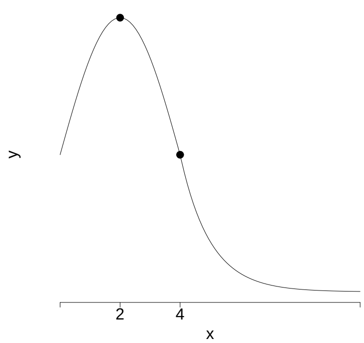
Week #6 - Taylor Series, Derivatives and Graphs

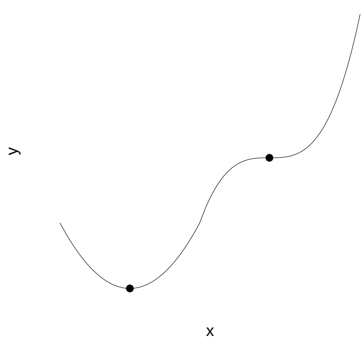
Section 4.1

SUGGESTED PROBLEMS

1. *Graph a function which has exactly one critical point, at* $x = 2$*, and exactly one inflection point, at* $x = 4$*.*



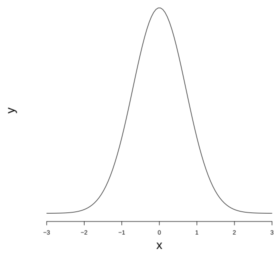
2. *Graph a function with exactly two critical points, one of which is a local minimum and the other is neither a local maximum nor a local minimum.*



3. (a) *Use a graph to estimate the* x*-values of any critical points and inflection points of* $f(x) = e^{-x^2}$ *.*

   (b) *Use derivatives to find the* x*-values of any critical points and inflection points exactly.*

   Part a) can be a bit challenging: it would be easier if doing this problem by hand to start with (b), then use that information to plot the graph. However, the graph of e^{-x^2} is one we've seen before, and so should be somewhat familiar:



It appears to have one critical point at $x = 0$, and two matched points of inflection on either side of $x = 0$.

Part b) can be done with differentiation.

$$\begin{aligned} f'(x) &= -2xe^{-x^2} \\ f''(x) &= -xe^{-x^2} - 2xe^{-x^2}(-2x) \\ &= -2e^{-x^2} + 4x^2e^{-x^2} \\ &= e^{-x^2}(4x^2 - 2) \end{aligned}$$

Because the e^{-x^2} term is always > 0, the only critical point, where $f'(x) = 0$ is where $x = 0$.

If we look at $f''(x) = 0$, those would be candidate points for the point of inflection. $f''(x) = 0$ when

$$\begin{aligned} 4x^2 - 2 &= 0 \\ x^2 &= 1/2 \\ x &= \pm\sqrt{1/2} \approx \pm 0.707 \end{aligned}$$

From the graph, we can see that these are in fact the points of inflection, as the graph changes from concave up to concave down, or vice-versa. We could verify that by checking the sign of $f''(x)$ on either side of $\pm\sqrt{1/2}$.

24. *Suppose f has a continuous derivative whose values are given in the following table.*

   (a) *Estimate the x-coordinates of critical points of f for $0 \leq x \leq 10$.*

   (b) *For each critical point, indicate if it is a local maximum of f, local minimum, or neither.*

x	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$f'(x)$	*5*	*2*	*1*	*-2*	*-5*	*-3*	*-1*	*2*	*3*	*1*	*-1*

   (a) For a function which has a continuous derivative, critical points occur when the derivative is zero, because there aren't any points where the derivative is undefined. If the derivative is continuous, then there must be at least one critical point between two points with different f' signs.

   Critical points occur between ($x = 2$ and $x = 3$), ($x = 6$ and $x = 7$), and ($x = 9$ and $x = 10$). Since we have no information besides the table, estimates like $x = 2.5$, $x = 6.5$ and $x = 9.5$ would be sufficient.

(b) Near the critical point between $x = 2$ and $x = 3$, the derivatives are going from positive to negative, so it's a local maximum (first derivative test).

Near the critical point between $x = 6$ and $x = 7$, the derivatives are going from negative to positive, so it's a local minimum.

Near the critical point between $x = 9$ and $x = 10$, the derivatives are going from positive to negative, so it's a local maximum.

