $g_\epsilon(x) > 0$ for all $x \in \mathbb{R}^n$ and $\epsilon > 0$. Let χ_A denote the indicator function of the event A. Since

$$\begin{aligned} \boldsymbol{E}\big[d(X, Y_D)\chi_{\{\|r(X) - Y_D\| > \epsilon\}} \mid X = x\big] \\ \ge g_\epsilon(x)\boldsymbol{P}(\|X - Y_D\| > \epsilon \mid X = x) \end{aligned}$$

denoting $\boldsymbol{P}(\|X - Y_D\| > \epsilon | X = x) = h_{D,\epsilon}(x)$, we have

$$D \ge \int g_\epsilon(x)h_{D,\epsilon}(x)\mu(dx) \tag{C.1}$$

where μ denotes the probability measure induced by X. For all $\delta > 0$

$$\begin{aligned} \int h_{D,\epsilon}(x)\mu(dx) \\ = \int_{\{g_\epsilon(x)\le\delta\}} h_{D,\epsilon}(x)\mu(dx) + \int_{\{g_\epsilon(x)>\delta\}} h_{D,\epsilon}(x)\mu(dx) \\ \le \mu(\{g_\epsilon(x) \le \delta\}) + \frac{1}{\delta}\int g_\epsilon(x)h_{D,\epsilon}(x)\mu(dx) \end{aligned}$$

and, therefore, by (C.1)

$$\limsup_{D\to 0}\int h_{D,\epsilon}(x)\mu(dx) \le \mu(\{g_\epsilon(x) \le \delta\}).$$

Since $g_\epsilon(x) > 0$ for all x, we have $\lim_{\delta\to 0}\mu(\{g_\epsilon(x) \le \delta\}) = 0$ and thus

$$\lim_{D\to 0}\boldsymbol{P}(\|X - Y_D\| > \epsilon) = \lim_{D\to 0}\int h_{D,\epsilon}(x)\mu(dx) = 0$$

for all $\epsilon > 0$. □

Proof That $\varphi(z,y)$ Is Invertible in z on $S_{m,T}(y)$: We have

$$\varphi(z,y) = W(g(z))(z-y)$$

for all $z \in G = r(U)$ and $y \in \mathbb{R}^n$, where U is defined in (38) and $g = r^{-1}$ is continuously differentiable and invertible on the open set G and has nonsingular derivative there. By Lemma 1, $W(x)$ is nonsingular and it is continuously differentiable on U. We obtain that

$$\Phi(z,y) \stackrel{\text{def}}{=} \frac{\partial}{\partial z}\varphi(z,y) = R(z,y) + W(g(z))$$

where $R(x,y) = \{r_{ik}(x,y)\}$ and

$$r_{ik}(z,y) = \sum_{j=1}^{n}\left(\frac{\partial}{\partial z_k} w_{ij}(g(z))\right)(z_j - y_j).$$

Note that $\Phi(y,y) = W(g(y))$ is nonsingular if $y \in G$. Let $\phi : G \to \mathbb{R}^n$ be a continuously differentiable mapping with derivative $\phi'(z)$, and assume that $A = \phi'(z_0)$ is nonsingular for a given $z_0 \in G$. Then by the proof of the inverse function theorem (see [30]), if

$$||\phi'(z) - A|| \le \frac{1}{2||A^{-1}||}$$

on an open ball centered at z_0, then $\phi(z)$ is invertible on this ball. Fix $y \in G$. It follows that $\varphi(z,y)$ is an invertible function of z on an open ball centered at y and contained in G, if on this ball

$$||\Phi(z,y) - \Phi(y,y)|| \le \frac{1}{2||\Phi^{-1}(y,y)||}$$

or

$$||\Phi(z,y) - \Phi(y,y)|| \le \frac{1}{2||W^{-1}(g(y))||}. \tag{C.2}$$

Recall that $V_m \subset U$ is an open set such that its closure is compact and $\overline{V_m} \subset U$. Let $U_m = r(V_m) = \{z : g(z) \in V_m\}$. Then $\overline{U_m} \subset G$ is compact. For any $E, E' \subset \mathbb{R}^n$ and $z \in \mathbb{R}^n$ let

