Week #7 - Maxima and Minima, Concavity, Applications

Section 4.4

From "Calculus, Single Variable" by Hughes-Hallett, Gleason, McCallum et. al.

SUGGESTED PROBLEMS

1. *Total cost and revenue are approximated by the functions* $C = 5000 + 2.4q$ *and*$R = 4q$, *both in dollars. Identify the fixed cost, marginal cost per item, and the price at which this commodity is sold.*

   The fixed cost (part not depending on units produced) is $5,000. The marginal cost per item is $2.4 per item, and since the revenue is $R = 4q$, the items are being sold at $4 per item.

5. *Figure 4.56 shows cost and revenue. For what production levels is the profit function positive? Negative? Estimate the production at which profit is maximized.*

   Figure 4.56

   The production levels corresponding to a profit are those points where the $C(q)$ curve lies above the $R(q)$ curve. This occurs for $\sim 6 < q \sim 13$. The company loses money for $0 < q <\sim 6$ and $\sim 13 < q < 15$.

   From a visual inspection of the graphs, the separation between the $C(q)$ and $R(q)$ curves is maximized when $q \simeq 9$.

9. *Table 4.1 shows cost,* $C(q)$*, and revenue,* $R(q)$*,*

   (a) *At approximately what production level,* q*, is profit maximized? Explain your reasoning.*

   (b) *What is the price of the product?*

   (c) *What are the fixed costs?*

   Table 4.1

q	0	500	1000	1500	2000	2500	3000
$R(q)$	0	1500	3000	4500	6000	7500	9000
$C(q)$	3000	3800	4200	4500	4800	5500	7400

   (a) The most systematic way to find the maximum profit production is to compute $P = R - C$ over the table:

q	0	500	1000	1500	2000	2500	3000
$P(q)$	-3000	-2300	-1200	0	1200	2000	1600

   From this calculation, the profit will be maximized for production of $q \approx 2500$ units.

   (b) The price only affects the revenue, and every 500 units increases revenue by $1,500. Therefore, the units are being sold for $3 each.

(c) The fixed costs are the costs incurred before the first unit is sold. Clearly this is \$3,000.

13. *The marginal revenue and marginal cost for a certain item are graphed in Figure 4.59. Do the following quantities maximize profit for the company? Explain your answer.*

    (a) $q = a$

    (b) $q = b$

    Figure 4.59

    (a) Up to $q = a$, the marginal cost of each unit has outweighed the marginal revenue. In other words, up to this point, every unit sold has been money lost. This cannot be the point of maximum profit, but rather of maximum loss.

    (b) Between $q = a$ and $q = b$, the company has been making profits on each unit. After $q = b$, the company begins to lose money for additional units made, so this point is the point of maximum profit.

16. *(a) A cruise line offers a trip for \$1000 per passenger. If at least 100 passengers sign up, the price is reduced for all the passengers by \$5 for every additional passenger (beyond 100) who goes on the trip. The boat can accommodate 250 passengers. What number of passengers maximizes the cruise line's total revenue? What price does each passenger pay then?*

    (b) The cost to the cruise line for q passengers is $40,000 + 200q$. *What is the maximum profit that the cruise line can make on one trip? How many passengers must sign up for the maximum to be reached and what price will each pay?*

    (a) We have two scenarios: fewer than 100 passengers, or more than 100 passengers. Clearly, if there are fewer than 100 passengers, the revenue is maximized if as many people buy tickets as possible, or 100 passengers $\cdot$ \$1000 $=$ \$100,000 revenue.
    If more than 100 people buy tickets, say q tickets, the revenue would be given by

$$\begin{aligned} R(q) &= (\$ \text{ per ticket}) \cdot (\# \text{ tickets sold}) \\ &= (1000 - 5 \underbrace{(q - 100)}_{\text{num above 100 sold}})(q) \\ &= 1000q - 5q^2 + 500q = 1500q - 5q^2 \end{aligned}$$

To maximize, we take the derivative of R with respect to q and set it equal to zero:

$$\begin{aligned} R'(q) &= 1500 - 10q \\ \text{set equal to zero: } 1500 - 10q &= 0 \\ q &= 150 \end{aligned}$$

Again, because R is a parabola opening downwards, this single critical point is a local maximum for revenue, and leads to a revenue of $R(150) = \$112,500$. This is clearly larger than the revenue for selling 100 tickets at a fixed price. So the boat line will maximize its ticket revenue by selling 150 tickets at $1000 - 5(150 - 50) = \$750$ per ticket.