34. *If water is flowing at a constant rate (i.e., constant volume per unit time) into the Grecian urn in Figure 4.20, sketch a graph of the depth of the water against time. Mark on the graph the time at which the water reaches the widest point of the urn.*

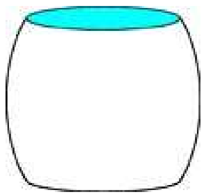
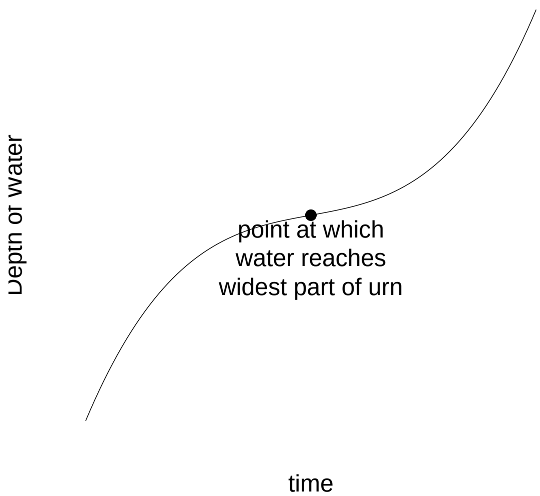


Figure 4.20



35. *Graph f given that:*

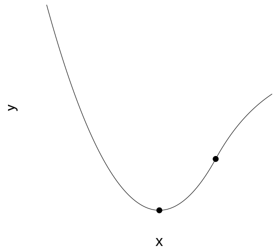
$f'(x) = 0$ *at* $x = 2$, $f'(x) < 0$ *for* $x < 2$, $f'(x) > 0$ *for* $x > 2$.

$f''(x) =$ *at* $x = 4$, $f''(x) > 0$ *for* $x < 4$, $f''(x) < 0$ *for* $x > 4$.

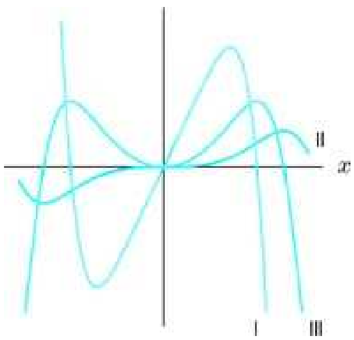
The key information tells you that

- $x = 2$ is a local minimum (first derivative test)
- $x = 4$ is a point of inflection, from concave up to concave down

A possible graph of this function is shown below.

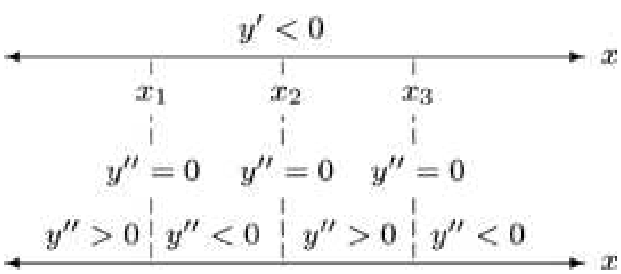


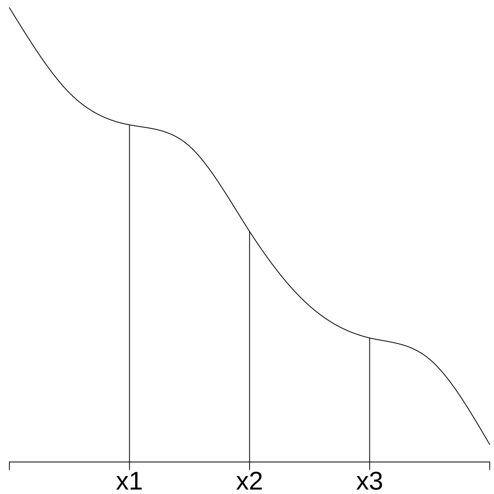
Problems 40-41 show graphs of f, f', f''. Each of these three functions is either odd or even. Decide which functions are odd and which are even. Use this information to identify which graph corresponds to f, which to f', and which to f''.

41. 

Since the derivative of an even function is odd, and the derivative of an odd function is even, f and f'' are either both odd or both even, and f' is the opposite. Graph I and II represent odd functions; III represents an even function, so III is f'. Since the maxima and minima of III occur where I crosses the x-axis, I must be the derivative of f', that is, f''. In addition, the maxima and minima of II occur where III crosses the x-axis, so II is f.

For Problems 42-45, sketch a possible graph of $y = f(x)$, using the given information about the derivatives $y' = f'(x)$ and $y'' = f''(x)$. Assume that the function is defined and continuous for all real x.

43. 



QUIZ PREPARATION PROBLEMS

Classify the critical points of the functions in Exercises 6-7 as local maxima or local minima.

7. $h(x) = x + 1/x$.