$$\rho(E,z) = \inf_{x \in E} ||x - z||$$

and

$$\rho(E,E') = \inf_{z \in E'} \rho(E,z).$$

Since G^c is closed, it follows that $\rho(\overline{U_m}, G^c) = \delta > 0$. Let

$$U_{m,\epsilon} = \{z : \rho(U_m, z) < \epsilon\}.$$

Then $U_{m,\epsilon}$ is open and $\overline{U_{m,\epsilon}} \subset G$ if $\epsilon < \delta$. Moreover, $\overline{U_{m,\epsilon}}$ is compact for all ϵ. For $T < \delta/4$ define

$$\hat{S}_{m,T} = \{(z,y) : y \in U_{m,\delta/4}, ||y - z|| < T\}.$$

Since $||W^{-1}(g(z))||$ is continuous on the compact set $\overline{U_{m,\delta/4}}$, we have

$$\sup_{y \in \overline{U_{m,\delta/4}}} ||W^{-1}(g(y))|| < \infty. \tag{C.3}$$

On the other hand, $\Phi(z,y)$ is (uniformly) continuous on the compact closure of $\hat{S}_{m,T}$ which is contained in $G \times G$, since $T < \delta/4$. It follows that for $T > 0$ small enough (C.2) is satisfied on $\hat{S}_{m,T}$. By definition, the set

$$\hat{S}_{m,T}(y) = \{z : (z,y) \in \hat{S}_{m,T}\}$$

is the open ball $\{z : ||z - y|| < T\}$ for all $y \in \overline{U_{m,\delta/4}}$, and $\varphi(z,y)$ is an invertible and continuously differentiable function of z in $\hat{S}_{m,T}(y)$. Since $S_{m,T} \subset \hat{S}_{m,T}$, it follows that for each fixed y, the function $\varphi(z,y)$ is invertible and continuously differentiable as a function of z on the open set

$$S_{m,T}(y) = \{z : (z,y) \in S_{m,T}\}$$

for any y such that $S_{m,T}(y)$ is nonempty. □

Appendix D

Properties of $r'(x)$ in Theorem 2: We have

$$r(x) = \frac{\int u\hat{m}(u)\varphi_\sigma(u - ax)\,du}{\int \hat{m}(u)\varphi_\sigma(u - ax)\,du}.$$

Since $\hat{m}$ is bounded, we can exchange the order of integration and differentiation with respect to x by standard arguments. After some calculus we obtain that the derivative of $r(x/a)$ is given by

$$(r(x/a))' = \frac{\int u^2\hat{m}(u)\varphi_\sigma(u-x)\,du}{\sigma^2\int \hat{m}(u)\varphi_\sigma(u-x)\,du} - \left(\frac{\int u\hat{m}(u)\varphi_\sigma(u-x)\,du}{\sigma^2\int \hat{m}(u)\varphi_\sigma(u-x)\,du}\right)^2.$$

Clearly, $r'(x)$ is continuous. By the change of variable $v = u - x$

$$\begin{aligned}(r(x/a))' &= \frac{\int (v+u)^2\hat{m}(v+u)\varphi_\sigma(v)\,dv}{\sigma^2\int \hat{m}(v+u)\varphi_\sigma(v)\,dv} - \left(\frac{\int (v+u)\hat{m}(v+u)\varphi_\sigma(v)\,dv}{\sigma^2\int \hat{m}(v+u)\varphi_\sigma(v)\,dv}\right)^2 \\ &= \frac{\mathrm{Var}\,(V_x)}{\sigma^2} > 0\end{aligned} \tag{D.1}$$

where $\mathrm{Var}\,(V_x)$ denotes the variance of the random variable V_x whose density is

$$f_x(v) = \frac{\hat{m}(v+x)\varphi_\sigma(v)}{\int \hat{m}(u+x)\varphi_\sigma(u)\,du}.$$