(b) We again have two cases: Ticket sales ≤ 100 tickets:

$$P(q) = \underbrace{(\$1,000q)}_{\text{revenue}} - \underbrace{(40,000 + 200q)}_{\text{cost}}$$
$$P(100) = 1000(100) - 40,000 - 200(100) = \$40,000 \text{ profit}$$

If ticket sales are above 100,

$$\begin{aligned} P(q) &= \underbrace{(\$1,000 - 5(q-100)q)}_{\text{revenue}} - \underbrace{(40,000 + 200q)}_{\text{cost}} \\ &= -5q^2 + 1300q - 40000 \\ P' &= 1300 - 10q \\ \text{Setting } P' = 0,\ 1300 - 10q &= 0 \\ q &= 130 \end{aligned}$$

Since $P(q)$ is an inverted parabola, this critical point is a global maximum of $P(q)$. The optimal number of tickets to sell is 130, at \$850 per ticket. This gives a profit of \$44,500, which is better than the optimal fixed-ticked-price profit of \$40,000.

22. *A reasonably realistic model of a firm's costs is given by the short-run Cobb-Douglas cost curve*

$C(q) = Kq^{1/a} + F$

where a is a positive constant, F is the fixed cost, and K measures the technology available to the firm.

(a) Show that C is concave down if $a > 1$.

(b) Assuming that $a < 1$ and that average cost is minimized when average cost equals marginal cost, find what value of q minimizes the average cost.

(a) To determine concavity, we need to find C'':

$$\begin{aligned} C(q) &= Kq^{1/a} + F \\ C'(q) &= K\left(\frac{1}{a}\right) q^{(1/a)-1} \\ C''(q) &= K\left(\frac{1}{a}\right)\left(\frac{1}{a} - 1\right) q^{(1/a)-2} \end{aligned}$$

- Since $K > 0$, the first factor is positive
- Since $a > 1$, the second factor is positive
- Since $a > 1$, (1/a ¡ 1) so the third factor is negative
- q^x is positive for any exponent x

Overall, the sign of C'' is negative, so $C(q)$ is concave down everywhere if $a > 1$.

(b)

$$\text{Average cost } a(q) = \frac{C(q)}{q} = Kq^{(1/a)-1} + Fq^{-1}$$

$$\text{Marginal cost } MC(q) = C'(q) = K\left(\frac{1}{a}\right)q^{(1/a)-1}$$

Setting $a(q) = MC(q)$:

$$Kq^{(1/a)-1} + Fq^{-1} = K\left(\frac{1}{a}\right)q^{(1/a)-1}$$

Solve for q: (Careful about the exponents!)

$$q^{-1}\left(q^{(1/a)} + \frac{F}{K}\right) = q^{-1}\left(\frac{1}{a}\right)q^{(1/a)}$$

Cancel q^{-1}, group $q^{1/a}$ terms:

$$\left(\frac{1}{a} - 1\right)q^{(1/a)} = \frac{F}{K}$$

$$\left(\frac{1-a}{a}\right)q^{(1/a)} = \frac{F}{K}$$

$$q^{(1/a)} = \left(\frac{a}{1-a}\right)\frac{F}{K}$$

$$q = \left(\frac{Fa}{K(1-a)}\right)^{a}$$

This number of items will generate the optimal average cost (occurs when the marginal cost = average cost).

