Week #3 - Exponential Functions and Logarithms; The Derivative

Section 2.2

SUGGESTED PROBLEMS

1. The table shows values of \( f(x) = x^3 \) near \( x = 2 \) (to three decimal places). Use it to estimate \( f'(2) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.998</th>
<th>1.999</th>
<th>2.000</th>
<th>2.001</th>
<th>2.002</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^3 )</td>
<td>7.976</td>
<td>7.988</td>
<td>8.000</td>
<td>8.012</td>
<td>8.024</td>
</tr>
</tbody>
</table>

The estimate will be most accurate if we use two points closest to \( x = 2 \). For example:

\[
f'(2) \approx \frac{f(2.001) - f(2)}{2.001 - 2}
= \frac{8.012 - 8}{0.001}
= 12
\]

We could also estimate it using the closest point before \( x = 2 \):

\[
f'(2) \approx \frac{f(2) - f(1.999)}{2 - 1.999}
= \frac{8 - 7.988}{0.001}
= 12
\]

QUIZ PREPARATION PROBLEMS

6. For the function \( f(x) = \log(x) \), estimate \( f'(1) \). From the graph of \( \log(x) \), would you expect your estimate to be greater than or less than (the exact value of) \( f'(1) \)?

To estimate \( f'(1) \), we select \( x = 1 \) and a point nearby, say \( x = 1.0001 \), and compute the slope over this short interval:

\[
f'(1) \approx \frac{f(1.0001) - f(1)}{1.0001 - 1}
= \frac{\log(1.0001) - \log(1)}{0.0001}
\approx 0.4343
\]
The graph of $\log(x)$ is concave down, as shown in the graph above. This means that any instantaneous slope estimated by an interval slope will be an underestimation (less steep) compared to the instantaneous slope. So $f'(1) > 0.4343$.

7. Estimate $f'(2)$ for $f(x) = 3^x$. Explain your reasoning.

To estimate $f'(2)$, we select $x = 2$ and a point nearby, say $x = 2.0001$, and compute the slope over this short interval:

$$f'(2) \approx \frac{f(2.0001) - f(2)}{2.0001 - 2}$$

$$= \frac{3^{(2.0001)} - 3^2}{0.0001}$$

$$\approx 9.888$$

14. Show how to represent the following on Figure 2.23.

(a) $f(4)$
(b) $f(4) - f(2)$
(c) $f(5) - f(2)$
(d) $f'(3)$
(a) Height of the function at $x = 4$. 

(b) Difference in height between $f(4)$ and $f(2)$

(c) Slope/average rate between the points at $x = 2$ and $x = 5$.

(d) Slope of the tangent line at $x = 3$

15. For each of the following pairs of numbers, use Figure 2.23 to decide which is larger. Explain your answer.

(a) $f(3)$ or $f(4)$?
(b) $f(3) - f(2)$ or $f(2) - f(1)$?
or or ?

(a) These both represent \( y \) values, and \( f(4) \) is clearly higher than \( f(3) \).

(b) These represent differences in \( y \) values between \( x = 2 \) and \( 3 \), compared to \( x = 1 \) and 2. Since the graph is getting flatter further to the right, that means the graph is growing less quickly, or the same \( x \) distance results in a smaller \( y \) change. Thus,

\[
f(2) - f(1) > f(3) - f(2)
\]

(c) These represent the slopes between \( x = 1 \) and 2, compared to the slope between \( x = 1 \) and 3. By sketching those lines on the graph, the fact that the curve is flattening to the right means that the slope over the longer interval will be lower than the slope over the shorter interval. Thus,

\[
\frac{f(2) - f(1)}{2 - 1} > \frac{f(3) - f(1)}{3 - 1}
\]

(d) These are the instantaneous rates of change / slopes of tangent lines at \( x = 1 \) and \( x = 4 \). Clearly, the graph has a slower rate of change further to the right, so

\[
f'(1) > f'(4)
\]