Since $\lim_{x\to\infty} \hat{m}(u+x)/\hat{m}(x) = 1$ for all u by (58), we have that

$$\lim_{x\to\infty}\int \frac{\hat{m}(u+x)}{\hat{m}(x)}\varphi_\sigma(u)\,du = 1$$

if we can prove that

$$\lim_{x\to\infty}\int \frac{\hat{m}(u+x)}{\hat{m}(x)}\varphi_\sigma(u)\,du = \int \left(\lim_{x\to\infty}\frac{\hat{m}(u+x)}{\hat{m}(x)}\right)\varphi_\sigma(u)\,du. \tag{D.2}$$

Let $x > 0$ and $M = \hat{m}(0) = \max_u \hat{m}(u)$. If $-\infty < u \le -x/2$, then

$$\frac{\hat{m}(u+x)}{\hat{m}(x)} \le \frac{M}{\hat{m}(x)} \le \frac{M}{\hat{m}(2u)} \tag{D.3}$$

since $\hat{m}(u)$ is monotone decreasing on $[0, \infty)$. On the other hand, if $-x/2 < u < \infty$, then for all x large enough we have by (59) that

$$\frac{\hat{m}(u+x)}{\hat{m}(x)} \le \frac{\hat{m}(x/2)}{\hat{m}(x)} \le c \tag{D.4}$$

for some finite c. As we have observed before, (58) implies that there exists $\alpha > 0$ such that $\hat{m}(u) > e^{-\alpha|u|}$ if $|u| > 0$ is large enough. Thus $\hat{m}(2u)^{-1} < e^{\alpha 2|u|}$ and we obtain from (D.3) and (D.4) that there exists $g(u)$ such that for all $x > 0$ large enough and for all u

$$\frac{\hat{m}(u+x)}{\hat{m}(x)} \le g(u)$$

and

$$\int g(u)\varphi_\sigma(u)\,du < \infty.$$

Thus (D.2) holds by the dominated convergence theorem. It can be shown the same way that

$$\lim_{x\to\infty} \int u \frac{\hat{m}(u+x)}{\hat{m}(x)} \varphi_\sigma(u)\,du = \lim_{x\to\infty} \int u\varphi_\sigma(u)\,du = 0$$

and

$$\lim_{x\to\infty} \int u^2 \frac{\hat{m}(u+x)}{\hat{m}(x)} \varphi_\sigma(u)\,du = \lim_{x\to\infty} \int u^2\varphi_\sigma(u)\,du = \sigma^2.$$

Since the above holds if $x \to -\infty$ also by symmetry, we conclude from (D.1) that

$$\lim_{|x|\to\infty} r'(x) = \lim_{|x|\to\infty} \frac{a\,\mathrm{Var}\,(V_x)}{\sigma^2} = a. \qquad \square$$

Acknowledgment

The authors wish to thank an anonymous reviewer for pointing out a simpler proof (in Section II-B) of the asymptotic form of the rate-distortion function for stationary sources.