QUIZ PREPARATION PROBLEMS

3. *The revenue from selling q items is $R(q) = 500q - q^2$ and the total cost is $C(q) = 150 + 10q$. Write a function that gives the total profit earned, and find the quantity which maximizes the profit.*

Total profit = Revenue - Cost = $R(q) - C(q) = (500q - q^2) - (150 + 10q)$. Simplifying, we see that

$$\text{Profit} = P = -q^2 + 490q - 150$$

The maximum profit occurs when $\frac{dP}{dq} = 0$:

$$\frac{dP}{dq} = -2q + 490$$

$$\text{Setting } \frac{dP}{dq} = 0, \text{ we get}$$

$$-2q + 490 = 0$$

$$q = 245$$

Since the shape of the profit function is a parabola opening downwards (because of the minus sign in front of the q^2 term in P), this single critical is maximum. Therefore, the profit will be maximized when $q = 245$ units are produced.

7. *When production is 2000, marginal revenue is \$4 per unit and marginal cost is \$3.25 per unit. Do you expect maximum profit to occur at a production level above or below 2000? Explain.*

   Since the marginal cost is currently below the marginal revenue, selling a few more units will increase profits. Therefore, the optimal production level should be above 2,000 units.

15. *The total cost $C(q)$ of producing q goods is given by:*

    $C(q) = 0.01q^3 - 0.6q^2 + 13q$

    (a) *What is the fixed cost?*

    (b) *What is the maximum profit if each item is sold for \$7? (Assume you sell everything you produce.)*

    (c) *Suppose exactly 34 goods are produced. They all sell when the price is \$7 each, but for each \$1 increase in price, 2 fewer goods are sold. Should the price be raised, and if so by how much?*

    (a) The fixed cost is zero: $C(0) = 0$.

    (b) $P = R(q) - C(q) = (7q) - (0.01q^3 - 0.6q^2 + 13q)$ To find the value of q that maximizes profit, we differentiate and set the derivative equal to zero:

    $$\begin{aligned} \frac{dP}{dq} &= 7 - 0.03q^2 + 1.2q - 13 \\ &= -6 + 1.2q - 0.03q^2 \end{aligned}$$

    Set this equal to zero, and solve using the quadratic formula:

    $$\begin{aligned} -6 + 1.2q - 0.03q^2 &= 0 \\ q &= \frac{-1.2 \pm \sqrt{1.44 - 4(-6)(-.03)}}{2(-0.03)} \\ q &\approx 34.1, \quad 5.9 \end{aligned}$$

    Using the second derivative test, with $\frac{d^2P}{dq^2} = P'' = 1.2 - 0.06q$,

    $$\begin{aligned} P''(5.9) &\approx 0.846, \text{ a local minimum} \\ P''(34.1) &\approx -0.846, \text{ a local maximum} \end{aligned}$$

    From this analysis, and the fact that the profit declines towards $-\infty$ as the production increases unboundedly, we see that the maximum profit occurs at $q \approx 34.1$.

Rounding to the nearest integer units, we should produce $q = 34$ units. This will give

$$\begin{aligned}
\text{Revenue: } & R(34) = 7(34) = 238 \\
\text{Costs: } & C(34) = (0.01)(34)^3 - 0.6(34)^2 + 13(34) = 141.44 \\
\text{Profit: } & R(34) - C(34) = 234 - 141.44 = \$92.56
\end{aligned}$$

(c) At $q = 34$, the net cost of production is \$141.44. The revenue gets more complicated in this situation. Let x be the amount of price increase above \$7, and find the revenue:

$$\begin{aligned}
R &= (\#\text{ units sold}) \cdot (\text{price}) \\
&= (34 - 2x)(7 + x) = 238 + 20x - 2x^2
\end{aligned}$$

Since the costs are fixed (we're producing 34 units regardless of the price), we want to maximize the revenue. Using derivatives,

$$\begin{aligned}
\frac{dR}{dx} &= 20 - 4x \\
\text{This equals zero when } 20 - 4x &= 0 \\
x &= 5
\end{aligned}$$

If $x = 5$, the revenue will be \$288, higher than the original revenue of \$238. Since $R(x)$ is an inverted parabola, this critical point must be a global maximum of revenue. Therefore, the units should be sold at \$(7+5) = \$12 per unit to maximize the profit.

