Week #5 - More About Derivatives

Section 3.1

SUGGESTED PROBLEMS

1. *Let $f(x) = 7$. Using the definition of the derivative, show that $f'(x) = 0$ for all values of x.*

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{7-7}{h}\\ &= \lim_{h\to 0} 0\\ &= 0\end{aligned}$$

For Exercises 3-44, find the derivatives of the given functions. Assume that a, b, c, and k are constants.

5. $y = x^{-12}$

$$y' = -12x^{-13}$$

15. $y = \sqrt{x}$

$$y' = \frac{1}{2}x^{-1/2}$$

19. $f(x) = x^e$

Careful: this is not an exponential function. You need to use the x^n formula for the derivative.

$$f'(x) = e \cdot x^{e-1} \simeq 2.7182 \cdot x^{1.7182}$$

27. $y = 3t^5 - 5\sqrt{t} + \dfrac{7}{t}$

$$y' = 15t^4 - \frac{5}{2}t^{-1/2} - \frac{7}{t^2}$$

33. $y = \dfrac{x^2+1}{x}$

Rewriting y before we differentiate, to avoid using the quotient rule,

$$\begin{aligned} y &= \frac{x^2}{x} + \frac{1}{x} \\ &= x + x^{-1} \end{aligned}$$

Differentiating,

$y' = 1 - x^{-2}$

39. $h(x) = \dfrac{ax+b}{c}$

Remember that a, b, and c are constants, so this is really a linear function, $h(x) = \dfrac{a}{c}x + \dfrac{b}{c}$.

$h'(x) = \dfrac{a}{c} + 0 = \dfrac{a}{c}$

43. $\dfrac{dy}{dx}$ *if* $y = ax^2 + bx + c$

$\dfrac{dy}{dx} = 2ax + b$

QUIZ PREPARATION PROBLEMS

53. *Find the equation of the line tangent to the graph of f at (1, 1), where f is given by $f(x) = 2x^3 - 2x^2 + 1$.*

$f'(x) = 6x^2 - 4x$, so the slope of the tangent line at $x = 1$ is $f'(1) = 6(1) - 4(1) = 2$

The tangent line has slope 2 and goes through (1, 1), so its equation is

$y = 2(x-1) + 1$.

55. *If $f(x) = 4x^3 + 6x^2 - 23x + 7$, find the intervals on which $f'(x) \geq 1$.*

$f'(x) = 12x^2 + 12x - 23$.

Searching for the intervals where $f'(x) > 1$, we set

$$\begin{aligned} 12x^2 + 12x - 23 &\geq 1 \\ 12x^2 + 12x - 24 &\geq 0 \\ 12(x^2 + x - 2) &\geq 0 \\ 12(x+2)(x-1) &\geq 0 \end{aligned}$$

This is true when $x \geq 1$ or $x \leq -2$. Thus, $f'(x) \geq 1$ when $x \in (-\infty, -2) \bigcup (1, \infty)$

57. *On what intervals is the function $f(x) = x^4 - 4x^3$ both decreasing and concave up?*

For 'decreasing', we need to know when $f'(x) < 0$:

$$\begin{aligned} f'(x) = 4x^3 - 12x^2 &< 0 \\ 4(x^3 - 3x^2) &< 0 \\ 4x^2(x-3) &< 0 \end{aligned}$$

Since x^2 is always positive, $f'(x) < 0$ when $x < 3$.

To determine when $f(x)$ is concave up, we set $f''(x) > 0$:

$$\begin{aligned} f''(x) = 12x^2 - 24x &> 0 \\ 12(x^2 - 2x) &> 0 \\ 12x(x-2) &> 0 \end{aligned}$$

Thus, $f''(x)$ is positive when $x > 2$ or $x < 0$.

For the function to be both decreasing and concave up, we must have both conditions satisfied which occurs for $x < 0$ or $2 < x < 3$.

63. *At a time t seconds after it is thrown up in the air, a tomato is at a height of* $f(t) = -4.9t^2 + 25t + 3$ *meters.*

(a) *What is the average velocity of the tomato during the first 2 seconds? Give units.*
(b) *Find (exactly) the instantaneous velocity of the tomato at $t = 2$. Give units.*
(c) *What is the acceleration at $t = 2$?*
(d) *How high does the tomato go?*
(e) *How long is the tomato in the air?*

(a) The average velocity between $t = 0$ and $t = 2$ is given by

$$\begin{aligned} \text{Average velocity} &= \tfrac{f(2)-f(0)}{2-0} = \tfrac{-4.9(2^2)+25(2)+3-3}{2-0} \\ &= \tfrac{33.4-3}{2} = 15.2\text{m/sec.} \end{aligned}$$

(b) Since $f'(t) = -9.8t + 25$, we have

$$\text{Instantaneous velocity} = f'(2) = -9.8(2) + 25 = 5.4\text{m/sec.}$$

(c) Acceleration is given by $f''(t) = -9.8$. The acceleration at $t = 2$ (and all other times) is the acceleration due to gravity, which is -9.8m/sec^2.

(d) We can use a graph of height against time to estimate the maximum height of the tomato. See the figure below. Alternatively, we can find the answer analytically. The maximum height occurs when the velocity is zero and $v(t) = -9.8t + 25 = 0$ when $t = 2.6$ seconds. At this time the tomato is at a height of $f(2.6) = 34.9$. The maximum height is therefore 34.9 meters.

(e) We see in the figure that the tomato hits ground at about $t = 5.2$ seconds. Alternatively, we can find the answer analytically. The tomato hits ground when

$$f(t) = 4.9t^2 + 25t + 3 = 0.$$

67. *What is the formula for V, the volume of a sphere of radius r? Find dV/dr. What is the geometrical meaning of dV/dr?*

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$. Notice that this is the same as the surface of a sphere.

One interpretation of dV/dr is that for a small increase in radius of Δr, the volume of the sphere will increase by $4\pi r^2 \Delta r$. Since $4\pi r^2$ is the surface area of a sphere, this is like the adding a thin layer Δr thick on outside of the sphere.

69. *Given a power function of the form $f(x) = ax^n$, with $f'(2) = 3$ and $f'(4) = 24$, find n and a.*

$f'(x) = nax^{n-1}$, so

$f'(2) = na2^{n-1} = 3$ and $f'(4) = na4^{n-1} = 24$. Dividing the second by the first, we get

$$\begin{aligned}
\frac{na4^{n-1}}{na2^{n-1}} &= \frac{24}{3} \\
\frac{4^{n-1}}{2^{n-1}} &= 8, \text{ but } 4^{n-1} = (2^2)^{n-1} = 2^{2n-2}, \\
\frac{2^{2n-2}}{2^{n-1}} &= 8 \\
2^{n-1} &= 8
\end{aligned}$$

and since $2^3 = 8$, n must be 4.

Given that $n = 4$, $na2^{n-1} = 3$ means

$$\begin{aligned}
4a2^{4-1} &= 3 \\
32a &= 3 \\
a &= 3/32
\end{aligned}$$

Thus, $f(x) = \frac{3}{32}x^4$.