References

[1] N. S. Jayant and P. Noll, *Digital Coding of Waveforms*. Englewood Cliffs, NJ: Prentice-Hall, 1984.
[2] W. R. Gardner and B. D. Rao, "Theoretical analysis of the high-rate vector quantization of LPC parameters," *IEEE Trans. Speech, Audio Processing*, vol. 3, pp. 367–381, Sept. 1995.
[3] T. Berger, *Rate Distortion Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1971.
[4] H. Gish and J. N. Pierce, "Asymptotically efficient quantizing," *IEEE Trans. Inform. Theory*, vol. IT-14, pp. 676–683, Sept. 1968.
[5] A. Gersho, "Asymptotically optimal block quantization," *IEEE Trans. Inform. Theory*, vol. IT-25, pp. 373–380, July 1979.
[6] R. Zamir and M. Feder, "On universal quantization by randomized uniform/lattice quantizers," *IEEE Trans. Inform. Theory*, vol. 38, pp. 428–436, Mar. 1992.
[7] T. Linder and K. Zeger, "Asymptotic entropy constrained performance of tessellating and universal randomized lattice quantization," *IEEE Trans. Inform. Theory*, vol. 40, pp. 575–579, Mar. 1994.
[8] T. Linder and R. Zamir, "On the asymptotic tightness of the Shannon lower bound," *IEEE Trans. Inform. Theory*, vol. 40, pp. 2026–2031, Nov. 1994.
[9] Y. N. Linkov, "Evaluation of epsilon entropy of random variables for small epsilon," *Probl. Inform. Transm.*, vol. 1, pp. 12–18, 1965. Translated from *Probl. Pered. Inform.*, vol. 1, pp. 18–28.
[10] P. Zador, "Asymptotic quantization error of continuous signals and the quantization dimension," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 139–149, Mar. 1982.
[11] Y. Yamada, S. Tazaki, and R. M. Gray, "Asymptotic performance of block quantizers with difference distortion measures," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 6–14, Jan. 1980.
[12] J. A. Bucklew and G. L. Wise, "Multidimensional asymptotic quantization theory with rth power distortion measures," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 239–247, Mar. 1982.
[13] S. Na and D. L. Neuhoff, "Bennett's integral for vector quantizers," *IEEE Trans. Inform. Theory*, vol. 41, pp. 886–900, July 1995.
[14] J. Li, N. Chaddha, and R. M. Gray, "Asymptotic performance of vector quantizers with the perceptual distortion measure," preprint, also in *Proc. IEEE Int. Symp. Information Theory* (Ulm, Germany, June 1997), p. 55.
[15] T. Linder, R. Zamir, and K. Zeger, "High-resolution source coding for non-difference distortion measures: Multidimensional companding," this issue, pp. 548–561.
[16] R. L. Dobrushin and B. S. Tsybakov, "Information transmission with additional noise," *IRE Trans. Inform. Theory*, vol. IT-18, pp. 293–304, 1962.
[17] J. K. Wolf and J. Ziv, "Transmission of noisy information to a noisy receiver with minimum distortion," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 406–411, July 1970.
[18] R. Zamir and M. Feder, "Rate distortion performance in coding band-limited sources by sampling and dithered quantization," *IEEE Trans. Inform. Theory*, vol. 42, pp. 141–154, Jan. 1995.
[19] A. Gersho and R. M. Gray, *Vector Quantization and Signal Compression*. Boston, MA: Kluwer, 1992.
[20] A. D. Wyner and J. Ziv, "Bounds on the rate-distortion function for stationary sources with memory," *IEEE Trans. Inform. Theory*, vol. IT-17, pp. 508–513, Sept. 1971.
[21] T. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.
[22] C. E. Shannon, "A mathematical theory of communication," *Bell Syst. Tech. J.*, vol. 27, pp. 379–423, 1948.
[23] I. Csiszár, "On an extremum problem of information theory," *Studia Scient. Math. Hungarica*, pp. 57–70, 1974.
[24] W. Rudin, *Real and Complex Analysis*, 3rd ed. New York: McGraw-Hill, 1987.
[25] H. S. Witsenhausen, "Indirect rate-distortion problems," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 518–521, Sept. 1980.
[26] R. M. Gray, J. C. Kieffer, and Y. Linde, "Locally optimum block quantizer design," *Inform. Contr.*, vol. 45, pp. 178–198, 1980.
[27] Y. Ephraim and R. M. Gray, "A unified approach for encoding clean and noisy sources by means of waveform and autoregressive model vector quantization," *IEEE Trans. Inform. Theory*, vol. 34, pp. 826–834, July 1988.
[28] D. J. Sakrison, "A geometric treatment of the source encoding of a Gaussian random variable," *IEEE Trans. Inform. Theory*, vol. IT-14, pp. 481–486, May 1968.
[29] W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. II. New York: Wiley, 1966.
[30] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed. New York: McGraw-Hill, 1976.