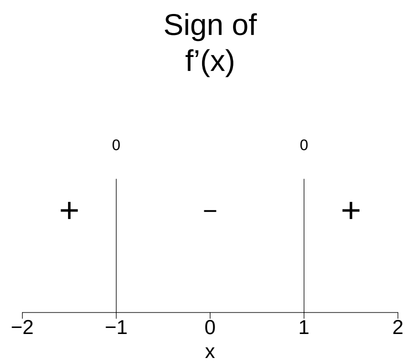
$$h'(x) = 1 - \frac{1}{x^2}$$

Critical points occur when the graph is defined and $h'(x) = 0$ or $h'(x)$ is undefined. The only place that $h'(x)$ is undefined is at $x = 0$, but $h(x)$ is undefined there as well, so we don't consider it.

Setting $h'(x) = 0$, we identify the critical points:

$$\begin{aligned} 1 - \frac{1}{x^2} &= 0 \\ \frac{1}{x^2} &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

There are two critical points, $x = -1$ and $x = 1$. Doing a sketch of the sign of $f'(x) = 1 - 1/x^2$ around these points, we get



From the first derivative test, $x = -1$ is a local maximum, while $x = 1$ is a local minimum.

17. *Indicate all critical points on the graph of f in Figure 4.15 and determine which correspond to local maxima of f, which to local minima, and which to neither.*

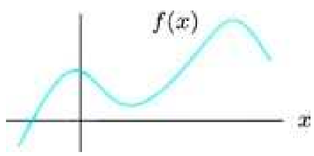
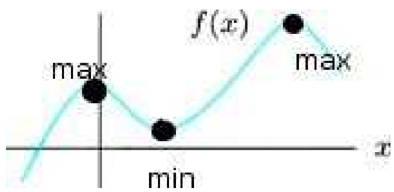


Figure 4.15



18. *Indicate on the graph of the derivative function $f'(x)$ in Figure 4.16 the x-values that are critical points of the function f itself. At which critical points does f have local maxima, local minima, or neither?*

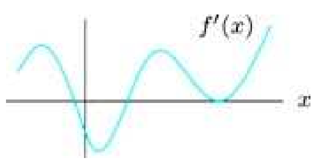
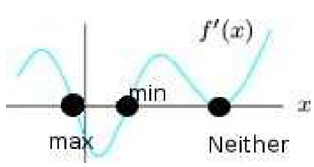


Figure 4.16

The critical points of f are the zeros of f'. Using the first derivative test, we can identify the nature of each critical point. See the graph below.



19. *Indicate on the graph of the derivative f' in Figure 4.17 the x-values that are inflection points of the function f.*

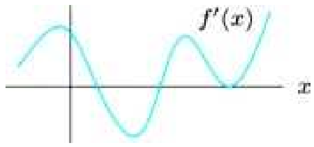
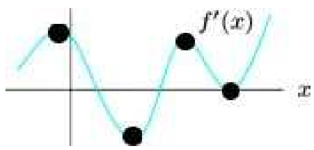


Figure 4.17

Remember that we are still dealing with the graph of $f'(x)$, not $f(x)$.

To find inflection points of the function f we must find points where f'' changes sign. However, because f'' is the derivative of f', any point where f'' changes sign will be a local maximum or minimum on the graph of f'. See the graph below.



20. *Indicate on the graph of the second derivative f'' in Figure 4.18 the x-values that are inflection points of the function f.*

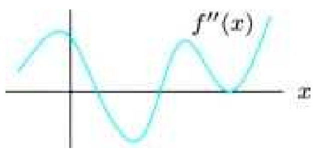
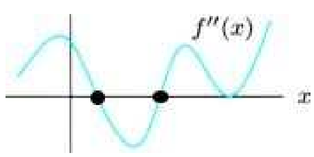


Figure 4.18

A point of inflection occurs where f'' changes sign, *not* just when it equals zero. The third point where $f''(x) = 0$ is not a point of inflection because f'' is positive on either side of it.)



27. *Find values of a and b so that the function $f(x) = x^2 + ax - b$ has a local minimum at the point (6, -5).*

First, we wish to have $f'(6) = 0$, since $f(6)$ should be a local minimum:

$$\begin{aligned} f'(x) &= 2x + a = 0 \\ x &= -\frac{a}{2} = 6 \\ a &= -12. \end{aligned}$$

Next, we need to have $f(6) = -5$, since the point (6,-5) is on the graph of $f(x)$. We can substitute $a = -12$ into our equation for $f(x)$ and solve for b:

$$f(x) = x^2 - 12x + b$$
$$f(6) = 36 - 72 + b = -5$$
$$b = 31.$$

Thus, $f(x) = x^2 - 12x + 31$.