17. *A company manufactures only one product. The quantity, q, of this product produced per month depends on the amount of capital, K, invested (i.e., the number of machines the company owns, the size of its building, and so on) and the amount of labor, L, available each month. We assume that q can be expressed as a Cobb-Douglas production function:*

$q = cK^\alpha L^\beta$

where c, α, β are positive constants, with $0 < \alpha < 1$ and $0 < \beta < 1$. In this problem we will see how the Russian government could use a Cobb-Douglas function to estimate how many people a newly privatized industry might employ. A company in such an industry has only a small amount of capital available to it and needs to use all of it, so K is fixed. Suppose L is measured in man-hours per month, and that each man-hour costs the company w rubles (a ruble is the unit of Russian currency). Suppose the company has no other costs besides labor, and that each unit of the good can be sold for a fixed price of p rubles. How many man-hours of labor per month should the company use in order to maximize its profits?

We want to maximize $profit$ by choosing L, given that c, α, β, and K are constants. We do so using our usual derivative approach:

$$\begin{aligned}
\text{Cost} &= wL \\
\text{Revenue} &= pq = p(cK^\alpha L^\beta) \\
\text{Profit} = P &= pcK^\alpha L^\beta - wL \\
P'(L) &= pcK^\alpha(\beta L^{(\beta-1)}) - w \\
\text{Solve for } P' = 0: \quad L^{\beta-1} &= \frac{w}{pc\beta K^\alpha} \\
L &= \left(\frac{w}{pc\beta K^\alpha}\right)^{\frac{1}{\beta-1}}
\end{aligned}$$

To show that this single critical point is a maximum, we use the first derivative test, reminding ourselves that $0 < \beta < 1$, so $(\beta - 1) < 0$:

$$P'(L) = pcK^\alpha(\beta L^{(\beta-1)}) - w$$

Since $(\beta - 1) < 0$, the L factor is essentially in the denominator. This means that

$$\begin{aligned}
L \to 0 &\Rightarrow P' \to +\infty \\
L \to \infty &\Rightarrow P' \to -w
\end{aligned}$$

Since there is only once critical point ($P' = 0$), and P' is positive to the left and negative to the right of this critical point, the critical point is a global maximum.

21. *The average cost per item to produce q items is given by* $a(q) = 0.01q^2 - 0.6q + 13$, *for* $q > 0$.

    (a) *What is the total cost,* $C(q)$, *of producing q goods?*
    (b) *What is the minimum marginal cost? What is the practical interpretation of this result?*
    (c) *At what production level is the average cost a minimum? What is the lowest average cost?*
    (d) *Compute the marginal cost at* $q = 30$. *How does this relate to your answer to part (c)? Explain this relationship both analytically and in words.*

    (a) Total cost = avg cost per item $\times$ number of items = $(0.01q^2 - 0.6q + 13)q$
    (b) From (a),

    $$C(q) = 0.01q^3 - 0.6q^2 + 13q$$

    $$\text{Marginal cost} = MC(q) = C'(q) = 0.03q^2 - 1.2q + 13$$

    To find minimial marginal cost, we differentiate $MC(q)$ and set it equal to zeroSet $MC'(q) = 0.0$

    $$q = 20$$

    This critical point must be a global minimum for $MC(q)$, because $MC(q)$ is a parabola opening upwards. The marginal cost associated with $q = 20$ is $MC(20) = 0.03(20)^2 - 1.2(20) + 13 = \$1$ per unit.

(c) Similar to (b), but finding the minimum of $a(t) = 0.01q^2 - 0.6q + 13$:

$$\begin{aligned} a'(q) &= 0.02q - 0.6 \\ \text{Set } a'(q) &= 0.02q - 0.6 = 0 \\ q &= 30 \end{aligned}$$

The minimal average cost occurs when $q = 30$ items, and is $a(30) = \$4$ per item.

(d) The marginal cost at $q = 30$ is $MC(30) = 0.03(30)^2 - 1.2 * 30 + 13 = 4$. The correspondance is that the *minimum* average cost will occur when the average cost equals the marginal (instantaneous average) cost. After this point, the marginal cost per item is higher than 4, so the average cost must increase. Before this point, the marginal cost is lower than 4, so it is worth producing more items to lower the average cost.