33. *If water is flowing at a constant rate (i.e., constant volume per unit time) into the vase in Figure 4.19, sketch a graph of the depth of the water against time. Mark on the graph the time at which the water reaches the corner of the vase.*

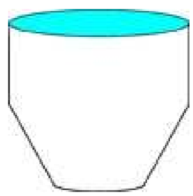
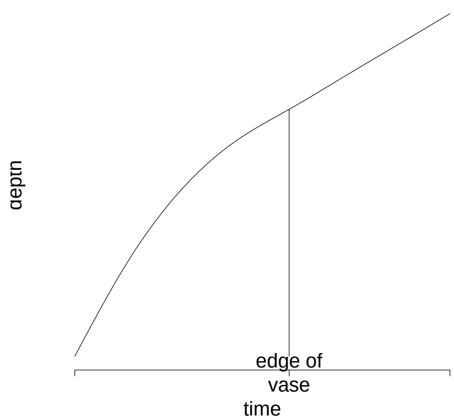


Figure 4.19

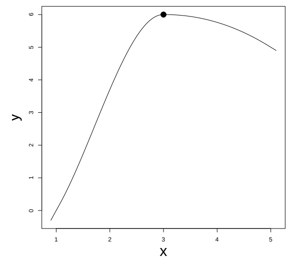
As the urn gets wider, the rate of change of height (slope on the depth vs time graph) will get lower, though still positive. Once the water reaches the corner of the vase, the size of the vase becomes constant, and so too would the rate of filling, and the slope of the graph.



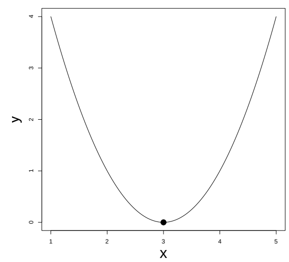
36. *Assume f is differentiable everywhere and has just one critical point, at $x = 3$. In parts (a)-(d), you are given additional conditions. In each case decide whether $x = 3$ is a local maximum, a local minimum, or neither. Explain your reasoning. Sketch possible graphs for all four cases.*

   (a) $f'(1) = 3$ *and* $f'(5) = -1$
   (b) $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = \infty$
   (c) $f(1) = 1$, $f(2) = 2$, $f(4) = 4$, $f(5) = 5$
   (d) $f'(2) = -1$, $f(3) = 1$, $\lim_{x\to\infty} f(x) = 3$

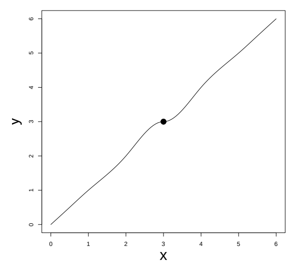
(a) $x = 3$ is a local maximum because $f(x)$ is increasing when $x < 3$ and decreasing when $x > 3$.



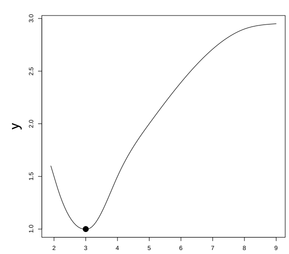
(b) $x = 3$ is a local minimum because $f(x)$ heads to infinity to either side of $x = 3$.



(c) $x = 3$ is neither a local minimum nor maximum, as $f(1) < f(2) < f(4) < f(5)$.



(d) $x = 3$ is a local minimum because $f(x)$ is decreasing to the left of $x = 3$ and must increase to the right of $x = 3$, as $f(3) = 1$, and eventually $f(x)$ must become close to 3.



47. *Let f be a function with $f(x) > 0$ for all x. Set $g = 1/f$.*

(a) *If f is increasing in an interval around x_0, what about g?*

(b) *If f has a local maximum at x_1, what about g?*

(c) *If f is concave down at x_2 , what about g?*

Using the chain rule,

$$\begin{aligned}
\text{if } g(x) &= [f(x)]^{-1} \\
\text{then } g'(x) &= -(f(x))^{-2} \cdot f'(x) \\
\text{and } g''(x) &= \left[2(f(x))^{-3} \cdot f'(x)\right] \cdot f'(x) - (f(x))^{-2} \cdot f''(x) \\
&= \frac{1}{(f(x))^2} \left(\frac{2(f'(x))^2}{f(x)} - f''(x) \right)
\end{aligned}$$

(a) If $f'(x_0) > 0$, and $f(x) > 0$, then $g'(x_0)$ must be negative.

(b) If $f'(x_1) = 0$, then $g'(x_1)$ must be also be zero, so it is a critical point. However, near x_1, if $f'(x)$ is going from negative to positive (making x_1 a local minimum for f), the $g'(x)$ must be going from positive to negative, making x_1 a local maximum for $g(x)$.

(c) Since f is concave down at x_2, $f''(x_2) < 0$. Looking factor by factor,

$$\begin{aligned}
(f(x_2))^2 &> 0 \\
\frac{2(f'(x_2))^2}{f(x_2)} &> 0 \\
\text{and } -f''(x_2) &> 0
\end{aligned}$$

so overall $g''(x_2) > 0$.